EXTENDING GAMES WITH LOCAL AND ROBUST LYAPUNOV EQUILIBRIUM AND STABILITY CONDITION

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Abstract: In this paper we extend the non-cooperative game theory including the Lyapunov’s equilibrium and stability criteria, locally and robustly. Equilibrium and stability conditions are obtained by defining the utility function as a Lyapunov function. In this sense, we present some properties about utility functions to show that an isomorphism can be determined between any utility function and the Lyapunov utility function. We introduce the Lyapunov equilibrium point as an alternative definition to the Nash equilibrium point for games. We prove that the concept of Lyapunov equilibrium coincides in this case with the concept of Nash equilibrium. The advantage of this approach is that fixed-point conditions for games are given by the definition of the Lyapunov function. We show that the game is asymptotically stable in the Lyapunov sense. A formal treatment leading to interesting mathematical results, and open problems

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in game theory are presented.

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1. Introduction

Non-cooperative game theory has been extensively used to analyze situations of interaction. The most important solution in non-cooperative games is the notion of Nash equilibrium. Formally, Nash equilibrium defines an equilibrium of a non-cooperative game with respect to a profile of strategies, one for each player in the game, such that each player’s strategy attempts to maximize that player’s expected utility opposed to the set of strategies of the other players ([8], [9], [10]). Then, players are in equilibrium if a change in strategies by any one of them would lead that player to earn less than if he stood by the current strategy. For games in which players randomize mixed strategies, the expected utility must be at least as large as that available by any one of the strategies ([3], [4], [5]).

In order to maximize the utility we propose to define the payoff functions as Lyapunov-like functions. The core idea of our approach uses a utility function that is non-negative and converges asymptotically to the equilibrium point. For instance, in the arms race the level of defense of a nation is non-negative. In economics models there are variables that correspond, e.g. with quantities of goods that remain non-negative.

In this equilibrium each player chooses a strategy with a utility equal to the utility that this strategy is a best reply to a strategy profile chosen by the opponents. The advantage of this approach is that fixed-point conditions for the game are given by the definition of the Lyapunov-like function, however formally it is not necessary for a fixed-point theorem to satisfy the Nash equilibrium conditions as usual. In addition, new properties of equilibrium and stability are consequently introduced for finite $n$-player non-cooperative games.

We show that the concept of Lyapunov equilibrium coincides with the concept of Nash equilibrium in a local and robust sense. Utility function properties are described with the purpose of showing that it is possible to establish an isomorphism between the preference order determined by any utility function and the Lyapunov-like function.

The rest of this paper is structured in the following manner. The next section presents the necessary mathematical background and terminology needed
to understand the rest of the paper. Section 3 describes the basic formalism of the game model. The issues associated to the utility functions isomorphisms are discussed in Section 4. Next, Section 5 presents the equilibrium and stability definition and theorems. Finally, Section 6, concludes the paper by giving future research directions.

2. Preliminaries (see [7])

In this section, we present some definitions and well-established properties which will be used later.

Let $\mathbb{R}_+ = [0, \infty)$. Given $x, y \in \mathbb{R}^k$, we usually denote the relation “$\leq$” to mean componentwise inequalities with the same relation, i.e., $x \leq y$ is equivalent to $x_i \leq y_i, \forall i$.

Consider systems of first ordinary differential equations given by:

$$
\dot{x}(t) = f(x(t), g(t)), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{R}^n, \quad g(t) \in \mathbb{R}^m,
$$

where we assume that $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is continuous and for each two compact subset $K \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^m$ there is some constant $l$ such that $\|f(x, g) - f(y, g)\| \leq l \|x - y\| \quad \forall x, y \in K$ and $\forall g \in D$, i.e. $f(x, g)$ is locally Lipschitz in $x$ and uniformly in $g$, and note that these properties are satisfied if $f$ extends to a continuously differentiable function on a neighborhood of $\mathbb{R}^n \times D$.

The perturbation functions $g$ are supposed to lie in the space $\mathcal{M}$ of measurable and locally bounded functions with values in $D$, where $D$ is an arbitrary subset of $\mathbb{R}^m$. For each $g \in \mathcal{M}$, we denote by $x(t, x_0, g)$ the solution at time $t$ of (1) with $x(0) = x_0$.

