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SIMILARITY SOLUTIONS FOR BOUNDARY LAYER FLOW ON A MOVING SURFACE IN AN OTHERWISE QUIESCENT FLUID MEDIUM

Liancun Zheng[1 §], Xiaohong Su[2], Xinxin Zhang[3]
[1,2]Department of Mathematics and Mechanics
University of Science and Technology Beijing
Beijing, 100083, P.R. CHINA
e-mail: liancunzheng@sina.com
[3]School of Mechanical Engineering
University of Science and Technology Beijing
Beijing, 100083, P.R. CHINA
e-mail: xxzhang@me.ustb.edu.cn

Abstract: This work considers the problem of boundary layer flow for a flat surface moving in an otherwise quiescent power law fluid medium. The laminar boundary layer equations are reduced to a singular nonlinear two-point boundary value problem by suitable similarity transformations. A rigorous proof of existence and uniqueness of solution to the boundary value problem is given and a theoretical estimation for skin friction coefficient is obtained. The reliability and efficiency of the proposed estimate formula are verified by numerical results with absolute errors obtained by lower bound no more than 0.0026 for power law exponent $n \geq 20$ and no more than 0.0004 for power law exponent $n \geq 200$.

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§Correspondence author

1. Introduction

Fluid dynamicists have long known that the appearance of boundary layers was not restricted to the canonical problem of the motion of a body through a viscous fluid. A technologically important source of the boundary layer phenomenon is the flow over a moving conveyor belt or the flow on a continuous moving surface [13]. Following the pioneering work of Sakiadis [11], [12], the problems of boundary layer over a continuous moving surface in an otherwise quiescent fluid medium have attracted considerable attention. The great majority of the theoretical investigations in this field described Newtonian fluid flow in the vicinity of the continuous moving surface with the aid of similarity solutions of the boundary layer equations. For a list of the key references of a vast literature concerning this subject in Newtonian fluids we refer to the recent papers [6], [16], [15], [8].

The drag force due to skin friction is a fluid dynamic resistive force, which is a consequence of the fluid viscous and the pressure distribution on the surface of the object [13]. Recently, a considerable attention has been devoted to the problem of how to predict the drag force behavior of non-Newtonian fluids. The main reason for this is probably that fluids (such as molten plastics, pulps, slurries, emulsions, etc.), which do not obey the Newtonian postulate that the tress tensor is directly proportional to the deformation tensor, are produced industrially in increasing quantities and are therefore in some cases just as likely to be pumped in a plant as the more common Newtonian fluids. Understanding the nature of this force by mathematical modeling with a view to predict the drag forces and the associated behavior of fluid flow has been the focus of research.

The theoretical analysis of an external boundary layer flow of non-Newtonian fluid was first performed by Schowalter [14] and Acrivos and Shah [1]. The boundary layer was formulated, and the condition for the existence of similarity solution was established in [14]. A similarity solution to the boundary layer equations for a power fluid flowing along a flat plate at zero degree of angle of attack was obtained in [1]. Later, Nachman and Callegari [9] discussed the nonlinear singular boundary value problem in the theory of pseudoplastic fluids when the Crocco variable was introduced, existence, uniqueness and analyticity are established. Howell [5] and Rao [10] examined momentum and heat transfer on a continuous moving surface in power law fluid. Recently, Zheng et al [17], [18], Lu and Zheng [7] studied a flat plate aligned with a uniform power law flow and continuous moving at constant speed in the direction or opposite to the direction to the mainstream in power fluid non-Newtonian, the existence, uniqueness or non-uniqueness of solutions to the problem are established by shooting technique.

This paper makes a theoretical analysis for the boundary layer flow for a flat surface moving in an otherwise quiescent non-Newtonian fluid medium. A special emphasis is given to the formulation of boundary layer equations, which provide similarity solutions. A rigorous proof of existence and uniqueness of similarity solutions to the laminar boundary layer equation and a theoretical estimation for skin friction coefficient is given, which is characterized by power law exponent.

