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AN ASYMPTOTIC INTERTWINING OF THE UNDELAYED AND DELAYED FIBONACCI NUMBERS

Miloslav Znojil

Nuclear Physics Institute of Czech Academy of Sciences
Řež near Prague, 250 68, CZECH REPUBLIC
e-mail: znojil@ujf.cas.cz

Abstract: The list of properties of Fibonacci numbers F_n (with multifaceted relevance in physics) is complemented by an empirical observation that in combination with the "next" family of the "delayed Fibonacci" numbers G_m (called, for convenience, "Gibonacci numbers" here), both sets exhibit certain remarkable and fairly unexpected asymptotic mutual-bracketing properties.

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1. Introduction

Our forthcoming considerations were inspired by the decoration of the northmost subway station "Holešovice" in Prague. Any passenger who waits there for the train to the city may wonder why the architects ornamented the walls by the regularly repeated lines of 2, 2, 3, 4, 5, and 6 tiles. In this sense, our present paper comes only too late when suggesting a replacement of such a pattern by an inessentially modified sextet of the integer numbers $G_3 = 2$, $G_4 = 2$, $G_5 = 3$, $G_6 = 4$, $G_7 = 5$ and $G_8 = 7$.

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2. Fibonacci-Type Recurrences with Zero- and One-Step Delay

Our deeper motivation stems from the fact that the above-mentioned segment of a sequence of G_n may be understood as one of the most natural "delayed" modifications of the famous Fibonacci numbers [1]. The latter sequence with conventional denotation $F(n+1) \equiv F_n$ [2] is defined by the well known and extremely elementary recurrent relations

$$F_n = F_{n-1} + F_{n-2}, \qquad n = 2, 3 \ldots, \qquad F_0 = F_1 = 1\,. \tag{1}$$

Similarly, their "delayed" version may be generated by the very similar "next-to-Fibonacci" recurrences with a one-step delay,

$$G_n = G_{n-2} + G_{n-3}, \qquad n = 3, 4, \ldots, \qquad G_0 = G_1 = G_2 = 1\,. \tag{2}$$

In our present note we intend to present a few arguments showing why both the sequences F_n and G_n might be considered comparably interesting.

Firstly, let us remind the reader that both these sequences are virtually equally easy to construct. Both difference equations (1) and (2) for F_n and G_m have constant coefficients and may be analyzed by the similar standard ansatzs, viz.,

$$F_n = a\,\left[\eta_{(a)}\right]^n + b\,\left[\eta_{(b)}\right]^n \tag{3}$$

and

$$G_n = A\,\left[\varrho_{(A)}\right]^n + B\,\left[\varrho_{(B)}\right]^n + C\,\left[\varrho_{(C)}\right]^n\,, \tag{4}$$

respectively. This procedure leads to the similar implicit definitions of the necessary quotients.

2.1. Fibonacci Sequences

In the former case which corresponds to the usual Fibonacci numbers F_n the implicit definition of the quotient is slightly simpler,

$$\left[\eta_{(a,b)}\right]^2 = \eta_{(a,b)} + 1\,. \tag{5}$$

This equation may be immediately assigned the following well known and complete explicit solution,

$$\eta_{(a)} = \frac{1+\sqrt{5}}{2}\,, \qquad \eta_{(b)} = \frac{1-\sqrt{5}}{2} = -\left(\eta_{(a)} - 1\right)\,. \tag{6}$$

In it, the former item coincides with the famous "golden mean" value.

2.2. "Gibonacci" Sequences

There is really no reason why one should be afraid of searching for the similar explicit solutions in the delayed case with the similar implicit definition

$$\left[\varrho_{(A,B,C)}\right]^3 = \varrho_{(A,B,C)} + 1 \tag{7}$$

of the quotients. Routinely, one arrives at just one real and positive root

$$\varrho_{(A)} = D_+ + D_- \,, \tag{8}$$

where we abbreviated

$$D_+ = \left(\frac{1+x_0}{2}\right)^{1/3} \approx 0.98699\,, \qquad D_- = \left(\frac{1-x_0}{2}\right)^{1/3} \approx 0.3377$$

and where

$$x_0 = \sqrt{\frac{23}{27}} \approx 0.9229582\,.$$

The other two complex conjugate roots possess the equally compact representation

$$\varrho_{(B)} = \left[\varrho_{(C)}\right]^* = e^{i\varphi}D_+ + e^{-i\varphi}D_- \ \left(= -\frac{1}{2}\,\varrho_{(A)} + i\,\frac{\sqrt{3}}{2}\,\varrho_{(D)}\right)\,, \tag{9}$$

where $\varphi = 2\pi/3$ and $\varrho_{(D)} = D_+ - D_- \approx 0.649264$.

