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# ON REGULAR MODULES, II

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**Abstract:** In this paper we study the relationship between the projectivity and regualarity. We shows that every projective module over a von Neumann regular ring is regular. We also show that a projective module $P$ is regular if and only if every cyclic submodule of $P$ is a direct summand of $P$ and we shall see that if $P$ is a projective regular module then $J(P) = Z(P) = 0$, where $J(P)$ and $Z(P)$ are the Jacobson and singular submodules of $P$ respectively. $P$ is regular and indecomposible iff $P$ is projective and hollow. In this paper $R$ will be a commutative ring with identity and all modules are unitary.

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**Definition.** Let $M$ be an $R$-module and element $x \in M$ is called regular if there is $\phi \in \mathrm{Hom}_R(M, R)$ such that $\phi(x)x = x$. An $R$-module $M$ is called regular if every element $x \in M$ is regular.

**Definition.** An $R$-module $M$ is called hollow if every submodule of $M$ is small in $M$.

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**Lemma 1.** *If $P$ is a regular $R$-module then for every $x \in P$, $Rx$ is a projective $R$-module.*

*Proof.* Since $P$ is regular, there is $f \in \operatorname{Hom}_R(P, R)$ such that $f(x)x = x$. If we write $e = f(x)$ then $ex = x$ and $e^2 = e$. Now we define $g : Rx \to Re$ by $g(rx) = re$, if $rx = r'x$, so $(rr')x = 0$, then $(r - r')f(x) = 0$ so $(r - r')e = 0$ and $re = r'e$, clearly $g$ is a well-defined epimorphism. Now if $rx \in \ker g$ so $g(rx) = re = 0, rf(x) = 0$. And, as $f(x)x = x$ we see that $rx = 0$, $\ker g = \{0\}$ so $Rx \cong Re$, and since $e$ is an idempotent, $Rx$ is a projective module. □

**Lemma 2.** *Let $R$ be a ring and $0 \to K \to P \to M \to 0$ an exact sequence of $R$-modules with $P$ projective. Then the following statement are equivalent:*

*(1) $M$ is flat.*

*(2) For $x \in K$, there exists a homomorphism $g : P \to K$ such that $g(x) = x$.*

*(3) For $x_1, x_2, \cdots, x_n$ in $K$ there exists a homomorphism $g : P \to K$ such that $g(x_i) = x_i$.*

*Proof.* Since $P$ is projective then $P$ is a direct summand of a free module $F$, so $F = P \oplus Q$, then $\frac{F}{K} \cong \frac{P}{K} \oplus Q \cong M \oplus Q$. As the sequence $0 \to K \to F \to M \oplus Q \to 0$ is exact and $Q$ is projective then $M \oplus Q$ is flat if and only if $M$ is flat. Chase in [4], Proposition 2.1, proved the lemma when $P$ is a free module.

(1) $\Rightarrow$ (3). Let $x_1, x_2, \cdots, x_n \in K$ and suppose $M$ is flat. Then $M \oplus Q$ is flat so there is $h : F \to K$ such that $h(x_i) = x_i$, let $g|p$. Then $g : P \to K$ and $g(x_i) = x_i, i = 1, \cdots, n$.

(3) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). Let $x \in K$. Then there is $g : P \to K$ such that $g(x) = x$, extend $g$ to $h : F \to K$ by $h(y) = 0$ for $y \in Q$ so by Chase's result we have $M \oplus Q$ is flat so $M$ is flat. □

**Theorem 3.** *Let $P$ be a projective $R$-module. Then the following statements are equivalent:*

*(1) $P$ is regular.*

*(2) Every homomorphic image of $P$ is flat.*

*(3) Every cyclic submodule of $P$ is a direct summand of $P$.*

*(4) Every finitely generated submodule of $P$ is a direct summand.*

*(5) Every submodule of $P$ is pure.*

*Proof.* We prove that (1) and (3) are equivalent (1) $\longleftrightarrow$ (3): Let $P$ be a regular module and $x \in P$ then there exists $\phi : P \to R$ such that $\phi(x)x = x$ so $P \cong Rx \oplus K$, where $K = \{y \in P | \phi(y)x = 0\}$.