2. Boundary Layer Governing Equations

The problem considered here is the steady boundary layer flow over a continuous moving flat surface in a an otherwise quiescent non-Newtonian fluid medium with a surface speed U_W. In the absence of body force, external pressure gradients and viscous dissipation, the laminar boundary layer equations expressing conservation of mass and momentum are governed by the equations:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{1}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{1}{\rho}\frac{\partial \tau}{\partial Y}, \tag{2}$$

where X, and Y axes are taken along and perpendicular to the plate, U and V are the velocity components parallel and normal to the plate, $\tau = K|\frac{\partial U}{\partial Y}|^{(n-1)}\frac{\partial U}{\partial Y}$ is the shear stress, and $\nu = \gamma|\frac{\partial U}{\partial Y}|^{n-1}(\gamma = K/\rho)$ is the kinematic viscosity. The case n=1 corresponds to a Newtonian fluid and for $0 < n < 1$ has been proposed as being descriptive of pseudoplastic non-Newtonian fluids and and $n > 1$ describes the dilatant fluid. The appropriate boundary conditions are:

$$U|_{Y=0} = U_w, \quad V|_{Y=0} = 0, \quad U|_{Y=+\infty} = 0. \tag{3}$$

Flows of this type are encounted in glacial advance, in transport of coal slurries down conveyor belts and in several other geophysical and industrial contexts.

3. Two-Point Boundary Value Problems

The following dimensionless variables are introduced [9], [5], [17], [18], [7]:

$$x = X/L, \quad y = (N_Re)^{1/(n+1)}Y/L, \quad u = U/U_W,$$

$$v = (N_{Re})^{\frac{1}{n+1}} V/U_W, \quad N_{Re} = \rho U_w^{2-n} L^n / K, \tag{4}$$

where L is a characteristic length and $U_W > 0$. Substituting (4) into equations (1)-(3) and introducing the stream function $\psi(x, y)$ and similarity variable η as [17], [18], [7]:

$$\psi = Ax^\alpha f(\eta), \quad \eta = Bx^\beta y, \tag{5}$$

where $A, B, \alpha,\ \beta$ are constants to be determined, and $f(\eta)$ denotes the dimensionless stream function. Thus the velocity components are respectively: $u = \frac{\partial \psi}{\partial y}$, and $v = -\frac{\partial \psi}{\partial x}$. In terms of existing similarity solutions, by choosing:

$$\alpha + \beta = 0, \quad AB = 1, \quad \alpha := \frac{1}{n+1}, \quad B := [(n+1)\gamma]^{-1/(n+1)},$$

we obtained the following nonlinear ordinary differential equations:

$$(|f''(\eta)|^{n-1} f''(\eta))' + f(\eta) f''(\eta) = 0, \tag{6}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f'(+\infty) = 0. \tag{7}$$

Equations (6)-(7) imply $f''(\eta) < 0$ in $(0, +\infty)$ and $f''(+\infty) = 0$. Define the variables transformation as $g(z)$ by

$$g(z) := [-f''(\eta)]^n, \quad z = f'(\eta), \quad z \in [0, 1), \tag{8}$$

where $z (= f')$ is the dimensionless tangential velocity as the independent variable and $g(z) (= (-f'')^n)$ is the shear stress as the dependent variable are what we call the generalized Crocco variable. Inserting equation (8) into equations (6)-(7) and applying the chain rule, we obtain the following singular nonlinear boundary value problems

$$g''(z) = -z g^{-1/n}(z), \quad 0 < z < 1, \tag{9}$$

$$g(0) = 0, \quad g'(1) = 0. \tag{10}$$

Clearly, it may be seen from the derivation process that only the positive solutions of the problems (9)-(10) are physically of significance.

4. Existence and Uniqueness of Solutions

In this section we establish the existence and uniqueness to the singular nonlinear two-point boundary problems (9)-(10). Letting $t = -z, w(t) = g(-z)$, equations (9)-(10) become:

$$w''(t) = tw^{-1/n}(t), \quad -1 < t < 0, \tag{11}$$

$$w'(-1) = 0, \quad w(0) = 0. \tag{12}$$

Since the problem is singular at $t = 0$, it is convenient by considering equation (11) subjet to the boundary conditions without sigularities

$$w'(-1) = 0, \quad w(0) = \xi, \tag*{$(12)_\xi$}$$

where ξ is nonnegative parameters.

Denote the solution of equations (11)-$(12)_\xi$ by $w_\xi(t)$, we first show the following lemmas.

Lemma 1. *If $\xi_1 > \xi_2 > 0$, then $w_{\xi_1}(t) \geq w_{\xi_2}(t)$.*

Proof. If the inequality is not true, then there exists a point $t_0 \in [-1, 0)$ such that $w_{\xi_1}(t_0) < w_{\xi_2}(t_0)$.

Case (i). $w_{\xi_1}(-1) < w_{\xi_2}(-1)$.

