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ON A STOCHASTIC INTEREST RATE MODEL FOR BONDS

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Abstract: Assuming a certain stochastic evolution of the interest rate B_t we consider the problem of how to prize a security whose payoff at maturity time T is a function which depends only on B_T. We show how to generalize the theory of Cox et al [1] to a situation with time dependent coefficients and arbitrary payoff. The solution to this problem is given in terms of "killed" Brownian motion.

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1. Introduction

In this paper we aim to generalize the classical theory of Cox et al [1] regarding price of bonds in stochastic markets (see also Karatzas and Shreve [4] and Hull [2]). We shall consider a market in which the money lending interest rate $B(t, \omega)$ evolves randomly according to the stochastic differential equation (SDE)

$$B_t = B_0 + \int_{t_0}^{t} a(s, B_s) ds + \int_{t_0}^{t} b(s, B_s) dW_s \,, \tag{1}$$

where W_t is the classical Brownian motion (BM) process. Here the coefficients $a(t, x)$ and $b(t, x)$ are the drift and variance coefficients – which represent, respectively, the underlying tendency and the volatility of B_t.

We shall consider the problem of how to prize a security whose payoff at

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maturity time T is a function $\Theta(B_T)$ which depends only on B_T. Here $\Theta : \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous function. Setting $\Theta = 1$ yields the price at t, $p(T/t, x)$, of a bond with maturity T; for *options* on bonds at T with exercise price k and maturity τ, $\Theta(x) = (p(\tau/T, x) - k)^+$. Let $v(T/t, B_t) \equiv v(t, B_t)$ represent the actual time $t \le T$ price of this financial instrument given that at (actual) time t we observe B_t. $v(t, x) \equiv v(t, B_t = x)$ is given by

$$v(t, x) = E\big(\Theta(B_T) e^{-\int_t^T B(s)ds} | B_t = x\big) . \tag{2}$$

The simplest examples is the bond in which individuals lend money to institutions and receive a payoff \$ $\Theta = 1$ at T. To understand how the latter formula comes about, notice that any initial amount of money \$ x_0 deposited at time t in an instant "savings account" will yield \$ $x_0 e^{\int_t^T B(s)ds} \equiv x_0 e^{Z_t^T}$ at T. The price of the bond must be $p(T/t, B_t) = E\big(e^{-Z_t^T} | B_t\big)$ since \$ $e^{-Z_t^T}$ at t produces, in an instant "savings account", \$ 1 at T.

Bond pricing aims to determine the basic pair of processes $(B_t, v(T/t,\, B_t))$. Unfortunately, very seldom it is possible to obtain such a strong representation; hence one usually sets out to obtain $v(T/t, B_t)$ and the transition density $f(T, x|s, z)dx \equiv P(B_T \in dx | B_s = z)$ of B_t when the process starts from z at time s. Notice that the latter object solves the Kolmogorov's forward equations

$$\frac{\partial f}{\partial T} = \frac{1}{2} \frac{\partial^2 (b^2 f)}{\partial x^2} - \frac{\partial (af)}{\partial x} , \tag{3}$$

with initial condition $\lim_{T \to s} f(T, x|s, z) = \delta(z - x)$.

To recover $v(t, x) \equiv v(T/t, x)$ we note that, if Θ is continuous and depends only on B_T, then Feynman-Kac formula (see for example Karatzas and Shreve [4]) establishes that the right hand side of (2) solves the final value problem

$$\frac{b^2}{2} \frac{\partial^2 v}{\partial x^2} + a(t, x) \frac{\partial v}{\partial x} - xv + \frac{\partial v}{\partial t} = 0 , \quad t < T, \; v(T, x) = \Theta(x) . \tag{4}$$