We say that the closed set $\mathcal{I}$ is an invariant set for (1) if

$$
\forall x_0 \in \mathcal{I}, \quad \forall g \in \mathcal{M} \quad x(t, x_0, g) \in \mathcal{I}, \quad \forall t \geq 0.
$$

For each nonempty subset $\mathcal{I} \subseteq \mathbb{R}^n$, and each $\rho \in \mathbb{R}^n$, we denote by

$$
\|\rho\|_{\mathcal{I}} = g(\rho, \mathcal{I}) = \inf_{\tau \in \mathcal{I}} g(\rho, \tau),
$$

which is the common point-to-set distance, and $\|\rho\|_{\{0\}} = \|\rho\|$ is the usual norm of $S$.

**Definition 2.1.**

\footnote{The expression $\|\|$ denotes the usual Euclidian norm}
A continuous function \( \alpha : \mathbb{R}_+ \to \mathbb{R}_+ \) is called a \( \mathcal{K} \)-function if \( \alpha(0) = 0 \), and it is strictly increasing for all \( s \geq 0 \).

A continuous function \( \alpha : \mathbb{R}_+ \to \mathbb{R}_+ \) is called a \( \mathcal{K}_\infty \)-function if it is a \( \mathcal{K} \)-function and \( \alpha(s) \to \infty \) when \( s \to \infty \).

A continuous function \( \beta : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+ \) is called a \( \mathcal{KL} \)-function if \( \beta(s, t) \) is a \( \mathcal{K} \)-function in \( s \) \( \forall t \in \mathbb{N} \) and it is strictly decreasing in \( t \) \( \forall s \geq 0 \) (note that \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \)).

**Definition 2.2.** The solution of equation (1) is called “robust” with respect to the closed invariant set \( \mathcal{I} \) if the following properties hold:

1. There exists a \( \mathcal{K}_\infty \)-function \( \delta(\cdot) \) such that for any \( \varepsilon \geq 0 \),
   \[
   \|x(t, x_0, g)\|_{\mathcal{I}} \leq \varepsilon \quad \forall g \in \mathcal{M},
   \]
   whenever \( \|x_0\| \leq \delta(\varepsilon) \) and \( t \geq 0 \), and

2. for any \( r, \varepsilon > 0 \), there is a \( T > 0 \), such that \( \forall g \in \mathcal{M}, \)
   \[
   \|x(t, x_0, g)\| < \varepsilon,
   \]
   whenever \( \|x_0\|_{\mathcal{I}} < r \) and \( t \geq T \).

**Proposition 2.1.** The equation (1) is uniformly globally asymptotically stable with respect to a closed, invariant set \( \mathcal{I} \) if and only if there exists a \( \mathcal{KL} \) function \( \beta \) such that, given any initial state \( x_0 \), the solution \( x(t, x_0, g) \) satisfies
   \[
   \|x(t, x_0, g)\|_{\mathcal{I}} \leq \beta(\|x_0\|, t), \quad \forall t \geq 0 \text{ and } \forall g \in \mathcal{M}.
   \]

For any differentiable function \( V : \mathbb{R}^n \to \mathbb{R} \), we use the standard Lie derivative notation
   \[
   L_{f_g}V(\rho) = \frac{\partial V(\rho)}{\partial x} \cdot f_g(\rho),
   \]
where for each \( g \in D, f_g(\cdot) \) is the vector field defined by \( f(\cdot, g) \). By “smooth” we always mean infinitely differentiable.

**3. Game Model**

A decision problem \( (S, \leq) \) consists of a finite set \( S = \{s_1, \ldots, s_n\} \) of outcomes and a preference relation \( \leq \). It is assumed for \( \leq \) to establish a poset on \( S \), i.e., given \( x, y, z \in S \) we expect the preference relation \( \leq \) to be fulfilled, and the
following axioms hold: reflexivity \((x \leq x)\), antisymmetry \((x \leq y \text{ and } y \leq x \implies x = y)\), transitivity \((x \leq y \text{ and } y \leq z \implies x \leq z)\).

Although the preference relation is the basic primitive of any decision problem (and generally observable) it is much easier to work with a consistent utility function \(u : S \to R\) because we only have to use \(n\) real numbers \(u = \{u_1, \ldots, u_n\}\).