2.3. Specific Initial Boundary Conditions

Once we require the compatibility of formula (3) with the Fibonacci's boundary conditions $F_0 = F_1 = 1$, we have to extract the corresponding values of a and b from the two equations

$$1 = a + b = a\,\frac{1+\sqrt{5}}{2} + b\,\frac{1-\sqrt{5}}{2}\,. \tag{10}$$

This gives the well known explicit formula for Fibonacci numbers,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]\,. \tag{11}$$

In the similar manner, the delayed-Fibonacci general solution (4) complemented by the boundary conditions $G_0 = G_1 = G_2 = 1$ will define our present Gibonacci numbers G_m. First of all we must find the correct values of the coefficients in equation (4) by solving the the triplet of the linear equations

$$\begin{aligned} 1 = A + B + C = A\,\varrho_{(A)} + B\,\varrho_{(B)} + C\,\varrho_{(C)} \\ = A\,\left[\varrho_{(A)}\right]^2 + B\,\left[\varrho_{(B)}\right]^2 + C\,\left[\varrho_{(C)}\right]^2\,. \end{aligned} \tag{12}$$

Its numerical solution gives the approximate form of the result,

$$\begin{array}{c} A = 0.72212441830311284114\ldots\,, \\ B = 0.13893779084844357942\ldots - i \cdot 0.20225012409895253966\ldots\,, \\ C = 0.13893779084844357942\ldots + i \cdot 0.20225012409895253966\ldots\,. \end{array} \tag{13}$$

Now, the same result will be derived in closed form, completing in this way the analogy with the previous "non-delayed" formula (11).

Firstly, we suppress all temptations to use a computerized symbolic manipulations and put $B = K - i\,L = C^*$. The first item in equation (12) then offers the direct elimination of $A = 1 - 2K$. Moreover, after an abbreviation

$$1 + 2\,K\,(\cos\varphi - 1) + 2\,L\,\sin\varphi = \Sigma\,, \quad 1 + 2\,K\,(\cos\varphi - 1) - 2\,L\,\sin\varphi = \Delta\,, \tag{14}$$

we may re-write the remaining two lines of equation (12) in a particularly friendly form,

$$\begin{array}{c} \Sigma\,D_+ + \Delta\,D_- = 1\,, \\ 3\,\Sigma\,D_-^2 + 3\,\Delta\,D_+^2 = 1\,. \end{array} \tag{15}$$

This two-by-two matrix equation is readily solvable,

$$\Sigma = \frac{1}{3x_0}\,\left(3\,D_+^2 - D_-\right)\,, \qquad \Delta = \frac{1}{3x_0}\,\left(-3\,D_-^2 + D_+\right)\,.$$

The final backward insertion in equation (14) is trivial and gives the final answer,

$$6\,K = 2 - \frac{1}{3x_0}\,(D_+ - D_-)\,(3D_+ + 3D_- + 1)\,,$$

$$2\,\sqrt{3}\,L = \frac{1}{3x_0}\,\left(3D_+^2 + 3D_-^2 - D_+ - D_-\right)\,,$$

which is compatible with its numerical check (13).

3. Intertwining Behavior of the Fibonacci-Type Sequences

3.1. Asymptotics of F_n and G_m at the Large Indices

As long as we observe that $|\eta_{(a)}| = 1.618\ldots > 1$ and $|\eta_{(b)}| = 0.618\ldots < 1$, while $|\varrho_{(A)}| = 1.3247\ldots > 1$ and $|\varrho_{(B)}| = |\varrho_{(C)}| = |-0.662\ldots \pm i\,0.562\ldots|$, i.e., $|\varrho_{(B)}| = .8688\ldots < 1$, we may conclude that irrespectively of the initial boundary conditions the solutions of both the difference equations (2) and (1) will always exhibit a similar asymptotic behavior. The reason is that both F_n and G_m are dominated by the single power-law term, i.e.,

$$F_n = a\,\left[\eta_{(a)}\right]^n + +\mathcal{O}\left\{\left|\eta_{(b)}\right|^n\right\}, \qquad n \gg 1\,, \tag{16}$$

while

$$G_m = A\,\left[\varrho_{(A)}\right]^m + \mathcal{O}\left\{\left|\varrho_{(B)}\right|^m\right\}, \qquad m \gg 1\,. \tag{17}$$