We suppose that $a(t, x) = q(t) - 2r(t)x$ and $b(t, x) = \gamma(t)\sqrt{x}$, where $q(t), r(t)$ and $\gamma(t)$ are arbitrary functions of time. In the case when γ, q, r and Θ are constants (bond pricing with time homogeneous differential interest rates), the solution to these equations was found in [1]. It is remarkable that when $q(t) = \gamma^2(t)/4$ but $\gamma(t), r(t)$ and $\Theta(x)$ are otherwise arbitrary it is possible to recover with all generality both $v(T/t, B_t)$ and also B_t. Thus, the generalization considered here allows valuing *arbitrary* interest rate securities in random markets where the drift and volatility change in time. We note that numerical integration procedures for studying the problem of bond pricing within the framework of the generalized time dependent model of Cox et al have been carried out in [3]. For the generalized Vasicek model – where the basic assumption is that the interest rate evolves as a time dependent Ornstein-Uhlenbeck process, some previous analytical results appear in [4, 5] .

2. Determination of the Interest Rate and Valuation Processes

2.1. Time Dependent CIR Model

As it has been pointed out we aim to solve the equations (1), (4) which determine the relevant valuation process. We have found that if $q(t) = \gamma^2(t)/4$ and $\gamma(t), r(t)$ and $\Theta(x)$ are arbitrary there exists a transformation which permits solving these equations. Suppose then that the interest rate B_t evolves according to the nonlinear Ito's SDE

$$dB_t = \left(\gamma^2(t)/4 - 2r(t)B_t\right) dt + \gamma(t)\sqrt{B_t}dW_t\,, \quad B(t_0) = B_0\,. \tag{5}$$

Given the functions $r(t), \gamma(t)$ we also define

$$R(t) \equiv \int^t r(z)dz \quad \text{and} \quad \tilde{W}_t \equiv \int_{t_0}^t \gamma(z)e^{R(z)}dW_z \tag{6}$$

and new coordinates $(x,t) \to (\hat{x},\hat{t})$ as follows

$$\begin{aligned} \hat{x} &\equiv h(t,x) = 2\sqrt{x}e^{R(t)}, \\ \hat{t} &\equiv \varphi(t) = \int^t \gamma^2(z)e^{2R(z)}dz \;\; \text{and also} \;\; \hat{T} \equiv \varphi(T)\,. \end{aligned} \tag{7}$$

For any function of time $\psi(t)$, set $\hat{\psi}(\hat{t}) \equiv \psi(\varphi^{-1}(\hat{t}))$ and, more generally, define the function $\hat{v}$ by

$$\hat{v}(\hat{t},\hat{x}) \equiv \hat{v}(\varphi(t), h(t,x)) = v(t,x) \tag{8}$$

To continue further we first recall a definition: Given $k : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, we call BM killed with rate $k(t,x)$ to $\breve{W}_t = \partial$ if $t \geq \mathcal{T}$ and $\breve{W}_t = W_t$ if $t < \mathcal{T}$. Here ∂ is a "death state", W_t is standard BM and, given the path $W_s, s \leq t$, $\mathcal{T}$ is exponentially distributed:

$$P(\breve{W}_t \neq \partial) = P(\mathcal{T} > t | W_s, s \leq t) = e^{-\int_0^t k(s,W_s)ds}\,.$$

Feynman-Kac's formula shows again that $P(\breve{W}_T \in dX|\breve{W}_t = x) = G(t, X|t, x)dX$, where $G(T, X|t, x)$ solves

$$\begin{aligned} &\left(\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t} - k(t,x)\right) G(T,X|t,x) = 0, \\ &\lim_{t\uparrow T} G(T,X|t,x) = \delta(X-x) \end{aligned} \tag{9}$$

(in this regard see for example Karatzas and Shreve [4] or Karlin and Taylor [5]). Here $\delta(X - x)$ is the Dirac unit mass located at $x = X$. One has the following theorem.