A utility function \(u : S \to R\) is consistent with the preference relationship of a decision problem \((S, \leq)\) if \(\forall x, y \in S: x \leq u_i y \text{ if and only if } u_i(x) \leq u_i(y)\).

The preference relation \(\leq u_i\) induce a von Neumann-Morgenstern ([13]) utility function.

Given any set \(Q \subseteq S\) of actions that are viable in some specific case, a rational decision maker chooses an action \(s' \in Q\) that is viable and optimal in the sense that \(u(s) \leq u(s')\) for all \(s \in Q\) solving the problem \(\max_{s \in S} u(s)\). A rational decision maker who deals with a decision problem selects a strategy \(s \in Q\) which maximizes his utility. An assumption upon which the efficiency of this model of decision making depends is that an individual makes use of the same preference relation \(\leq u_i\) when choosing from different sets \(Q\).

**Definition 3.1.** A non-cooperative local game is a tuple \(G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle\), where:

1. \(N = \{1, 2, \ldots, n\}\) denotes a finite set of players,
2. each player \(i \in N\) has a finite set \(S_i\) of pure strategies, henceforth called actions, and
3. \(u_i = \prod_{i \in N} S_i \to R_+\) denotes a continuous utility functions such that:
   
   a) \(\exists s^* \in \prod_{i \in N} S_i\) such that \(u_i(s^*) = 0\),
   
   b) \(u_i(s) > 0\) for \(s \neq s^*\), where \(s \in \prod_{i \in N} S_i\),
   
   c) \(u_i(s)\) approaches to a maximum when \(s\) is large,
   
   d) \(\forall s, t \in \prod_{i \in N} S_i\) such that \(s \leq u_i t\) then \(\Delta u = u_i(t) - u_i(s) < 0\)
      for \(s, t \neq s^*\), or it is equivalent to say that \(\exists \epsilon > 0\) such that \(|u_i(t) - u_i(s)| > \epsilon\) for \(s, t \neq s^*\).

**Proposition 3.1.** The utility functions \(u_i = \prod_{i \in N} S_i \to R\) are Lyapunov functions.

**Proof.** The proof is straightforward from the definition and therefore it is omitted. \(\square\)

**Definition 3.2.** A non-cooperative robust game is a tuple \(G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle\), where:
1. $N = \{1, 2, \ldots, n\}$ denotes a finite set of players,

2. each player $i \in N$ has a finite set of mixed strategies,

3. $u_i : \prod_{i \in N} S_i \to \mathbb{R}_+$ denotes the utility functions with respect to a nonempty, closed, invariant set $\mathcal{I} \subset \mathbb{R}^n$ such that:
   
   (a) $u_i$ are smooth on $\mathbb{R}^n \setminus \mathcal{I}$,
   
   (b) there exist two $\mathcal{K}_\infty$-functions $\alpha_1$ and $\alpha_2$ such that for any $\rho \in \mathbb{R}^n$
       
       \[ \alpha_1(\|\rho\|_I) \leq u_i \leq \alpha_2(\|\rho\|_I), \]
   
   (c) there exists a continuous, positive definite function $\alpha_3$ such that for any $\rho \in \mathbb{R}^n \setminus \mathcal{I}$, and any $g \in \mathcal{D}$,
       
       \[ L_{\mathcal{F}_g} u_i(\rho) \leq -\alpha_3(\|\rho\|_I). \]

**Remark 3.1.** Continuity of $u_i$ on $\mathbb{R}^n \setminus \mathcal{I}$ in the previous definition imply:

- $u_i$ is continuous everywhere in all $\mathbb{R}^n$,
- $u_i(s^\ast) = 0 \iff s^\ast \in \mathcal{I}$,
- $u_i : \mathbb{R}^n \to \mathbb{R}_+$,
- $\forall s, t \in \prod_{i \in N} S_i$ such that $s \leq u_i t$ we have that $\exists \varepsilon > 0$ such that $\|u_i(t) - u_i(s)\| > \varepsilon$ for $s, t \notin \mathcal{I}$.

**Proposition 3.2.** The utility functions $u_i = \prod_{i \in N} S_i \to R$ are robust Lyapunov functions.

**Proof.** The proof is straightforward from the definition and therefore it is omitted. \(\square\)

Note that by definition, the utility function $u_i$ defined as a Lyapunov function maximizes the utility ([2], [12]).