(3) $\Rightarrow$ (1). Let $x \in P$. By (3), there exists a submodule $K$ of $P$ such that $P = Rx \oplus K$, and since $P$ is projective, $Rx$ is projective and hence the diagram

$$\begin{array}{ccccc} & & Rx & & \\ & \phi \swarrow & \downarrow & & \\ R & \underset{f}{\longrightarrow} & Rx & \rightarrow & 0 \end{array}$$

is commutative, where $f(r) = rx$, let $\phi(x) = r$ then $f(\phi(x)) = i(x) = x$ which implies that $rx = x$ and hence $\phi(x)x = x$. Now we define $\rho : P \rightarrow R$ by $\rho(rx + k) = \phi(rx)$ and hence $\rho(x) = \phi(x)x = x$ so $x$ is regular.

(2) $\Rightarrow$ (5). Let $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ be exact sequence. By [3], p. 122, $M$ is flat iff for every ideal $I$ of $R, IP \cap K = IK$.

(2) $\Rightarrow$ (4). Suppose $K = Rx_1 + Rx_2 + \cdots + Rx_n$ is a finitely generated submodule of $P$, so we have an exact sequence $0 \rightarrow K \rightarrow P \rightarrow \frac{P}{K} \rightarrow 0$ and since $\frac{P}{K}$ is flat by Lemma 2 there is an $R$-module homomorphism $g : P \rightarrow K$ such that $g(x_i) = x_i$, then $K$ is a direct summand of $P$.

(4) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (2). Let $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ be an exact sequence. Then for each $x \in K, Rx$ is a direct summand of $P$. So, there is $g : P \rightarrow K$ such that $g(x) = x$. Therefore, $M$ is flat, by Lemma 2. □

**Proposition 4.** *Any projective module $P$ over a von Neumann regular ring is regular.*

*Proof.* By Theorem 1.11, [5], every finitely generated submodule of $P$ is direct summand and hence by Theorem 3, $P$ is a regular $R$-module. □

**Remark.** Any semisimple projective $R$-module is regular, since every submodule of semisimple $R$-module is a direct summand.

**Lemma 5.** *If $P$ is a projective regular $R$-module then $J(P) = 0$.*

*Proof.* Let $K$ be a submodule of $P$ and $x \in K$, then $P = Rx \oplus T$, where $T = \{y \in P | \phi(y)x = 0\}, \phi \in \operatorname{Hom}_R(P, R)$. As $Rx \oplus T \subset K + T, P = K + T$. Since $T \neq P, K$ is not an small submodule of $P$, so $P$ has no non-zero small submodule and since $J(P) = \sum K$, where $K$ is small, $J(P) = 0$.

**Lemma 6.** *If $P$ is a projective regular $R$-module then $\operatorname{Ann}_R(P)$ is an intersection of maximal ideals.*

*Proof.* $\operatorname{Ann}(P) = \bigcap\limits_{x \in P} \operatorname{ann}(x)$, if $x \in P$ then $Rx$ is a direct summand of $P$, so is projective then $J(Rx) = 0$, and since $\frac{R}{\operatorname{ann}(x)} \simeq Rx$, so $J(Rx) = J(\frac{R}{\operatorname{ann}(x)}) = \operatorname{ann}(x) = \cap m$, where $m$ is a maximal ideal in $R, \operatorname{Ann}(x) \subset m$ so $\operatorname{Ann}(P) = \cap m$. □

**Lemma 7.** *If $P$ is a projective regular $R$-module then $Z(P) = 0$.*

*Proof.* Let $x \in Z(P)$ since $Rx$ is projective, the exact sequence $0 \to \text{ann}\,(x) \to R \to Rx \to 0$ spilits, which implies that $\text{ann}_R(x)$ is a direct summand of $R$ and so is not essential, hence $Z(P) = 0$. □

**Remark.** If $P$ is a projective regular $R$-module then $J(P) = Z(P)$.

**Lemma 8.** *Suppose $P$ is a Noetherian regular projective module over a hereditary ring $R$, then $P$ is isomorphic to a finite directs sum of minimal ideals of $R$.*