From these relations we may deduce that the size of the numbers F_N and G_M cannot remain comparable unless the indices $N \gg 1$ and $M \gg 1$ obey the following rule,

$$\frac{N}{M} \approx \frac{\ln \varrho_{(A)}}{\ln \eta_{(a)}} = 0.584\ldots\,. \tag{18}$$

This means that the replacement of the Fibonacci recurrences (1) by their one-step-delayed modification (2) slows down the asymptotic growth of the new sequence, $G_M \approx F_{entier\{0.584\,M\}}$ at $M \gg 1$.

3.2. Inequalities between F_n and G_m at the Finite Indices

An identity

$$\frac{\ln \varrho_{(A)}}{\ln \eta_{(a)}} = 0.58435715765740408667...$$

$$= \frac{7}{12} + \frac{3}{10^3}\left(\frac{7}{12}\right)^2 + \frac{3}{10^6} - \frac{9}{10^9} + \mathcal{O}\left(\frac{1}{10^{11}}\right) \tag{19}$$

complements equation (18) and indicates that the ratio 7/12 may play a key role in our present analysis. Indeed, once we tentatively re-index all the very large Fibonacci numbers,

$$F_N = f(j,k), \qquad N = N(j,k) = 7\,j + k, \qquad 0 \le k \le 6, \tag{20}$$

and once we re-write their delayed-generated alternative in the similar form,

$$G_M = g(J, K), \qquad M = M(J, K) = 12\,J + K, \qquad 0 \le K \le 11, \tag{21}$$

we may re-interpret the above "asymptotic comparability rule" (18) as a requirement

$$\frac{7\,j + k}{12\,J + K} \approx \frac{7}{12}\,, \qquad j, J \gg 1\,. \tag{22}$$

In the other words, we achieve the approximative asymptotic coincidence of $G_M = g_{J,K}$ with $F_N = f_{j,k}$ if and only if the new auxiliary indices j and J do not differ too much.

3.3. Inequalities Between F_n and G_m at the Small Indices

A priori, there is no reason to believe that the similar rule could be extended to the domain of the small indices. Nevertheless, we may take the union set of *all* the numbers in equations (20) and (21) and *order* this family in a way starting at the very first subscripts. In this way the first seven lines of inequalities are revealed,

$$\begin{array}{l} g(0,-1)\ (=0)\ < f(0,0)\ (=1)\ \le g(0,0)\ (=1),\\ g(0,1)\ (=1)\ \le f(0,1)\ (=1)\ \le g(0,2)\ (=1),\\ g(0,3)\ (=2)\ \le f(0,2)\ (=2)\ \le g(0,4)\ (=2),\\ g(0,5)\ (=3)\ \le f(0,3)\ (=3)\ < g(0,6)\ (=4),\\ g(0,6)\ (=4)\ < f(0,4)\ (=5)\ \le g(0,7)\ (=5),\\ g(0,8)\ (=7)\ < f(0,5)\ (=8)\ < g(0,9)\ (=9),\\ g(0,10)\ (=12)\ < f(0,6)\ (=13)\ < g(0,11)\ (=16)\,. \end{array} \tag{23}$$

Encouraged by the smoothness of this pattern we may verify, with an utterly unexpected success, the existence and validity of its next-step continuation

$$\begin{array}{l} g(0,11)\ (=16)\ < f(1,0)\ (=21)\ \le g(1,0)\ (=21),\\ g(1,1)\ (=28)\ < f(1,1)\ (=34)\ < g(1,2)\ (=37),\\ g(1,3)\ (=49)\ < f(1,2)\ (=55)\ < g(1,4)\ (=65),\\ g(1,5)\ (=86)\ < f(1,3)\ (=89)\ < g(1,6)\ (=114),\\ g(1,6)\ (=114)\ < f(1,4)\ (=144)\ < g(1,7)\ (=151),\\ g(1,8)\ (=200)\ < f(1,5)\ (=233)\ < g(1,9)\ (=265),\\ g(1,10)\ (=351)\ < f(1,6)\ (=377)\ < g(1,11)\ (=465)\,. \end{array} \tag{24}$$