For notational convenience we write $S = \prod_{i \in N} S_i$ (the pure strategies profile), and $S_{-i} = \prod_{j \in N \setminus \{i\}} S_j$ (the pure strategies profile of all the players but for player $i$). For an action tuple $s = (s_1, \ldots, s_n) \in S$ we denote $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ and, with an abuse of notation, $s = (s_i, s_{-i})$.

We denote

\[ \Gamma_i = \left\{ \sigma_i : S_i \to \mathbb{R}_+ : \forall s_i \in S_i : \sigma_i(s_i) \geq 0, \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\} \]
as the set of mixed strategies, henceforth called strategies, for player $i$, where
$
\sigma_i$
the relative frequency of players that take strategy $i$. Note that in our game
each player can either adopt a pure strategy $S_i$ or a mixed strategy $\Gamma_i$.

Analogously to the action case, we use notations $\Gamma = \prod_{i \in N} \Gamma_i$ (the mixed
strategies profile), and $\Gamma_{-i} = \prod_{j \in N \setminus \{i\}} \Gamma_j$ (the mixed
strategies profile of all the
players except for player $i$). For a strategy tuple $\sigma = (\sigma_1, ..., \sigma_n) \in \Gamma$ we denote
$\sigma_{-i} = (\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_n)$ and, with a abuse of notation, $\sigma = (\sigma_i, \sigma_{-i})$.
The state $\sigma = (\sigma_1, ..., \sigma_n)$ represents the distribution vector of strategy fre-
quencies and can only move on $\Gamma$. For a strategy profile $\sigma_{-i}$, we write
$\sigma_{-i} = \prod_{j \in N \setminus \{i\}} \sigma_j$, the probability that the opponents of player $i$
play action profile $s_{-i} \in S_{-i}$. We restrict attention to independent strategy profiles.

Note that the game $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a game
$G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle$ if we consider
$\sigma(s_i) = 1$. So, from now we will consider only the game $G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle$.

**Example 3.1.** The entropy $H(p, u) = -\sum_{s \in S} (\prod_{j=1}^n \sigma_j(s_j)) u(s) \ln[(\prod_{j=1}^n \sigma_j(s_j)) u(s)]$ is a Lyapunov function that maximizes the utility (see
[1])

**Theorem 3.1.** Let the set $S$ finite and the utility functions $u_i$ are Lyapunov functions, then the utility functions $u_i$ are consistent with the preference relation.

Proof. Let $\equiv_u$ be the equivalence relation on $S$ induced by $u_i$
\[
\forall s, t \in S : s \equiv_u t \iff u_i(s) = u_i(t). \tag{2}
\]
then the collection of equivalence classes $\{S/ \equiv_u\} = \bigcup_{i \in N} S/ \equiv_u = \{\pi_i(s) | s \in S\}$
is a poset isomorphic to a subset of $R$. Thus, $\{S/ \equiv_u\}$ is linearly ordered and,
consequently, it is a lattice. The structure $\{S/ \equiv_u\}$ is indeed trivial: all ele-
ments in $S$ giving the same value under $u_i$ are identified in this quotient set.

On the other hand, let us consider the relation $\leq_u$ as follows:
\[
\forall s, t \in S : s \leq_u t \iff u_i(s) \leq u_i(t). \tag{3}
\]

This relation is reflexive and transitive, and it is antisymmetric because $u_i$
is a Lyapunov function therefore it is one-to-one. Thus, $\leq_u$ is an ordering in $S$. \qed
4. Utility Functions Isomorphism

A problem arises at this point. In the original definition the utility functions \( u \) can take positive or negative values, but defined as a Lyapunov functions they only can take positive values.

At this point let us introduce some notation on partial order. For any \( s \in S \) let successors of \( s \):

\[
t \in \text{suc}(s) \iff s \neq t, s \leq_{u_i} t
\]

and \( \forall q : (s \leq_{u_i} q \leq_{u_i} t) \implies (q = s) \lor (q = t) \),

predecessors of \( s \):

\[
t \in \text{pre}(s) \iff t \neq s, t \leq_{u_i} s
\]

and \( \forall q : (t \leq_{u_i} q \leq_{u_i} s) \implies (q = t) \lor (q = s) \).