*Proof.* Since $P$ is projective and Noetherian, every submodule of $P$ is finitely generated and by Theorem 3 is a direct summand of $P$. Let $K$ be a maximal submodule of $P$, then $P = K \oplus N_1, N_1$ is a simple submodule of $P$. We applied the argument for $K$, for $K = N_2 \oplus T$, where $N_2$ is simple. Eventually as $P$ is Noetherian we can write $P = N_1 \oplus N_2 \oplus \cdots \oplus N_k$, where $N_i$ is a simple $R$-module. Since $N_i$ is regular there exists a homomorphism $f_i : N_i \to R$ such that $f_i(x_i)x_i = x_i$, where $N_i = Rx_i, x_i \in P$ we show that $N_i \simeq < f_i(x_i) >$. If we define $g_i : N_i \to < f_i(x_i) >$ by $g_i(rx_i) = rf_i(x_i)$, then $\ker g_i = \{rx_i \in N_i | rf(x_i) = 0\}$. Since $f_i(x_i)x_i = x_i$ so $f_i(rx_i) = rx_i = 0$, i.e., $\ker g_i = \{0\}$. Also $g_i$ is an epimorphism, so $N_i \simeq < f_i(x_i) >$ and hence

$$P = N_1 \oplus N_2 \oplus \cdots N_k \simeq f_1(x_1) > \oplus \cdots \oplus < f_k(x_k) >,$$

where $< f_i(x_i) >$ is a simple ideal of $R$. □

**Corollary 9.** *If $P$ is a projective Noetherian regular $R$-module over a hereditary ring $R$, then $P = \text{Soc}\,(P)$ and hence $P$ is semisimple.*

*Proof.* By above proposition $P \simeq \bigoplus_{i=1}^{k} N_i$, where $N_i$ is a simple $R$-module so $P = \text{Soc}\,(P)$ and hence $P$ is semisimple. □

**Lemma 10.** *If $P$ is a projective regular $R$-module and if $S = \text{End}\,(P)$ then $J(S) = 0$.*

*Proof.* We have $J(S) = \text{Hom}_R(P, J(P))$ and since $P$ is regular so $J(P) = 0$ and hence $J(S) = 0$. □

**Proposition 11.** *If $P$ is a finitely generated regular projective $R$-module then $\text{Soc}\,(P)$ is flat.*

*Proof.* Since $P$ is finitely generated projective regular so $\text{End}\,(P)$ is a regular ring so by [2], 1.2, $\text{Soc}\,(P)$ is $\text{End}\,(P)$-module so is flat. □

**Theorem 12.** *The following statements are equivalent for a module $P$ with $J(P) = 0$.*

*(1) $P$ is regular and indecomposable.*

*(2) $P$ is projective and* End $(P)$ *is local ring.*

*(3) $P$ is projective and local.*

*(4) $P$ is projective and hollow.*

*Proof.* (1) $\Rightarrow$ (2). Let $0 \neq x \in P$. Then $P = Rx \oplus W$ where $W = \{w \in P | \phi(w)x = 0\}$ and $\phi \in \text{Hom}_R(P, R)$. As $P$ is indecomposable, $W = 0$ and $P = Rx$, so $Rx \cong Re$ where $e = \phi(x)$. Therefore $P$ is projective and hence End $(P)$ is local, by [2, 17.19].

(2) $\Rightarrow$ (3). See [2], 17.19.

(3) $\Rightarrow$ (4). Let $N \subset P$, we show that $N$ is small. If $N + K = P$ since $P$ is projective and local, there is a maximal submodule $N_0$ such that $N \subset N_0$, if $K \neq P$ then $K \subset N_0$, so $N_0 = P$, that is contradiction.

(4) $\Rightarrow$ (1). Since $P$ is projective, then there is a maximal submodule $N_0$ of $P$ which is also small and since $J(P) = 0, N_0 = 0$. Hence $P = Rx$ for every $x \neq 0$, so the diagram

$$\begin{array}{ccccc} & & Rx & & \\ & \swarrow & \downarrow & & \\ R & \rightarrow & Rx & \rightarrow & 0 \end{array}$$

is commutative, then $x$ is regular. Hence $P$ is regular, since $P$ is hollow so $P$ is indecompasible. □

## References

[1] N. Amiri, M. Ershad, On regular modules, *IJPAM*, **19**, No. 1 (2005).

[2] F.W. Anderson, K.R. Fuller, *Ring and Categories of Module*, Springer-Verlag (1974).

[3] H. Cartan, S. Eilenberg, *Homological Algebra*, Princton Univ. Press, Princeton, N.J. (1956).

[4] S.U. Chase, Direct product of modules, *Trans. Amer. Math. Soc.*, **97** (1960), 457-473.

[5] Goodearl, *Von Neumann Regular Ring*, Pitman Publishing Limited (1979).