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EXTENDING GAMES WITH LOCAL AND ROBUST LYAPUNOV EQUILIBRIUM AND STABILITY CONDITION

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Abstract: In this paper we extend the non-cooperative game theory including the Lyapunov's equilibrium and stability criteria, locally and robustly. Equilibrium and stability conditions are obtained by defining the utility function as a Lyapunov function. In this sense, we present some properties about utility functions to show that an isomorphism can be determined between any utility function and the Lyapunov utility function. We introduce the Lyapunov equilibrium point as an alternative definition to the Nash equilibrium point for games. We prove that the concept of Lyapunov equilibrium coincides in this case with the concept of Nash equilibrium. The advantage of this approach is that fixed-point conditions for games are given by the definition of the Lypunov function. We show that the game is asymptotically stable in the Lyapunov sense. A formal treatment leading to interesting mathematical results, and open problems

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in game theory are presented.

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1. Introduction

Non-cooperative game theory has been extensively used to analyze situations of interaction. The most important solution in non-cooperative games is the notion of Nash equilibrium. Formally, Nash equilibrium defines an equilibrium of a non-cooperative game with respect to a profile of strategies, one for each player in the game, such that each player's strategy attempts to maximize that player's expected utility opposed to the set of strategies of the other players ([8], [9], [10]). Then, players are in equilibrium if a change in strategies by any one of them would lead that player to earn less than if he stood by the current strategy. For games in which players randomize mixed strategies, the expected utility must be at least as large as that available by any one of the strategies ([3], [4], [5]).

In order to maximize the utility we propose to define the payoff functions as Lyapunov-like functions. The core idea of our approach uses a utility function that is non-negative and converges asymptotically to the equilibrium point. For instance, in the arms race the level of defense of a nation is non-negative. In economics models there are variables that correspond, e.g. with quantities of goods that remain non-negative.

In this equilibrium each player chooses a strategy with a utility equal to the utility that this strategy is a best reply to a strategy profile chosen by the opponents. The advantage of this approach is that fixed-point conditions for the game are given by the definition of the Lypunov-like function, however formally it is not necessary for a fixed-point theorem to satisfy the Nash equilibrium conditions as usual. In addition, new properties of equilibrium and stability are consequently introduced for finite n-player non-cooperative games.

We show that the concept of Lyapunov equilibrium coincides with the concept of Nash equilibrium in a local and robust sense. Utility function properties are described with the purpose of showing that it is possible to establish an isomorphism between the preference order determined by any utility function and the Lyapunov-like function.

The rest of this paper is structured in the following manner. The next section presents the necessary mathematical background and terminology needed to understand the rest of the paper. Section 3 describes the basic formalism of the game model. The issues associated to the utility functions isomorphisms are discussed in Section 4. Next, Section 5 presents the equilibrium and stability definition and theorems. Finally, Section 6, concludes the paper by giving future research directions.

2. Preliminaries (see [7])

In this section, we present some definitions and well-established properties which will be used later.

Let $\mathbb{R}_+ = [0, \infty)$. Given $x, y \in \mathbb{R}^k$, we usually denote the relation "$\leq$" to mean componentwise inequalities with the same relation, i.e., $x \leq y$ is equivalent to $x_i \leq y_i, \forall i$.

Consider systems of first ordinary differential equations given by:

$$\dot{x}(t) = f(x(t), g(t)), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{R}^n, \quad g(t) \in \mathbb{R}^m\,, \tag{1}$$

where we assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and for each two compact subset $K \subset \mathbb{R}^n$ and $\mathcal{D} \subset \mathbb{R}^m$ there is some constant l such that $\|f(x,g) - f(y,g)\| \leq l\,\|x-y\|$ [1] $\forall x, y \in K$ and $\forall g \in \mathcal{D}$, i.e. $f(x,g)$ is locally Lipschitz in x and uniformly in g, and note that these properties are satisfied if f extends to a continuously differentiable function on a neighborhood of $\mathbb{R}^n \times \mathcal{D}$.