Theorem 1. *(i) Suppose γ, r are functions of class C^1 and C^0, respectively. Then, a strong solution to (5) is given by*

$$B_t = \frac{e^{-2R(t)}}{4}\left(\tilde{W}_t + 2e^{R(t_0)}\sqrt{B_0}\right)^2 . \tag{10}$$

(ii) The value of a claim with exercise value $\Theta(B_T)$ is given by

$$v(t,x) = e^{R(T)} \int G(T,X|t,x)\Theta(X)\frac{dX}{\sqrt{X}}\,, \tag{11}$$

where $G(T, X|t, x) \equiv \hat{G}(\varphi(T), h(T, X)|\varphi(t), h(t, x))$ and $G(\hat{T}, .|\hat{t}, \hat{x})$ is the transition density of BM killed at a rate

$$\hat{k}(\hat{t},\hat{x}) \equiv \frac{\hat{\rho}(\hat{t})\hat{x}^2}{2}\,, \quad \text{where} \quad \hat{\rho} \equiv \frac{e^{-4\hat{R}(\hat{t})}}{2\hat{\gamma}^2}\,. \tag{12}$$

Proof. (i) Apply Ito's rule to $B_t = g(t, \tilde{W}_t)$, where $\tilde{W}_t$ is given in (6) and

$$g(t,y) = h(t,.)^{-1}(y + 2e^{R(t_0)}\sqrt{B_0}) \equiv \frac{e^{-2R(t)}}{4}\left(y + 2e^{R(t_0)\sqrt{B_0}}\right)^2 \tag{13}$$

is of class $C(1, \infty)$.

(ii) (4) yields that, upon transformation to coordinates given by (7,8), $\hat{v}(\hat{t}, \hat{x})$ solves

$$\left(\frac{1}{2}\frac{\partial^2}{\partial \hat{x}^2} + \frac{\partial}{\partial \hat{t}} - k(\hat{t},\hat{x})\right)\hat{v}(\hat{t},\hat{x}) = 0, \tag{14.1}$$

$$\hat{v}(\hat{T},\hat{x}) = \Theta(h^{-1}(T,\hat{x}))\,. \tag{14.2}$$

According to (9), $\hat{G}(\hat{T}, \hat{X}|\hat{t}, \hat{x})$ is a Green's function for this equation. Therefore, the solution to (14) is

$$\hat{v}(\hat{t},\hat{x}) = \int \hat{G}(\hat{T},\hat{X}|\hat{t},\hat{x})\Theta\left(h^{-1}(T,\hat{X})\right) d\hat{X}\,. \tag{15}$$

Letting $X = h^{-1}(T, \hat{X})$ formula (11) follows. $\square$

Remark. We note that if $q(t) \neq \gamma^2(t)/4$ then it can be proven that (1) has no strong solution with the representation $B_t = g\big(t, \int_{t_0}^t f(s)dW_s\big)$ for some $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $f : [t_0, \infty) \to \mathbb{R}$. Note that if $\psi(t) \equiv \int_{t_0}^t f^2(s)ds$ then, according to the well known Levy's Characterization Theorem, there exists a BM $\hat{W}_t$ such that $\int_{t_0}^t f(s)dW_s = \hat{W}_{\psi(t)}$.

Theorem 2. *Let $\hat{\lambda}_j(\hat{T}) \equiv \hat{\lambda}_j(\hat{T}/\hat{t}), j = 1, 2,$ be solutions of the linear ordinary differential equation*

$$\frac{d^2\hat{\lambda}}{d\hat{T}^2} - \hat{\rho}(\hat{T})\hat{\lambda}(\hat{T}) = 0 \tag{16}$$

satisfying, at $\hat{T} = \hat{t}$, the initial conditions (here $\hat{\lambda}' \equiv \frac{d\hat{\lambda}}{d\hat{T}}$)

$$\hat{\lambda}_1 = \hat{\lambda}_2'|_{\hat{T}=\hat{t}} = 1; \quad \hat{\lambda}_1' = \hat{\lambda}_2|_{\hat{T}=\hat{t}} = 0\,.$$