Now, there comes one of our main empirical observations. Against all odds, the same scheme works during unexpectedly many iterations numbered by the integer K which appears as the first argument in the functions $g(K,*)$ and $f(K,*)$

and which was equal to zero in equation (23) and to one in the subsequent set of the fourteen inequalities (24). We arrive at the formidably extensive set of the empirical inequalities

$$\begin{array}{lll} g(K-1,11) < f(K,0) < g(K,0)\ , & K < 48, \\ g(K,1) < f(K,1) < g(K,2)\ , & K < 35, \\ g(K,3) < f(K,2) < g(K,4)\ , & K < 21, \\ g(K,5) < f(K,3) < g(K,6)\ , & K < 7, \\ g(K,6) < f(K,4) < g(K,7)\ , & K < 41, \\ g(K,8) < f(K,5) < g(K,9)\ , & K < 27, \\ g(K,10) < f(K,6) < g(K,11)\ , & K < 14\,. \end{array} \tag{25}$$

The existence and structure of the upper limits of their validity reminds us of the fact that the next correction to equation (22) (given in equation (19)) does not vanish. This means that the range of the allowed K in equation (25) cannot be unlimited.

3.4. Inequalities Between F_n and G_m at the Growing Indices

It is remarkable that the inequalities (25) are violated *so extremely slowly* and in such an unbelievably regular manner. This is one of the main consequences of the smallness of the absolute value of the second correction $\approx 0.003 \cdot (7/12)^2$ to the rule $N/M = 7/12$ because it will enable us to extend our inequality pattern beyond its limits listed in equation (25).

As long as the next correction to the asymptotic comparability rule $N/M = 7/12$ is positive, we know *in advance* that the middle terms $f(K,J)$ in equation (25) will all move to the left and violate their lower estimates at a critical K. Due to this expectation (confirmed by the explicit calculations in *MAPLE*), the next-step bracketing law acquires the mere following shifted form

$$\begin{array}{ll} g(K-1,10) < f(K,0) < g(K-1,11)\ , & 48 \le K < 96, \\ g(K,0) < f(K,1) < g(K,1)\ , & 35 \le K < 82, \\ g(K,2) < f(K,2) < g(K,3)\ , & 21 \le K < 70, \\ g(K,4) < f(K,3) < g(K,5)\ , & 7 \le K < 55, \\ g(K,5) < f(K,4) < g(K,6)\ , & 41 \le K < 89, \\ g(K,7) < f(K,5) < g(K,8)\ , & 27 \le K < 75, \end{array}$$

$$g(K,9) < f(K,6) < g(K,10)\ , \quad 14 \le K < 61, \quad \text{etc.}\,. \tag{26}$$

One should notice that due to the fact that all the K's are already very large, all the separate intervals of validity of the innovated rule (26) are perceivably longer than their respective predecessors in equation(25). This seems to indicate a general tendency, the detailed analysis of which would require much more space than available here.

4. Outlook

4.1. Towards Combinatorial Applications

Fibonacci recurrences (1) (without any delay) are extremely popular and find (perhaps, unexpectedly) numerous practical applications. *Pars pro toto*, Fibonacci numbers F_n occurred recently as a sequence which numbers all the possible re-arrangements of the Born-Lanczos expansions of the scattering amplitudes in quantum mechanics [3]. The mathematical essence of this particular application lies, in a way illustrated by Table 1, in the following combinatorial representation of the Fibonacci numbers,

$$F_k = \begin{pmatrix} k \\ 0 \end{pmatrix} + \begin{pmatrix} k-1 \\ 1 \end{pmatrix} + \begin{pmatrix} k-2 \\ 2 \end{pmatrix} + \dots\ . \tag{27}$$

length	eligible structures				
1	♠				
2	♠♠	♡			
3	♠♠♠	♠♡	♡♠		
4	♠♠♠♠	♠♠♡	♠♡♠	♡♠♠	♡♡
5	♠♠♠♠♠ ♠♡♡	♠♠♠♡ ♡♠♡	♠♠♡♠ ♡♡♠	♠♡♠♠	♡♠♠♠
6	♠♠♠♠♠♠ ♡♠♠♠♠ ♡♠♡♠	♠♠♠♠♡ ♠♠♡♡ ♡♡♠♠	♠♠♠♡♠ ♠♡♠♡ ♡♡♡	♠♠♡♠♠ ♡♠♠♡	♠♡♠♠♠ ♠♡♡♠

Table 1: Chains of elements ♠ with pairwise confluences ♡ (cf. [3]).