Let \( G_{\{u_i\}} \) be the graph whose set of nodes is \( S \) and for each pair \((s, t) \in S^2 : (s, t) \) is an edge iff \( t \in \text{suc}(s) \), or equivalently, \( s \in \text{pre}(t) \). Let us say that \( u_i \) is consistent if \( G_{\{u_i\}} \) has no cycles. From now on, we will consider only consistent functions. Since \( R \) is linearly ordered we have

\[
\forall s, t \in S : (s <_{u_i} t) \lor (s \equiv_{u_i} t) \lor (t <_{u_i} s), \tag{4}
\]

thus, \( u_i \) induces a hierarchical structure on \( S \).

The minimal elements are those with no predecessors, i.e. nodes with null inner degree in \( G_{\{u_i\}} \). The maximal elements are those with no successors, i.e. nodes with null outer degree in \( G_{\{u_i\}} \).

Let us define the upper distance \( d^+ \) as follows:

\[
d^+(s, t) = 1 \iff t \in \text{suc}(s), \\
d^+(s, t) = 1 + r \iff \exists q : d^+(s, q) = r \land d^+(q, t) = 1.
\]

Similarly, the lower distance \( d^- \)

\[
d^-(s, t) = 1 \iff t \in \text{pre}(s), \\
d^-(s, t) = 1 + r \iff \exists q : d^-(s, q) = r \land d^-(q, t) = 1.
\]

Thus \( d^+(s, t) = d^-(t, s) \). The upper height of a node \( s \) is \( h^+(s) = \max \{d^+(s, t)|t \text{ is minimal}\} \). The lower height of a node \( s \) is \( h^-(s) = \max \{d^-(s, t)|t \text{ is maximal}\} \).

Let \( S \neq \emptyset \) and let \( u, w : S \to R \) be two real functions.

Let us say that \( u \) is an eq-refinement of \( w \) if
\[ \forall s_1, s_2 \in S : (u(s_1) = u(s_2)) \implies (w(s_1) = w(s_2)). \] 

(5)

In this case, \((S/ \equiv_u)\) is an homeomorphic image of \((S/ \equiv_w)\) (both are linearly ordered sets).

Let us say that \(u\) is an \textit{ineq-refinement} of \(w\) if

\[ \forall s_1, s_2 \in S : (u(s_1) \leq u(s_2)) \implies (w(s_1) \leq w(s_2)). \] 

(6)

In this case, the ordering \(\leq_u\) is included, as a set in \(S \times S\), in the ordering \(\leq_w\). Hence, it follows that \(G_w\) is an homomorphic image of \(G_u\), i.e. \(G_w\) can be realized as a subgraph of \(G_u\).

We may introduce a stronger notion to compare functions. For instance, let \(\text{Sgn} : \mathbb{R} \rightarrow \mathbb{R}\) be such that

\[ \forall x \in \mathbb{R} : x > 0 \implies \text{Sgn}(x) = 1; x = 0 \implies \text{Sgn}(x) = 0; x < 0 \implies \text{Sgn}(x) = -1 \]

Let us say that \(u\) is an \textit{tonal-refinement} of \(w\) if

\[ \forall s_1, s_2 \in S : \text{Sgn} (u(s_1) - u(s_2)) = \text{Sgn} (w(s_1) - w(s_2)). \] 

(7)

In this case, \(G_w\) is isomorphic to \(G_u\).

The von Neumann and Morgenstern utility assessments called a preference probability determined by a preference relation \(\leq_u\) that establish a poset.

**Proposition 4.1.** If \(u\) and \(w\) are utility functions related by a positive affine transformation, then \(u\) and \(w\) produce the same ranking of payoffs, i.e. \(G_u\) is isomorphic to \(G_w\).

**Proof.** The proof is straightforward from the previous definitions and therefore omitted.

**Example 4.1.** Let us consider the entropy as a utility function. We must do the analogy between probability and utility which appears naturally in the probabilistic equivalence used in the von Neumann and Morgenstern utility assessments.

We assume that there is at least one strategy, which has strict preference (to exclude the case of absolute indifference between the \(n\) strategies). Then, without loss generality, we assign a utility value of zero to the least preferred strategy and a utility value of one to the most preferred strategy.