Then, the transition density for BM with killing rate (12) is given by

$$\hat{G}(\hat{T}, \hat{X}|\hat{t}, \hat{x}) = \frac{1}{\sqrt{2\pi\hat{\lambda}_2}} \exp[-\frac{\hat{\lambda}_2'}{2\hat{\lambda}_2}\left(\hat{X} - \frac{\hat{x}}{\hat{\lambda}_2'}\right)^2 - \frac{\hat{\lambda}_1'}{2\hat{\lambda}_2'}\hat{x}^2]\,. \tag{17}$$

Proof. $\hat{G}(\hat{T}, \hat{X}|\hat{t}, \hat{x})$ solves, in addition to (14.1), the equation in forward variables

$$\left(\frac{1}{2}\frac{\partial^2}{\partial\hat{X}^2} - \frac{\partial}{\partial\hat{T}} - \hat{k}(\hat{T}, \hat{X})\right)\hat{G}(\hat{T}, \hat{X}|\hat{t}, \hat{x}) = 0 \tag{18}$$

and $\hat{G}(\hat{t}, \hat{X}|\hat{t}, \hat{x}) = \delta(X - x)$. The transformation

$$\hat{G}(\hat{T}, \hat{X}|\hat{t}, \hat{x}) = \sqrt{\hat{\lambda}_1} \exp\big(-\frac{\hat{\lambda}_1'}{2\hat{\lambda}_1}\hat{X}^2\big)\nu(\hat{T}, \hat{X}|\hat{t}, \hat{x}) \tag{19}$$

yields that $\nu(\hat{T}, \hat{X}|\hat{t}, \hat{x})$ solves the Cauchy problem

$$\frac{\partial\nu}{\partial\hat{T}} - \frac{1}{2}\frac{\partial^2\nu}{\partial\hat{X}^2} + \frac{\hat{\lambda}_1'}{\hat{\lambda}_1}\frac{\partial(\hat{X}\nu)}{\partial\hat{X}} = 0\,. \tag{20}$$

This equation is amenable to be solved via Fourier transformation. To its end let

$$\phi(\hat{T}, s|\hat{t}, \hat{x}) = \int_{-\infty}^{\infty} e^{isx}\nu(\hat{T}, \hat{X}|\hat{t}, \hat{x})d\hat{X}\,. \tag{21}$$

Assuming that $\nu \in L_1$ and decays quickly enough to permit two differentiations under the integral sign, we find that $\phi(\hat{T}, s)$ solves

$$\frac{\partial \phi}{\partial \hat{T}} + \frac{s^2}{2}\phi - \frac{\hat{\lambda}_1'}{\hat{\lambda}_1} s \frac{\partial \phi}{\partial s} = 0;\ \phi(\hat{t}, s|\hat{t}, \hat{x}) = e^{i\hat{x}s}\,. \tag{22}$$

Solving through characteristics one can prove that

$$\phi(\hat{T}, s|\hat{t}, \hat{x}) = \exp\{i\hat{x}s\hat{\lambda}_1(\hat{T}) - \frac{s^2}{2}\hat{\lambda}_1(\hat{T})c(\hat{T}\},$$
$$\text{where}\ \ c(\hat{T}) \equiv \hat{\lambda}_1(\hat{T}) \int_{\hat{t}}^{\hat{T}} \frac{1}{\hat{\lambda}_1^2(l)} dl\,. \tag{23}$$

(17) is recovered upon inversion noticing that $c(\hat{T})$ satisfies (16) with initial conditions $\hat{\lambda}_2'|_{\hat{T}=\hat{t}} = 1; \hat{\lambda}_2|_{\hat{T}=\hat{t}} = 0$ and that $-\hat{\lambda}_2\hat{\lambda}_1' + \hat{\lambda}_1\hat{\lambda}_2' = 1$. □

The reader may consult Villarroel [7] for applications of these ideas.