The perturbation functions g are supposed to lie in the space $\mathcal{M}$ of measurable and locally bounded functions with values in $\mathcal{D}$, where $\mathcal{D}$ is an arbitrary subset of $\mathbb{R}^m$. For each $g \in \mathcal{M}$, we denote by $x(t, x_0, g)$ the solution at time t of (1) with $x(0) = x_0$.

We say that the closed set $\mathcal{I}$ is an *invariant set* for (1) if

$$\forall x_0 \in \mathcal{I}, \ \ \forall g \in \mathcal{M} \quad x(t, x_0, g) \in \mathcal{I}, \quad \forall t \geq 0.$$

For each nonempty subset $\mathcal{I} \subseteq \mathbb{R}^n$, and each $\rho \in \mathbb{R}^n$, we denote by

$$\|\rho\|_{\mathcal{I}} = g(\rho, \mathcal{I}) = \inf_{\tau \in \mathcal{I}} g(\rho, \tau),$$

which is the common point-to-set distance, and $\|\rho\|_{\{0\}} = \|\rho\|$ is the usual norm of S.

Definition 2.1.

[1] The expression $\|\cdot\|$ denotes the usual Euclidian norm

- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{K}$-function if $\alpha(0) = 0$, and it is strictly increasing for all $s \geq 0$.
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and $\alpha(s) \to \infty$ when $s \to \infty$.
- A continuous function $\beta : \mathbb{R}_+ \times Z_+ \to \mathbb{R}_+$ is called a $\mathcal{KL}$-function if $\beta(s,t)$ is a $\mathcal{K}$-function in s $\forall t \in \mathbb{N}$ and it is strictly decreasing in t $\forall s \geq 0$ (note that $Z_+ = \mathbb{N} \cup \{0\}$).

Definition 2.2. The solution of equation (1) is called "robust" with respect to the closed invariant set $\mathcal{I}$ if the following properties hold:

1. There exists a $\mathcal{K}_\infty$-function $\delta(\cdot)$ such that for any $\varepsilon \geq 0$,

$$\|x(t, x_0, g)\|_{\mathcal{I}} \leq \varepsilon \quad \forall g \in \mathcal{M},$$

whenever $\|x_0\| \leq \delta(\varepsilon)$ and $t \geq 0$, and

2. for any r,$\varepsilon > 0$, there is a $T > 0$, such that $\forall g \in \mathcal{M}$,

$$\|x(t, x_0, g)\| < \varepsilon,$$

whenever $\|x_0\|_{\mathcal{I}} < r$ and $t \geq T$.

Proposition 2.1. *The equation (1) is uniformly globally asymptotically stable with respect to a closed, invariant set $\mathcal{I}$ if and only if there exists a $\mathcal{KL}$ function β such that, given any initial state x_0, the solution $x(t, x_0, g)$ satisfies*

$$\|x(t, x_0, g)\|_{\mathcal{I}} \leq \beta(\|x_0\|_{\mathcal{I}}, t), \qquad \forall t \geq 0 \text{ and } \forall g \in \mathcal{M}.$$

For any differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, we use the standard Lie derivative notation

$$L_{f_g} V(\rho) = \frac{\partial V(\rho)}{\partial x} \cdot f_g(\rho),$$

where for each $g \in \mathcal{D}$, $f_g(\cdot)$ is the vector field defined by $f(\cdot, g)$. By "smooth" we always mean infinitely differentiable.

3. Game Model

A decision problem $(S, \leq)$ consists of a finite set $S = \{s_1, ..., s_n\}$ of outcomes and a preference relation $\leq$. It is assumed for $\leq$ to establish a poset on S, i.e., given $x, y, z \in S$ we expect the preference relation $\leq$ to be fulfilled, and the following axioms hold: reflexivity ($x \leq x$), antisymmetry ($x \leq y$ and $y \leq x$ implies that $x = y$), transitivity ($x \leq y$ and $y \leq z$ implies that $x \leq z$).