We will extend \(u_i\) to a function defined for all events \(S' \subset S\) as follows

\[ u_i(S') = \sum_{s \in S'} u_i(s) \left( \prod_{j=1}^{n} \sigma_j(s_j|S') \right). \] 

We will normalize the utility functions
such that \( u_i(s) \geq 0 \) and \( u_i(S) = 1 \), the the set of functions defined by \( f(s) := u_i(s) \left( \prod_{j=1}^{n} \sigma_j(s_j) \right) \) is a probability measure, i.e. \( f : S \to [0, 1) \).

**Example 4.2.** Let us consider a positive utility function linear transformation as a utility function. A positive utility function’s linear transformation is a function \( u(w) = aw + b \), where \( a \) is positive and \( b \) is the equilibrium point. A linear transformation \( u(w) = aw + b \) means that there is an isomorphism between \( u \) and \( w \), which maps points from \( u \) to corresponding points of \( w \), and vice versa.

The utility functions specified up to a positive linear transformation are called interval scales. Our definition of a utility functions require an interval scale to be invariant only under positive linear transformations.

## 5. Equilibrium Point and Stability

Consider the game \( G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle \). Denote for each player \( i \in N \) and each profile \( \sigma_{-i} \in \Gamma_{-i} \) of actions of his opponent the set of best replies, i.e. the actions that player \( i \) cannot improve upon \( B_i(\sigma_{-i}) \):

\[
B_i(\sigma_{-i}) := \{ \sigma_i \in \Gamma_i | \forall \sigma'_i \in \Gamma_i : u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}) \}.
\]

An action \( s_i \in S_i \) is weakly dominated by a strategy \( \sigma_i \in \Gamma_i \) if

\[
\forall s_{-i} \in S_{-i} : u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i}),
\]

with strict inequality for at least one \( s_{-i} \) and strictly dominated if all inequalities are strict. A strictly dominated action is clearly never a best reply.

**Definition 5.1.** Let \( G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle \) be a local game. A strategy profile \( \sigma \in \Gamma \) is an equilibrium point in the Lyapunov sense if for every player \( i \in N \) and for every \( \sigma_i \in \Gamma_i \) the utility functions \( u_i(\sigma_{-i}^*, \sigma_i) = 0 \).

**Theorem 5.1.** Every local game \( G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle \) has a local strategy Nash/Lyapunov equilibrium point.

**Proof.** Nash equilibrium point exists in a non-empty, compact, convex subset of a Euclidian space by Kakutani’s Fixed-Point Theorem ([8]). On the other hand, by the definition of local Lyapunov function \( u_i \) we have that \( \exists s^* \in \prod_{i \in N} S_i \) such that \( u_i((s^*) = 0) = 0 \). \( \square \)

**Remark 5.1.** The potentiality of the previous theorem remains in its formal simplicity proof for the existence of an equilibrium point.
Theorem 5.2. The local equilibrium point in the Lyapunov sense of a game \( G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle \) coincides with the Nash equilibrium.

Proof. (\( \implies \)) The equilibrium point in the Nash sense of a game \( G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle \) is satisfied by a Lyapunov function \( v = \sigma^T A \sigma \) ([11]), where the payoff matrix is described by \( A \), the increment of \( v \) is negative definite, considering that \( \Delta v \) satisfies the following condition

\[
A^T P A - P = -Q,
\]

where \( P \) and \( Q \) are negative definite and the eigenvalues of \( A \), \( \{\lambda_i\} \), are bounded in \([0,1)\).

(\( \impliedby \)) The equilibrium point in the Lyapunov sense of a game \( G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle \) is satisfied by \( v = \sigma^* T A \sigma^* \). In agreement with Lyapunov \( |\sigma^* T A \sigma^* - \sigma^T A \sigma^*| > \varepsilon \), it is equivalent to \( \sigma^* T A \sigma^* > \varepsilon + \sigma^T A \sigma^* \). When \( \sigma \) becomes \( \sigma^* \) we have that \( \varepsilon \to 0 \), then \( \sigma^* T A \sigma^* \geq \sigma^T A \sigma^* \) and this is the Nash equilibrium condition.

Corolary 5.1. The local game \( G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle \) is asymptotically stable in the Lyapunov sense.