One may feel inspired to re-interpret the latter property as a definition. In the next step one then could modify this type of definition, obtaining the following "higher" Fibonacci numbers,

$$\begin{aligned} F_k^{(2)} &= \sum_{j\geq 0 \text{ while } 3j\leq k} \begin{pmatrix} k-2j \\ j \end{pmatrix} \\ &= \begin{pmatrix} k \\ 0 \end{pmatrix} + \begin{pmatrix} k-2 \\ 1 \end{pmatrix} + \begin{pmatrix} k-4 \\ 2 \end{pmatrix} + \dots \end{aligned} \tag{28}$$

(cf. Table 2) or

length	eligible structures					
1	♠					
2	♠♠					
3	♠♠♠	♣				
4	♠♠♠♠	♠♣	♣♠			
5	♠♠♠♠♠	♠♠♣	♠♣♠	♣♠♠		
6	♠♠♠♠♠♠	♠♠♠♣	♠♠♣♠	♠♣♠♠	♣♠♠♠	♣♣
7	♠♠♠♠♠♠♠ ♣♠♠♠♠	♠♠♠♠♣ ♠♣♣	♠♠♠♣♠ ♣♠♣	♠♠♣♠♠ ♣♣♠	♠♣♠♠♠	
8	♠♠♠♠♠♠♠♠ ♠♣♠♠♠♠ ♠♣♣♠	♠♠♠♠♠♣ ♣♠♠♠♠♠ ♣♠♣♠	♠♠♠♠♣♠ ♠♠♣♣ ♣♣♠♠	♠♠♠♣♠♠ ♠♣♠♣	♠♠♣♠♠♠ ♣♠♠♣	

Table 2: Chains of elements ♠ with confluences in triplets ♣.

$$F_k^{(3)} = \sum_{j\geq 0 \ \text{while} \ 4j\leq k} \begin{pmatrix} k-3j \\ j \end{pmatrix}$$

$$= \begin{pmatrix} k \\ 0 \end{pmatrix} + \begin{pmatrix} k-3 \\ 1 \end{pmatrix} + \begin{pmatrix} k-6 \\ 2 \end{pmatrix} + \dots \quad (29)$$

(cf. Table 3), etc. From this background, one can always return to the recurrent approach, discovering that it starts from a general initial $\ell-$plet of values

$$F_k^{(\ell)} = 1, \quad k = 0, 1, \dots, \ell-1, \ell, \quad \ell = 1, 2, \dots . \quad (30)$$

Moreover, the old combinatorial definitions (28) or (29), etc., become replaced by the corresponding Fibonnaci-type new three-term-like recurrences

$$F_n^{(2)} = F_{n-1}^{(2)} + F_{n-3}^{(2)}, \quad n = 3, 4, \dots \ , \quad (31)$$

or

$$F_n^{(3)} = F_{n-1}^{(3)} + F_{n-4}^{(3)}, \quad n = 4, 5, \dots \ , \quad (32)$$

etc. In this way we would obtain a new series of generalized Fibonacci numbers. The analysis of the influence of the delay in the underlying recurrences lies already beyond the scope of our present short communication but it is worth mentioning that it would proceed precisely along the lines applied here to the simplest delayed case (2).