Although the preference relation is the basic primitive of any decision problem (and generally observable) it is much easier to work with a consistent utility function $u : S \to R$ because we only have to use n real numbers $u = \{u_1, ..., u_n\}$. A utility function $u : S \to R$ is consistent with the preference relationship of a decision problem $(S, \leq)$ if $\forall x, y \in S : x \leq_{u_i} y$ if and only if $u_i(x) \leq u_i(y)$. The preference relation $\leq_{u_i}$ induce a von Neumann-Morgenstern ([13]) utility function.

Given any set $Q \subseteq S$ of actions that are viable in some specific case, a rational decision maker chooses an action $s^{'} \in Q$ that is viable and optimal in the sense that $u(s) \leq u(s^{'})$ for all $s \in Q$ solving the problem $\max_{s \in S} u(s)$. A rational decision maker who deals with a decision problem selects a strategy $s \in Q$ which maximizes his utility. An assumption upon which the efficiency of this model of decision making depends is that an individual makes use of the same preference relation $\leq_{u_i}$ when choosing from different sets Q.

Definition 3.1. A non-cooperative local game is a tuple $G = \left\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$, where:

1. $N = \{1, 2, ..., n\}$ denotes a finite set of players,
2. each player $i \in N$ has a finite set S_i of pure strategies, henceforth called actions, and
3. $u_i = \prod_{i \in N} S_i \to R_+$ denotes a continuous utility functions such that:
   (a) $\exists s^* \in \prod_{i \in N} S_i$ such that $u_i(s^*) = 0$,
   (b) $u_i(s) > 0$ for $s \neq s^*$, where $s \in \prod_{i \in N} S_i$,
   (c) $u_i(s)$ approaches to a maximum when s is large,
   (d) $\forall s, t \in \prod_{i \in N} S_i$ such that $s \leq_{u_i} t$ then $\Delta u = u_i(t) - u_i(s) < 0$ for $s, t \neq s^*$, or it is equivalent to say that $\exists \epsilon > 0$ such that $|u_i(t) - u_i(s)| > \epsilon$ for $s, t \neq s^*$.

Proposition 3.1. *The utility functions $u_i = \prod_{i \in N} S_i \to R$ are Lyapunov functions.*

Proof. The proof is straightforward from the definition and therefore it is omitted. ☐

Definition 3.2. A non-cooperative robust game is a tuple $G = \left\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$, where:

1. $N = \{1, 2, ..., n\}$ denotes a finite set of players,
2. each player $i \in N$ has a finite set of mixed strategies,
3. $u_i : \prod_{i \in N} S_i \rightarrow \mathbb{R}_+$ denotes the utility functions with respect to a nonempty, closed, invariant set $\mathcal{I} \subset \mathbb{R}^n$such that:
   (a) u_i are smooth on $\mathbb{R}^n \backslash \mathcal{I}$,
   (b) there exist two $\mathcal{K}_\infty$-functions α_1 and α_2 such that for any $\rho \in \mathbb{R}^n$
   $$\alpha_1(\|\rho\|_{\mathcal{I}}) \leq u_i \leq \alpha_2(\|\rho\|_{\mathcal{I}}),$$
   (c) there exists a continuous, positive definite function α_3 such that for any $\rho \in \mathbb{R}^n \backslash \mathcal{I}$, and any $g \in \mathcal{D}$,
   $$L_{f_g} u_i(\rho) \leq -\alpha_3(\|\rho\|_{\mathcal{I}}).$$

Remark 3.1. Continuity of u_i on $\mathbb{R}^n \backslash \mathcal{I}$ in the previous definition imply:

- u_i is continuous everywhere in all $\mathbb{R}^n$,
- $u_i(s^*) = 0 \Longleftrightarrow s^* \in \mathcal{I}$,
- $u_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$,
- $\forall s, t \in \prod_{i \in N} S_i$ such that $s \leq_{u_i} t$ we have that $\exists \varepsilon > 0$ such that $\|u_i(t) - u_i(s)\| > \varepsilon$ for $s, t \notin \mathcal{I}$.

Proposition 3.2. *The utility functions* $u_i = \prod_{i \in N} S_i \rightarrow R$ *are robust Lyapunov functions.*

Proof. The proof is straightforward from the definition and therefore it is omitted. □

Note that by definition, the utility function u_i defined as a Lyapunov function maximizes the utility ([2], [12]).