Choose $t = -1$ as $t = t_0$. Since $w_{\xi_1}(0) > w_{\xi_2}(0) > 0$, then there exists a maximal interval $[-1, k](k < 0)$, such that $w_{\xi_1}(t) < w_{\xi_2}(t)$, for $t \in [-1, k)$ and $w_{\xi_1}(k) > w_{\xi_2}(k) = m > 0$. $w_{\xi_1}(t)$ and $w_{\xi_2}(t)$ are both the positive of the integral equation:

$$w(t) = m + \int_{-1}^{k} G_1(t, s) s w^{-1/n}(s) ds, \tag{13}$$

where the Green's function

$$G_1(s, t) := \begin{cases} (t - k), & \text{if} \quad -1 \leq s \leq t \leq k < 0, \\ (s - k), & \text{if} \quad -1 \leq t \leq s \leq k < 0. \end{cases}$$

Equation (13) implies

$$0 < w_{\xi_2}(t) - w_{\xi_1}(t) = \int_{-1}^{k} G_1(t, s)[w_{\xi_2}^{-1/n}(s) - w_{\xi_1}^{-1/n}(s)] s ds < 0,$$

which ia a contradiction.

Case (ii). $w_{\xi_1}(-1) \geq w_{\xi_2}(-1)$.

Since $w_{\xi_1}(0) > w_{\xi_2}(0) > 0$, then there exists a maximal interval $[a, b](-1 \le a < b < 0)$, which contains the point t_0 such that $w_{\xi_1}(a) = w_{\xi_2}(a)$ and $w_{\xi_1}(b) = w_{\xi_2}(b)$, and $w_{\xi_1}(t) < w_{\xi_2}(t)$ for $t \in (a, b)$. Let $w_{\xi_1}(a) = w_{\xi_2}(a) = \alpha$ and $w_{\xi_1}(b) = w_{\xi_2}(b) = \beta$, then for $t \in [a, b]$, $w_{\xi_1}(t)$ and $w_{\xi_2}(t)$ are the positive solutions of integral equation:

$$w(t) = \frac{a\beta - b\alpha}{a - b} + \frac{\alpha - \beta}{a - b}t + \int_a^b G_2(t, s)sw^{-1/n}(s)ds, \tag{14}$$

where the Green's function

$$G_2(s, t) := \begin{cases} \dfrac{(b - t)(s - a)}{a - b}, & \text{if } \; -1 \le a \le s \le t \le b < 0, \\ \dfrac{(b - s)(t - a)}{a - b}, & \text{if } \; -1 \le a \le s \le t \le b < 0. \end{cases}$$

Equation (14) implies,

$$0 < w_{\xi_2}(t) - w_{\xi_1}(t) = \int_a^b G_2(t, s)[w_{\xi_2}^{-1/n}(s) - w_{\xi_1}^{-1/n}(s)]sds < 0.$$

Which also is a contradiction. This completes the proof. □

Lemma 2. *For each fixed* $\xi > 0$, *equations (11)-*$(12)_\xi$ *have at most one positive solution* $w_\xi(t)$.

Proof. Suppose that equations (11)- $(12)_\xi$ have two different soutions $w_{\xi_1}(t)$ and $w_{\xi_2}(t)$ for each fixed $\xi > 0$. Without loss of generality, we may assume that there exists a point $t_0 \in [-1, 0)$ such that $w_{\xi_1}(t_0) > w_{\xi_2}(t_0)$. Since $w_{\xi_1}(0) = w_{\xi_2}(0) = \xi$, then there exists a maximal interval $[a_1, b_1] \subseteq [-1, 0]$ such that $w_{\xi_1}(t) \ge w_{\xi_2}(t)$ for $t \in (a_1, b_1)$.

(i) If $a_1 = -1$, then $w_{\xi_1}(t) \ge w_{\xi_2}(t)$ for $t \in [-1, b_1] \subseteq [-1, 0]$ and $w_{\xi_1}(b) = w_{\xi_2}(b)$.

(ii) If $a \ne -1$, then $w_{\xi_1}(a_1) = w_{\xi_2}(a_1)$ for $t \in [a_1, b_1] \subset [-1, 0]$, $w_{\xi_1}(b_1) = w_{\xi_2}(b_1)$, and $w_{\xi_1}(t) > w_{\xi_2}(t)$ for $t \in (a_1, b_1)$.

It follows along the same lines as the Case (i) and Case (ii) in Lemma 1, we may show which is impossible. □

Lemma 3. *For each fixed* $\xi > 0$, *equations (11)-*$(12)_\xi$ *have one positive solution* $w_\xi(t)$.