length	eligible structures				
1	$\spadesuit$				
2	$\spadesuit\spadesuit$				
3	$\spadesuit\spadesuit\spadesuit$				
4	$\spadesuit\spadesuit\spadesuit\spadesuit$	$\diamondsuit$			
5	$\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit$	$\spadesuit\diamondsuit$	$\diamondsuit\spadesuit$		
6	$\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit$	$\spadesuit\spadesuit\diamondsuit$	$\spadesuit\diamondsuit\spadesuit$	$\diamondsuit\spadesuit\spadesuit$	
7	$\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit$	$\spadesuit\spadesuit\spadesuit\diamondsuit$	$\spadesuit\spadesuit\diamondsuit\spadesuit$	$\spadesuit\diamondsuit\spadesuit\spadesuit$	$\diamondsuit\spadesuit\spadesuit\spadesuit$
8	$\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit$ $\diamondsuit\spadesuit\spadesuit\spadesuit\spadesuit$	$\spadesuit\spadesuit\spadesuit\spadesuit\diamondsuit$ $\diamondsuit\diamondsuit$	$\spadesuit\spadesuit\spadesuit\diamondsuit\spadesuit$	$\spadesuit\spadesuit\diamondsuit\spadesuit\spadesuit$	$\spadesuit\diamondsuit\spadesuit\spadesuit\spadesuit$
9	$\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit$ $\spadesuit\diamondsuit\spadesuit\spadesuit\spadesuit\spadesuit$	$\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit\diamondsuit$ $\diamondsuit\spadesuit\spadesuit\spadesuit\spadesuit\spadesuit$	$\spadesuit\spadesuit\spadesuit\spadesuit\diamondsuit\spadesuit$ $\spadesuit\diamondsuit\diamondsuit$	$\spadesuit\spadesuit\spadesuit\diamondsuit\spadesuit\spadesuit$ $\diamondsuit\spadesuit\diamondsuit$	$\spadesuit\spadesuit\diamondsuit\spadesuit\spadesuit\spadesuit$ $\diamondsuit\diamondsuit\spadesuit$

Table 3: Chains of elements $\spadesuit$ with quadruplets $\diamondsuit$.

4.2. An Appeal of Linearity

An apparent ambiguity of an initialization of the delayed Fibonacci recurrences (2) is just fictitious. Indeed, although a more general choice of the initialization appears admissible,

$$G_0(a) = G_2(a) = 1, \qquad G_1(a) = a \in (-\infty, \infty) \tag{33}$$

the question has an elementary answer since the innovated initialization (33) merely produces the sequence with elements $G_3(a) = 1 + a$, $G_4(a) = 1 + a$, $G_5(a) = 2 + a$, $G_6(a) = 2 + 2a$, $G_7(a) = 3 + 2a$, $G_8(a) = 4 + 3a$, etc. We immediately see that we have

$$G_n(a) = G_{n-2}(1) + a\,G_{n-3}(1)\,, \tag{34}$$

so that all the variations of $a \neq 1$ do not induce any real gain in generality.

The rule of this type may be understood as one of the manifestations of the linearity of our present example. This property opens a path towards applications of the similar models in statistics where certain generalized Fibonacci numbers proved related to the close-packed dimers on non-orientable surfaces [4], to the local temperature distribution on quasiperiodic chains [5], to the existence of fractional statistics in quantum gases of quasiparticles [6], to the statistics born by the stacking of squares on a staircase [7] and to the electron and phonon excitations in quasicrystals [8].

Closely related use of the linearity of the three-term Fibonacci-like recurrences helped, in [9], to introduce randomness directly in the coefficients a in a way resembling equation (34), with possible impact ranging from the modelling of chaos (e.g., in quantum gate networks and quantum Turing machines with the Turing head controlled by a Fibonacci-like sequence of rotation angles [10]) till explicit models of the critical level-spacing distributions, with eigenvectors lying between extended and localized [11]. Last but not least, a strong appeal of all these models (which may all be interpreted as various discretized versions of Schrdinger equations) lies in their generic non-Hermiticity (see [12]) which became subject to an intensive study recently (see [13] or all papers in the dedicated issue [14]).

4.3. Numerical "Miracles" and Open Questions

An exceptional role of our *most elementary* modification (2) of the *most popular* Fibonacci's recurrences has been illustrated here with a particular emphasis on some of the *most interesting numerical features* of the "Gibonacci" sequence G_n. In this sense, our key message has been based, in essence, on the remarkably quick convergence of the asymptotic series (19).