For notational convenience we write $S = \prod_{i \in N} S_i$ (the pure strategies profile), and $S_{-i} = \prod_{j \in N | \{i\}} S_j$ (the pure strategies profile of all the players but for player i). For an action tuple $s = (s_1, ..., s_n) \in S$ we denote $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$ and, with an abuse of notation, $s = (s_i, s_{-i})$.

We denote

$$\Gamma_i = \left\{ \sigma_i : S_i \rightarrow R_+ | \forall s_i \in S_i : \sigma_i(s_i) \geq 0, \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\}$$

as the set of mixed strategies, henceforth called strategies, for player i, where σ_i the relative frequency of players that take strategy i. Note that in our game each player can either adopt a pure strategy S_i or a mixed strategy Γ_i.

Analogously to the action case, we use notations $\Gamma = \prod_{i \in N} \Gamma_i$ (the mixed strategies profile), and $\Gamma_{-i} = \prod_{j \in N|\{i\}} \Gamma_j$ (the mixed strategies profile of all the players except for player i). For a strategy tuple $\sigma = (\sigma_1, ..., \sigma_n) \in \Gamma$ we denote $\sigma_{-i} = (\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_n)$ and, with a abuse of notation, $\sigma = (\sigma_i, \sigma_{-i})$. The state $\sigma = (\sigma_1, ..., \sigma_n)$ represents the distribution vector of strategy frequencies and can only move on Γ. For a strategy profile σ_{-i}, we write $\sigma_{-i} = \prod_{j \in N|\{i\}} \sigma_j$, the probability that the opponents of player i play action profile $s_{-i} \in S_{-i}$. We restrict attention to independent strategy profiles.

Note that the game $G = \left\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$ is a game $G = \left\langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$ if we consider $\sigma(s_i) = 1$. So, from now we will consider only the game $G = \left\langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$.

Example 3.1. The entropy $H(p, u) = -\sum_{s \in S} (\prod_{j=i}^{n} \sigma_j(s_j)) u(s) \ln[(\prod_{j=i}^{n} \sigma_j(s_j)) u(s)]$ is a Lyapunov function that maximizes the utility (see [1])

Theorem 3.1. *Let the set S finite and the utility functions u_i are Lyapunov functions, then the utility functions u_i are consistent with the preference relation.*

Proof. Let $\equiv_{u_i}$be the equivalence relation on S induced by u_i

$$\forall s, t \in S : s \equiv_{u_i} t \Longleftrightarrow u_i(s) = u_i(t). \tag{2}$$

then the collection of equivalence classes $\{S/\equiv_{u_i}\} = \bigcup_{i \in N} S/\equiv_{u_i} = \{\pi_i(s) | s \in S\}$ is a poset isomorphic to a subset of R. Thus, $\{S/\equiv_{u_i}\}$ is linearly ordered and, consequently, it is a lattice. The structure $\{S/\equiv_{u_i}\}$ is indeed trivial: all elements in S giving the same value under u_i are identified in this quotient set.

On the other hand, let us consider the relation $\leq_{u_i}$as follows:

$$\forall s, t \in S : s \leq_{u_i} t \Longleftrightarrow u_i(s) \leq u_i(t) \, . \tag{3}$$

This relation is reflexive and transitive, and it is antisymmetric because u_i is a Lyapunov function therefore it is one-to-one. Thus, $\leq_{u_i}$is an ordering in S. $\square$

4. Utility Functions Isomorphism

A problem arises at this point. In the original definition the utility functions u_i can take positive or negative values, but defined as a Lyapunov functions they only can take positive values.

