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KdV EQUATION AND SINGULAR STURM-LIOUVILLE PROBLEMS

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Abstract: In this paper, we obtain KdV equation by using lie groups and give existence of translation operators for Sturm-Liouville problem having singularity $x = 0$ by using solution of the KdV equation.

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1. Introduction

Korteweg-de Vries [2] first derived it in the study of long water waves in a channel of finite depth. Other fluid dynamical applications have been studied by [3], [4]. On the other hand, it is obtained a special cases of the structure equation of Lie Group SL(2;R) (the special linear group of all (2×2)-real unimodular matrices) by S.S. Chern and C.K. Peng [8]. mKdV equations are characterized as relations between local invariants of certain foliations on a surface of constant Gaussian curvature by S.S. Chern and K. Tenenblat [9].

In the Sturm-Liouville problems potential function $q(x)$ corresponds to the initial data $u(x,t) = u_0(x)$ for the Korteweg-de Vries (KdV) equation, where $u(x,t)$ is solution of the KdV equation. This relation is studied by Gardener [5], Levitan [1], Lax [6].

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We consider Korteweg-de Vries equation as defined by

$$u_t + u_{xxx} = 6uu_x\,,$$

which was obtained by Chern and Peng, it is solved Sturm-Liouville problem obtained by solutions of KdV equation in second section.

2. Lie Groups and KdV Equation

In this section, we will recall the notations and terminology used in [8]. Let

$$SL(2;R) = \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; ad - bc = 1 \right\} \tag{1.1}$$

be the group of all ($2x2$)-real unimodular matrices. Its right-invariant Maurer-Cartan form is

$$w = dXX^{-1} = \begin{pmatrix} w_1^1 & w_1^2 \\ w_2^1 & w_2^2 \end{pmatrix}, \tag{1.2}$$

where

$$w_1^1 + w_2^2 = 0. \tag{1.3}$$

The structure (Maurer-Cartan) equation of $SL(2;R)$ is

$$dw_1^1 = w_1^2 \Lambda w_2^1, \quad dw_1^2 = 2w_1^1 \Lambda w_1^2, \quad dw_2^1 = 2w_2^1 \Lambda w_1^1. \tag{1.4}$$

Let u be a neighbourhood in the (x,t)-plane and consider the smooth mapping

$$f : u \rightarrow SL(2;R). \tag{1.5}$$

The Pull-backs of the Maurer-Cartan forms can be written

$$w_1^1 = \eta dx + Adt, \quad w_1^2 = qdx + Bdt, \quad w_2^1 = rdx + Cdt\,, \tag{1.6}$$

where the coefficients are functions of x, t. The forms in (1.6) satisfy the equations (1.4). This gives

$$\begin{aligned} -\eta_t + A_x - qC + rB = 0, \\ -q_t + B_x - 2\eta B + 2qA = 0, \\ -r_t + C_x - 2rA + 2\eta C = 0. \end{aligned} \tag{1.7}$$

We consider the special case that r $= \pm 1$ and η is a parameter indepent of x, t. Writing $q = u(x,t)$. We get from (1.7)

$$A = +\eta C + \frac{1}{2} C_x, \quad B = uC - \eta C_x - \frac{1}{2} C_{xx}.$$

Substitution into the second equation of (1.7) gives

$$u_t = K(u), \tag{1.8}$$

where $K(u) = u_x C + 2uC_x + 2\eta^2 C_x - \frac{1}{2}C_{xxx}$. As an example we take $C = -4\eta^2 + 2u$. Then, (1.9) becomes

$$u_t + u_{xxx} = 6uu_x\,, \tag{1.9}$$

which is well-known KdV equation.