We did not pay attention to all the properties of G_m of the similar numerical type. For example, we did not throw any light on an alternative numerical relation between our asymptotic quotient $\varrho_{(A)}$ and the golden mean $\eta_{(a)}$ which is based on the evaluation of the logarithm of their ratio,

$$\ln\left(\frac{\varrho_{(A)}}{\eta_{(a)}}\right) = 0.20001225073664160\ldots\,. \tag{35}$$

We see that it has an exceptional form (with several zero digits in it) as well as a remarkably compact approximate representations with higher precision, e.g.,

$$\frac{1}{5} + \frac{1}{8\cdot 10^4} - \frac{1}{4\cdot 10^6} + \frac{2}{27\cdot 10^8} - \frac{41}{10^{13}} = 0.20001225073664074\ldots\,. \tag{36}$$

The smallness of the subsequent corrections as well as the use of the natural logarithm in equation (35) do not have in fact *any* natural explanation. In the other words, the comparatively high reliability of the estimate

$$\varrho_{(A)}^5 = e\left(\frac{1+\sqrt{5}}{2}\right)^5 + \ldots \tag{37}$$

represents an unclarified numerical mystery. Why the two quotients $\varrho_{(A)}$ and $\eta_{(a)}$ should be related at all? And even if "yes", why are they related just to the base $e = \exp(1) \approx 2.718$ of natural logarithms?

Marginally, we may add a remark that in the next possible study of the doubly delayed recurrent relations of the above Fibonacci type,

$$H_n = H_{n-3} + H_{n-4}\,, \tag{38}$$

one could ask why precisely the one-step delay in equation (2) should be considered exceptional. The first answer could be purely pragmatic, stating that algebraic equations (5) and (7) seem to be the only sufficiently easily manageable pair of definitions of quotients. Indeed, in the next case with a double-step delay, even the straightforward asymptotic analysis could be marred by the less transparent solution of the quartic analogue of equations (5) and (7),

$$\tau^4 = \tau + 1\,. \tag{39}$$

Its two real roots

$$-0.72449195900051561159\ldots\,, \qquad 1.22074408460575947536\ldots \tag{40}$$

and their two complex conjugate partners

$$\tau_{(\pm)} = -0.24812606280262193189\ldots \pm i\,1.03398206097596775\ldots$$

can be hardly expressed via reasonably compact formulae. Moreover, on a deeper level one reveals that the absolute value $|\tau_{(\pm)}| = 1.0633$ of the two complex roots is bigger than one. This implies that in the doubly delayed case, the subdominant component of the asymptotics would not decrease anymore, with all the possible related complications which were not encountered *just* in the two above-listed cases, viz., in sequences generated by equations (1) and (2). This adds a further background to our belief that only the famous Fibonacci numbers F_n *and* their present one-step-delayed "Gibonacci" numbers G_m deserve really an *exceptional* attention.

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References

[1] Leonardo Pisano *alias* Fibonacci, *Liber Abaci*, Manuscript (1228); N.N. Vorobiev, *Fibonacci Numbers*, Nauka, Moscow (1978, In Russian.

[2] See the standard notation conventions available on the web address http://www.mathpropress.com/problemColumns/fq/fqDefinitions.html

[3] M. Znojil, *Phys. Lett. A*, **211** (1996), 319.

[4] W.T. Lu, F.Y. Wu, *Mod. Phys. Lett. B*, **16** (2002), 1177.

[5] M. Torikai, T. Odagaki, *J. Phys.: Condens. Matter*, **14** (2002), L503.

[6] M. Rachidi, E.H. Saidi, J. Zerouaoui, *Int. J. Mod. Phys.*, **A18** (2003), 159.

[7] L. Turban, *J. Phys. A: Math. Gen.*, **33** (2000), 2587.

[8] M.T. Velhinho, I.R. Pimentel, *Phys. Rev. B*, **61** (2000), 1043.

[9] C. Sire, P.L. Krapivsky, *J. Phys. A: Math. Gen.*, **34** (2001), 9065.

[10] I. Kim, G. Mahler, *Phys. Lett. A*, **263** (1999), 268.

[11] D.E. Katsanos, S.N. Evangelou, *Phys. Lett. A*, **289** (2001), 183.

[12] B. Derrida, J.L. Jacobsen, R. Zeitak, *J. Stat. Phys.*, **98** (2000), 31; D.E. Holz, H. Orland, A. Zee, *J. Phys. A: Math. Gen.*, **36** (2003), 3385.

[13] C.M. Bender, S. Boettcher, *Phys. Rev. Lett.*, **24** (1998), 5243; A. Ramirez, B. Mielnik, *Rev. Mex. Fis.*, **49S2** (2003), 130.

[14] cf. all Nr. 1 of *Czech. J. Phys.*, **54** (2004), 1-156.