Corollary 1. *Suppose that B_t evolves via (5) where γ, r are functions of class C^1 and C^0. Let*

$$Q(t) \equiv \frac{\partial_t \gamma}{\gamma} + r(t), f(t) = \gamma(t)e^{R(t)}\,,$$

and let $\lambda_j(T) \equiv \lambda_j(T/t) = \hat{\lambda}_j(\hat{T}/\hat{t})$, $j = 1, 2$;

$$\frac{d^2\lambda}{dT^2} - 2Q(T)\frac{d\lambda}{dT} = \frac{\gamma^2(T)}{2}\lambda(T)\,, \tag{24}$$

where $\dot{\lambda}_2|_{T=t} = f^2(t)$, $\lambda_1 - 1 = \dot{\lambda}_1 = \lambda_2|_{T=t} = 0$, $\dot{\lambda} \equiv \frac{d\lambda}{dT}$. Then:

(i) the value of claims with payoff $\Theta(x)$ is given by

$$v(t, x) = \int \frac{1}{\sqrt{2\pi\hat{\lambda}_2}} \exp\left[-\frac{\hat{\lambda}_2'}{2\hat{\lambda}_2}\left(\hat{X} - \frac{\hat{x}}{\hat{\lambda}_2'}\right)^2 - \frac{\hat{\lambda}_1'}{2\hat{\lambda}_2'}\hat{x}^2\right]$$
$$\times\, \Theta\left(\hat{X}^2 \frac{e^{-2\hat{R}(T)}}{4}\right) d\hat{X}\,. \tag{25}$$

(ii) The price of the bond is

$$p(T/t, B_t) = \frac{f(T)}{\sqrt{\dot{\lambda}_2(T)}} \exp\left[-2e^{2R(t)}\frac{\dot{\lambda}_1(T)}{\dot{\lambda}_2(T)}B_t\right]\,. \tag{26}$$

Remarks. (1) In Hull and White [3] the problem of pricing discount bonds is reduced to solving a system of *nonlinear* differential equations. In contrast, (26) gives the solution to this problem in an explicit way, while (25) gives the solution with arbitrary payoff.

(2) It is possible to prove that the functions $\lambda_j, j = 1, 2$ and their derivatives are positive and strictly increasing; furthermore they satisfy the identities

$$\dot{\lambda}_1/\dot{\lambda}_2 = \int_t^T \frac{\gamma^2 f^2(s)}{\dot{\lambda}_2^2} ds, \quad \lambda_1\dot{\lambda}_2 - \dot{\lambda}_1\lambda_2 = f^2(T)\,.$$

This implies that p satisfies $0 \le p(T/t, x) \le 1$. Besides, it decreases with the maturity time and is a convex, decreasing to 0, function of x.

2.2. Several Concrete Constructions

We next obtain in an explicit way all objects appearing in the construction above.

a) Suppose that $b(t,x) = \gamma(t)\sqrt{x}, a(t,x) = \gamma^2(t)/4 + \frac{\partial_t\gamma}{\gamma}x$, where $\gamma(t) \ge 0$. In this case we have

$$\hat{t} \equiv \varphi(t) = \frac{1}{\sqrt{2}} \int^t \gamma(s)ds, \quad R(t) = -\frac{1}{2}\log(\sqrt{2}\gamma(t)), \quad \hat{x} = 2^{3/4}\sqrt{\frac{x}{\gamma}} \qquad (27)$$

and (see (12)), $\hat{\rho} = 1$. Hence, the solutions $\hat{\lambda}_{1,2}$ of (16) are $\hat{\lambda}_1(\hat{t}) = \cosh(\hat{T} - \hat{t}), \hat{\lambda}_2(\hat{t}) = \sinh(\hat{T} - \hat{t})$. The value of claims follows from (25). In particular, we get from (26) the price of the bond as:

$$p(T/t, x) = \left(\cosh \int_t^T \frac{\gamma(s)ds}{\sqrt{2}}\right)^{-\frac{1}{2}} \exp\left[-\frac{\sqrt{2}x}{\gamma(t)} \tanh \int_t^T \frac{\gamma(s)ds}{\sqrt{2}}\right]. \qquad (28)$$

b) Suppose that there exists a C^1 function $\psi(t)$ such that

$$\gamma(t) = \sqrt{2}\psi^2 e^{-\int^t \psi^2}, \quad r(t) = \psi^2 - \dot{\psi}/\psi\,.$$