3. The Inverse Problem and KdV Equation

We consider the problem

$$Ly = -y'' + [u(x,t) + \frac{l(l+1)}{x^2}]y = \lambda y, \quad t \neq 1, \quad 0 < x \leq 1\,, \tag{2.1}$$

$$y\,(0) = 0, \tag{2.2}$$

$$y'\,(1,\lambda) + Hy\,(1,\lambda) = 0, \quad H \neq 0\,, \tag{2.3}$$

where

$$u\,(x,t) = \frac{x}{6\,(1-t)}$$

is solution of KdV equation and $u\,(x,t)|_{t=0} = u_0\,(x)$. It is real and continuous according to x. Hence, the operator L is self-adjoint, have discrete spectrum which consists of simple eigenvalues. As known, solution of the problem (2.1)-(2.3) is

$$\varphi(x,\lambda) = \frac{\sqrt{x}}{(\sqrt{\lambda})^{l+\frac{1}{2}}} J_l(x) + \int_0^x K(x,s)\frac{\sqrt{s}}{(\sqrt{\lambda})^{l+\frac{1}{2}}} J_l(s)ds. \tag{2.4}$$

On the other hand, if we solve (2.3) equation for λ, it is

$$\lambda_n = (n + \frac{l}{2})^2 + a + \frac{a_1}{n^2} + O\left(\frac{1}{n^2}\right), \tag{2.5}$$

where a and a_1 are real numbers. λ_n are called eigenvalues of problem (2.1)-(2.3). Using formula (2.5), we will obtain an asymptotic formula for the eigenfunctions

$$\varphi_n(x) = \cos(n\pi x - \frac{l\pi}{2} - \frac{\pi}{4}) + O(\frac{1}{n}). \tag{2.6}$$

Now, we obtain solution of (2.1)-(2.3) by translation operator. Solution of the problem

$$Ly = -y^{''} + [u(x,t) + \frac{l(l+1)}{x^2}]y = \lambda y, \quad t \neq 1, \quad 0 < x \leq 1\,, \tag{2.7}$$

$$y\,(0) = 0 \tag{2.8}$$

is $\varphi\,(x,\lambda)$ and solution of the problem

$$Ly = -y^{''} + \frac{l(l+1)}{x^2}y = \lambda y\,, \tag{2.9}$$

$$y\,(0) = 0 \tag{2.10}$$

is $J_l(x)$, where $J_l(x)$ is first kind a Bessel function. The relation of these solutions is following

$$\varphi(x,\lambda) = \chi[J_l(x)] = J_l(x) + \int_0^x K(x,s)J_l(s)ds\,, \tag{2.11}$$

where χ is a translation operator.

Theorem 2.1 *When the translation operator $X = X_{L_1,L_2}$ can be realized as*

$$\varphi(x,\lambda) = \chi[J_l(x)] = J_l(x) + \int_0^x K(x,s)J_l(s)ds\,. \tag{2.12}$$

The kernel $K\,(x,s)$ is solution of differential equation

$$\frac{\partial^2 K\,(x,s)}{\partial x^2} - \frac{l(l+1)}{x^2}K\,(x,s) = \frac{\partial^2 K\,(x,s)}{\partial s^2} - \left[\frac{l(l+1)}{s^2} + u_0\,(s)\right]K\,(x,s) \tag{2.13}$$

and satisfies conditions

$$K\,(x,0) = 0, \tag{2.14}$$

$$2\frac{dK\,(x,x)}{dx} = u_0\,(x)\,. \tag{2.15}$$

Conversely, if a function $K(x,s)$ is a solution of the problem (2.13)-(2.15) then the operator X_{L_1,L_2} defined by formula (2.12) is a translation operator for the pair L_1 and L_2.

Proof. We define two operators

$$L_1 = \frac{d^2}{dx^2} - \left[\frac{l(l+1)}{x^2}\right],\ L_2 = \frac{d^2}{dx^2} - \left[\frac{l(l+1)}{x^2} + u_0\left(x\right)\right].$$