Proof. For each fixed $\xi > 0$, if $w(t)$ is the positive solution of equations (11)-$(12)_\xi$, then $w(t)$ is convex on $[-1, 0]$ and must be a positive solution of the integral equation:

$$w(t) = \xi + \int_{-1}^0 G_3(t, s)sw^{-1/n}(s)ds, \tag{15}$$

where the Green's function

$$G_3(s,t) := \begin{cases} t, & \text{if } \ -1 \le t \le s \le 0, \\ s, & \text{if } \ -1 \le s \le t \le 0. \end{cases}$$

Defining a Mapping T :

$$T_{w(t)} = \xi + \int_{-1}^{0} G_3(t,s) s w^{-1/n}(s) ds.$$

Thus, $T_{w(t)} \ge \xi$; let $\xi := \underline{\omega}(t)$, and for $x(t) \ge \underline{\omega}(t)$,

$$T_{x(t)} \le \xi + \int_{-1}^{0} G_3(t,s) s \underline{w(s)}^{-1/n} ds := \overline{\omega}(t).$$

Define $\Omega = x(t) \in C[-1,0] : \underline{\omega}(t) \le x(t) \le \overline{\omega}(t)$. Then, T is a completely continuous mapping from $\Omega \to \Omega$. The Schauder Fixed Theorem [3], [2], [4] asserts that the mapping T has at least one fixed point $x_\xi(t)$ in Ω, this implies that $x_\xi(t)$ is a positive solution of equations (11)-(12)$_\xi$. □

Lemma 4. *For any fixed $\xi > 0$ and $n > 0$, the positive solution of equations (11)-(12)$_\xi$ satisfying: $w_\xi(-1) > (\frac{1}{\sqrt{3}})^{\frac{2n}{n+1}}$.*

Proof. We denote $w(-1) = \alpha$ and consider the initial value problem

$$\begin{cases} w''(t) = t/w^{-1/n}(t), & -1 < t < 0, \\ w(-1) = \alpha > 0, \quad w'(-1) = 0. \end{cases} \tag{16}$$

Equation (16) yields,

$$w(t) =\le \alpha + \frac{1}{6\alpha^{1/n}}(t+1)^2(t-2) \le \alpha - \frac{1}{3\alpha^{1/n}}(t+1)^2.$$

Let $f(t) = \alpha - \frac{1}{3\alpha^{1/n}}(t+1)^2$, the solution of equation (16) satisfies $w(t) \le f(t)$ for $t \in (-1,0)$. Let the function $f(t)$ intersects the x-axis at the point $t_0^\star$, then, $t_0^\star = -1 + \sqrt{3}\alpha^{\frac{2n}{n+1}}$. Especially, for $t_0^\star = 0$, we obtain $\alpha = (\frac{1}{\sqrt{3}})^{\frac{2n}{n+1}} > \frac{1}{3}$.

For $\alpha \le \frac{1}{3}$, the solution $w(t)$ cannot intersect the $x-$ axis at the point $x \ge 0$. This shows that for $\alpha \le \frac{1}{3}$, the solution of initial value problem (16) only holds for $w(0) < 0$. □

Theorem 5. *For any fixed $n > 1/2$, equations (9)-(10) have a unique positive solution $g(z)$, the solution satisfies the estimate formula:*

$$\left(\frac{1}{\sqrt{3}}\right)^{\frac{2n}{n+1}} < g(1) < \frac{n}{3^{-1/(n+1)}(2n-1)}.$$

Proof. Lemma 2 and Lemma 3 imply that for any $\xi > 0$, equations (11)-$(12)_\xi$ have a unique positive solution. Then, for any $\xi_2 > \xi_1 > 0$, it follows from equation (15) and Lemma 1 that

$$0 < w_{\xi_2}(t) - w_{\xi_1}(t) \\ = \xi_2 - \xi_1 + \int_{-1}^{0} G_3(t,s)[w_{\xi_2}^{-1/n}(s) - w_{\xi_1}^{-1/n}(s)]sds \leq \xi_2 - \xi_1.$$

It indicates the series of solutions $w_\xi(t)$ converges to a limit uniformly with ξ on $[-1, 0]$, denoted by $w_0(t)$. Then, letting $\xi \to 0$ in (15), we have

$$\lim_{\xi\to 0} w_\xi(t) = w_0(t),$$

uniformly on [-1, 0].