Proof. Since \( |u_i(\sigma') - u_i(\sigma)| > \varepsilon \) for all \( \sigma, \sigma' \in \Gamma \), where \( \sigma, \sigma' \neq \sigma^* \), the game is asymptotically stable (see [12]).

Definition 5.2. Let \( G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle \) be a robust game. A strategy profile \( \sigma \in \Gamma \) is a robust equilibrium point in the Lyapunov sense if for every player \( i \in N \) and for every \( \sigma_i \in \Gamma_i \) the utility functions \( u_i(\sigma^*_i, \sigma_i) = 0 \).

Theorem 5.3. Every robust game \( G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle \) has a robust Nash equilibrium point.

Proof. Since we are asking \( f \) in (1) to be Lipschitz, then by Picard (see [6]) there is a solution satisfying the initial conditions of (1). Then the set \( I \subseteq \mathbb{R}^n \) is not empty.

Theorem 5.4. Every robust game \( G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle \) has a robust Lyapunov equilibrium point.

Proof. For definition of robust Lyapunov function \( u_i \) the set \( I \subseteq \mathbb{R}^n \) is nonempty.

Remark 5.2. The potential of the previous theorem remains in its formal proof simplicity for the existence of an equilibrium point.
Theorem 5.5. The equilibrium point in the robust Lyapunov sense of a game $G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle$ coincides with a robust Nash equilibrium point.

Proof. ($\implies$) The equilibrium point, in the Nash sense, of a game $G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is satisfied by a Lyapunov function $v = \sigma^T A \sigma$ ([11]), where the payoff matrix is described by $A = \sum_{i=1}^{n} \omega_i A_i$, the increment of $v$ is negative, considering that $\Delta v$ satisfies the following condition $A^T P A - P = -Q$, where $P$ and $Q$ are positive definite and the eigenvalues of $A$, $\{\lambda_{ij}\}$, are bounded in $[0,1)$.

($\impliedby$) The equilibrium point in the Lyapunov sense of a game $G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is satisfied by $v = \sigma^* A \sigma^*$. In agreement with Lyapunov $|\sigma^T A \sigma^* - \sigma^T A \sigma^*| > \varepsilon$, it is equivalent to $\sigma^T A \sigma^* \geq \varepsilon + \sigma^T A \sigma^*$. When $\sigma$ becomes $\sigma^*$ we have that $\varepsilon \to 0$, then $\sigma^T A \sigma^* \geq \sigma^T A \sigma^*$ and this is the Nash equilibrium condition. $\Box$

Corollary 5.2. The robust game $G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is robust uniformly globally asymptotically stable in the Lyapunov sense.

Proof. Since $|u_i(\sigma') - u_i(\sigma)| > \varepsilon$ for all $\sigma, \sigma' \in \Gamma$, where $\sigma, \sigma' \neq \sigma^*$, the game is robust uniformly globally asymptotically stable (see [12]). $\Box$

Example 5.1. The entropy in a game $G = \langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \rangle$ maximizes the utility only in Nash’s equilibrium or in Lyapunov equilibrium point. The utility function satisfies that in the equilibrium point the standard deviation

$E_i = \sum_{i=1}^{n} (u_i(\sigma, \sigma_{-i}) - u_i(\sigma', \sigma_{-i})) \prod_{j=i}^{n} \sigma_j(s_j)) = 0.$

Outside the equilibrium point we find the standard deviation $E_i \neq 0$.

6. Conclusions and Future Works

The Lyapunov method induces a new equilibrium and stability concept in non-cooperative games. We proved that the equilibrium concept in local and robust Lyapunov sense coincides with the equilibrium concept of Nash, representing an alternative way to calculate the equilibrium and stability of the game. It is the most important contribution of this paper. We show that an utility function isomorphism can be attained.

We also have shown conditions under which the game is stable in local and robust Lyapunov sense. The Lyapunov method is based on the determination
of a function able to show stability and instability of a system. We believe that there are many applications in game theory where robust stability properties are of interest.

However, the main disadvantage of Lyapunov’s stability criterion is that it gives only the sufficient conditions for stability. Necessary conditions can attained under certain space restrictions, but we will leave the proof for a upcoming work. Furthermore, there are no unique methods of determining the function for a wide class of systems. As future work, it will also be of interest to formally tackle the time complexity of the game.

References