Translation operator satisfies $L_1 X = X L_2$, see [1]. Hence,

$$\begin{aligned} L_1 X\left[J_l\left(x,\lambda\right)\right] &= L_1\varphi\left(x\right) = \varphi\left(x\right)'' - \left[\frac{l(l+1)}{x^2}\right]\varphi\left(x\right) \\ &= J_l''\left(x,\lambda\right) + \frac{dK\left(x,x\right)}{dx}J_l\left(x,\lambda\right) + J_l'\left(x,\lambda\right)K\left(x,x\right) \\ &\quad + \frac{\partial K\left(x,x\right)}{\partial x}J_l\left(s\right)\Big|_{s=x} + \int_0^x \frac{\partial^2 K\left(x,t\right)}{\partial x^2}J_l\left(s,\lambda\right)ds \\ &\quad - \left[\frac{l(l+1)}{x^2}\right]J_l\left(x,\lambda\right) - \left[\frac{l(l+1)}{x^2}\right]\int_0^x K\left(x,s\right)J_l\left(s,\lambda\right)ds. \end{aligned} \tag{2.16}$$

On the other hand,

$$\begin{aligned} &X L_2\left[J_l\left(x,\lambda\right)\right] \\ &\quad = J_l''\left(x,\lambda\right) - \left[\frac{l(l+1)}{x^2} + u_0\left(x\right)\right]J_l\left(x,\lambda\right) + \int_0^x K\left(x,s\right)J_l''\left(s,\lambda\right)dt \\ &\qquad - \int_0^x K\left(x,s\right)\left[\frac{l(l+1)}{s^2} + u_0\left(s\right)\right]J_l\left(s,\lambda\right)ds\,. \end{aligned} \tag{2.17}$$

Integrating by parts to the third term, from (2.16) and (2.17) we get

$$\begin{aligned} &\frac{\partial^2 K\left(x,s\right)}{\partial x^2} - \left[\frac{l(l+1)}{x^2}\right]K\left(x,s\right) \\ &\qquad = \frac{\partial^2 K\left(x,s\right)}{\partial s^2} - \left[\frac{l(l+1)}{s^2} + u_0\left(s\right)\right]K\left(x,s\right)\,, \end{aligned} \tag{2.18}$$

$$K(x,0) = 0\,, \tag{2.19}$$

$$2\frac{dK\left(x,x\right)}{dx} = u_0\left(x\right)\,. \tag{2.20}$$

Thus, if an operator $X = X_{L_1,L_2}$ can be realized in form (2.12), then its kernel, i.e. function $K(x,s)$ is a solution of problems (2.18)-(2.20). Conversely, if a function $K(x,s)$ is a solution of problems (2.18)-(2.20), then operator $X = X_{L_1,L_2}$ constructed from it by formula (2.12) is a transformation operator for the pair L_1 and L_2. To verify this, it suffices to repeat the above computations in reverse order. This completes the proof. □

We consider translations

$$z = \frac{(x+s)^2}{4}, \quad r = \frac{(x-s)^2}{4}, \quad K(x,s) = (z-r)^{l+\frac{1}{2}} U(z,r) .$$

Taking the necessary derivatives in (2.18), this equation transforms to

$$\frac{\partial^2 U}{\partial z \partial r} - \frac{\alpha}{(z-r)} \frac{\partial U}{\partial z} + \frac{\alpha}{(z-r)} \frac{\partial U}{\partial s} = \frac{1}{4\sqrt{zr}} [u_0(x)] U , \tag{2.21}$$

where $\alpha = -l$. This equation is a Euler-Poisson-Darboux type equation. For $r = 0$ in (2.19) because of being $z = r$, equation (2.18) has a singularity on the characteristic $z = r$. So, for $\varepsilon > 0$ we consider characteristic $z = r + \varepsilon$. Hence, problem (2.18)-(2.20) transforms to the following one

$$\frac{\partial^2 U}{\partial z \partial r} - \frac{\alpha}{(z-r)} \frac{\partial U}{\partial z} + \frac{\alpha}{(z-r)} \frac{\partial U}{\partial r} = \frac{1}{4\sqrt{zr}} [u_0(x)] U , \tag{2.22}$$

$$\frac{dU}{dz} + \frac{\alpha}{z} U = \frac{1}{4} (u_0(x))^{-\alpha-\frac{1}{2}} , \; U(z, z-\varepsilon) = 0 . \tag{2.23}$$

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