Then one finds that

$$\hat{t} \equiv \int^t \psi^2(s)ds, \quad R(t) = -\log(\sqrt{2}\psi) + \int^t \psi^2(s)ds,$$

$$\hat{x} = \sqrt{\frac{2x}{\psi^2}} e^{\int^t \psi^2}, \quad \hat{\rho}(\hat{t}) = e^{-2\hat{t}}$$

and, upon solution of (16), that

$$\hat{\lambda}_1(\hat{T}/\hat{t}) = e^{-\hat{t}}\left(K_1(e^{-\hat{t}})I_0(e^{-\hat{T}}) + I_1(e^{-\hat{t}})K_0(e^{-\hat{T}})\right),$$

$$\hat{\lambda}_2(\hat{T}/\hat{t}) = I_0(e^{-\hat{t}})K_0(e^{-\hat{T}}) - K_0(e^{-\hat{t}})I_0(e^{-\hat{T}}), \tag{29}$$

where $I_n(z), K_n(z)$ are modified Bessel functions of first and second kind. The value of claims follows from (25). In particular, one finds that the price of bonds is given by

$$p(T/t,x) = e^{\frac{1}{2}\varphi(T)}\left(B(T/t)\right)^{-1/2}\exp\left[-\frac{e^{\varphi(t)}}{\psi^2(t)}\frac{A(T/t)}{B(T/t)}x\right], \tag{30}$$

where

$$A(T/t) \equiv K_1(e^{-\varphi(T)})I_1(e^{-\varphi(t)}) - I_1(e^{-\varphi(T)})K_1(e^{-\varphi(t)}),$$

$$B(T/t) \equiv I_0(e^{-\varphi(t)})K_1(e^{-\varphi(T)}) + K_0(e^{-\varphi(t)})I_1(e^{-\varphi(T)}),$$

$$\varphi(t) \equiv \int_0^t \psi^2(s)ds\,. \tag{31}$$

c) Suppose that γ and r are constants. Then, upon solution of (24) we find

$$\lambda_1(T/t) = \frac{\gamma^2(t)e^{2R_t}}{\omega_- - \omega_+}\left(\omega_- e^{\omega_+(T-t)} - \omega_+ e^{\omega_-(T-t)}\right),$$

$$\lambda_2(T/t) = \frac{\gamma^2(t)e^{2R_t}}{\omega_- - \omega_+}\left(e^{\omega_-(T-t)} - e^{\omega_+(T-t)}\right), \tag{32}$$

where $\omega_\pm = r \pm \eta/2$ and $\eta \equiv \sqrt{4r^2 + 2\gamma^2}$, $\xi \equiv e^{(T-t)}$. The value of an arbitrary claim is given by (25):

$$v(t,x) = \int \sqrt{D}\exp[-A\left(\sqrt{X} - C\sqrt{x}\right)^2 - Bx]\Theta(X)dX\,, \tag{33}$$

where

$$B \equiv \frac{2(\xi^\eta - 1)}{(2r+\eta)(\xi^\eta - 1) + 2\eta}, \quad A \equiv \frac{2}{\gamma^2 B},$$

$$C \equiv \frac{2\eta\xi^{\eta/2}}{(2r+\eta)(\xi^\eta - 1) + 2\eta}, \quad D \equiv \frac{\eta}{2\pi}\frac{\xi^{r+\eta/2}}{\xi^\eta - 1}\,. \tag{34}$$

In particular, if $\Theta = 1$ we recover the result of Cox et al [1]:

$$p(T/t,x) = \sqrt{C}\xi^{r/2}e^{-Bx}\,. \tag{35}$$

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