Lemma 4 implies $w_0(-1) > \frac{1}{3}$. For any $\xi > 0$, by the convexity of $w_\xi(t)$,

$$w_\xi(t) \geq (\xi - w_\xi(-1))t + \xi \geq (\xi - \frac{1}{3})t + \xi \geq -\frac{1}{3}t. \tag{17}$$

Thus,

$$G_3(t,s)\frac{s}{w_\xi^{1/n}(s)} \leq -3^{1/n} G_3(t,s)(-s)^{1-1/n}. \tag{18}$$

Letting $\xi \to 0^+$ in integral equation (15), by the Monotone Convergence Theorem [3], [2], [4], we obtain that for $n > 1/2$,

$$w_0(t) = \lim_{\xi\to 0} \int_{-1}^{0} G_3(t,s)sw_\xi^{-1/n}(s)ds = \int_{-1}^{0} \lim_{\xi\to 0} G_3(t,s)sw_\xi^{-1/n}(s)ds,$$

i.e.,

$$w_0(t) = \int_{-1}^{0} G_3(t,s)sw_0^{-1/n}(s)ds. \tag{19}$$

The above arguments indicate that equations (11)-$(12)_0$ have a unique positive solution $w_0(t)$. Further, the convexity of $w_\xi(t)$ and Lemma 4 imply that for any $\xi \geq 0$ and power law exponent $n (n > 1/2)$,

$$w_\xi(t) \geq (\xi - w_\xi(-1))t + \xi \geq (\xi - \frac{1}{3})t + \xi \geq (\xi - 3^{-\frac{n}{n+1}})t + \xi \geq -3^{-\frac{n}{n+1}}t.$$

Thus, for $n > 1/2$,

$$G_3(t,s)\frac{s}{w_\xi^{1/n}(s)} \leq -3^{1/(n+1)} G_3(t,s)(-s)^{1-1/n}. \tag{20}$$

It follows from (19) and (20) that,

$$(\sqrt{3})^{-\frac{2n}{n+1}} < w(-1) < 3^{\frac{1}{n+1}}\frac{n}{2n-1}.$$

This proves that equations (9)-(10) have a unique positive solution $g(z)$, which satisfies:

$$(\sqrt{3})^{-\frac{2n}{n+1}} < g(1) < 3^{\frac{1}{n+1}}\frac{n}{2n-1}. \tag{21}$$

5. Numerical Solutions and Discussions

In order to illustrate the reliability and efficiency of the proposed estimate formula (21), equations (9)-(10)are solved for values of power law exponent n ($n > 1/2$) by employing the shooting technique. As far as the skin friction coefficient $g(1)$ is concerned, we denote the nmerical results of $g(1)$ by $\sigma_{com.}$, the estimate results obtained by lower-bound of $\sigma_{lower-bound}(= \sqrt{3}^{-\frac{2n}{n+1}})$, and obtained by upper-bound of $\sigma_{upper-bound}(= 3^{\frac{1}{n+1}}\frac{n}{2n-1})$. The numerical results and estimated results are presented in Table 1.

Exponent	$\sigma_{lower-bound}$	$\sigma_{com.}$	$\sigma_{upper-bound}$
$n = 0.6$	0.6623	0.7344	5.9610
$n = 0.8$	0.6137	0.6715	2.454
$n = 1.0$	0.5773	0.6254	1.7320
$k = 2.0$	0.4807	0.5063	0.9614
$n = 3.0$	0.4386	0.4558	0.7896
$n = 5.0$	0.4003	0.4106	0.6671
$n = 8.0$	0.3766	0.3830	0.6025
$n = 10.0$	0.3683	0.3735	0.5815
$n = 20.0$	0.3512	0.3538	0.5403
$k = 30.0$	0.3453	0.3470	0.5268
$n = 50.0$	0.3405	0.3415	0.5160
$n = 80.0$	0.3378	0.3385	0.5100
$n = 100.0$	0.3369	0.3375	0.5080
$n = 200.0$	0.3351	0.3354	0.5040

Table 1: Skin friction for values of power law exponent n

The reliability and efficiency of the proposed estimate formula are verified by numerical results with absolute error obtained by lower bound no more than

0.0026 for power law exponent $n \geq 20$ and no more than 0.0004 for power law exponent $n \geq 200$. The lower bound presented in this paper can be successfully applied to give the value of skin friction coefficient for suitable large of power law exponent.

6. Conclusions

This paper presents a similarity analysis for the steady boundary layer flow over a continuous moving flat surface in a an otherwise quiescent non-Newtonian fluid medium with a surface speed U_W. The boundary layer equations are changed into a singular nonlinear two-point boundary value problem of ordinary differential equation when similarity variables and generalized Crocco variables were introduced. Sufficient conditions for existence and uniqueness of positive solutions are established. Furthermore, a theoretical estimate formula for skin friction coefficient is given, which is characterized by power law exponent. The reliability and efficiency of the proposed estimate formula are verified by numerical results. The lower bound presented in this paper can be successfully applied to give the value of skin friction coefficient for suitable large of power law exponent.

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