At this point let us introduce some notation on partial order. For any $s \in S$ let successors of s:

$$t \in \mathrm{suc}\,(s) \;\text{ iff }\; s \neq t, s \leq_{u_i} t \quad \text{and } \; \forall q : (s \leq_{u_i} q \leq_{u_i} t) \Longrightarrow (q = s) \vee (q = t),$$

predecessors of s:

$$t \in \mathrm{pre}\,(s) \;\text{ iff }\; t \neq s, t \leq_{u_i} s \quad \text{and } \; \forall q : (t \leq_{u_i} q \leq_{u_i} s) \Longrightarrow (q = t) \vee (q = s).$$

Let $G_{\{u_i\}}$be the graph whose set of nodes is S and for each pair $(s, t) \in S^2$: (s, t) is an edge iff $t \in \mathrm{suc}\,(s)$, or equivalently, $s \in pre(t)$. Let us say that u_i is consistent if $G_{\{u_i\}}$ has no cycles. From now on, we will consider only consistent functions. Since R is linearly ordered we have

$$\forall s, t \in S : (s <_{u_i} t) \vee (s \equiv_{u_i} t) \vee (t <_{u_i} s), \tag{4}$$

thus, u_i induces a hierarchical structure on S.

The minimal elements are those with no predecessors, i.e. nodes with null inner degree in $G_{\{u_i\}}$. The maximal elements are those with no successors, i.e. nodes with null outer degree in $G_{\{u_i\}}$

Let us define the upper distance d^+ as follows:

$$d^+(s, t) = 1 \Longleftrightarrow t \in \mathrm{suc}\,(s),$$
$$d^+(s, t) = 1 + r \Longleftrightarrow \exists q : d^+(s, q) = r \;\&\; d^+(q, t) = 1.$$

Similarly, the lower distance d^-

$$d^-(s, t) = 1 \Longleftrightarrow t \in \mathrm{pre}\,(s),$$
$$d^-(s, t) = 1 + r \Longleftrightarrow \exists q : d^-(s, q) = r \;\&\; d^-(q, t) = 1.$$

Thus $d^+(s, t) = d^-(t, s)$. The upper height of a node s is $h^+(s) = \max\{d^+(s, t) | t \text{ is minimal}\}$. The lower height of a node s is $h^-(s) = \max\{d^-(s, t) | t$ is maxima

Let $S \neq \emptyset$ and let $u, w : S \to R$ be two real functions.

Let us say that u is an *eq-refinement* of w if

$$\forall s_1, s_2 \in S : (u(s_1) = u(s_2)) \Longrightarrow (w(s_1) = w(s_2)). \tag{5}$$

In this case, $(S/\equiv_u)$ is an homeomorphic image of $(S/\equiv_w)$ (both are linearly ordered sets).

Let us say that u is an *ineq-refinement* of w if

$$\forall s_1, s_2 \in S : (u(s_1) \le u(s_2)) \Longrightarrow (w(s_1) \le w(s_2)). \tag{6}$$

In this case, the ordering $\le_u$ is included, as a set in $S \times S$, in the ordering $\le_w$. Hence, it follows that G_w is an homomorphic image of G_u, i.e. G_w can be realized as a subgraph of G_u.

We may introduce a stronger notion to compare functions. For instance, let $\mathrm{Sgn}: R \to R$ be such that

$\forall x \in R : x > 0 \Longrightarrow \mathrm{Sgn}(x) = 1; x = 0 \Longrightarrow \mathrm{Sgn}(x) = 0; x < 0 \Longrightarrow \mathrm{Sgn}(x) = -1$

Let us say that u is an *tonal-refinement* of w if

$$\forall s_1, s_2 \in S : \mathrm{Sgn}(u(s_1) - u(s_2)) = \mathrm{Sgn}(w(s_1) - w(s_2)). \tag{7}$$

In this case, G_w is isomorphic to G_u.

The von Neumann and Morgenstern utility assessments called a preference probability determined by a preference relation $\le_{u_i}$that establish a poset.

Proposition 4.1. *If u and w are utility functions related by a positive affine transformation, then u and w produce the same ranking of payoffs, i.e. G_u is isomorphic to G_w*

Proof. The proof is straightforward from the previous definitions and therefore omitted. ☐

Example 4.1. Let us consider the entropy as a utility function. We must do the analogy between probability and utility which appears naturally in the probabilistic equivalence used in the von Neumann and Morgenstern utility assessments.

We assume that there is at least one strategy, which has strict preference (to exclude the case of absolute indifference between the n strategies). Then, without loss generality, we assign a utility value of zero to the least preferred strategy and a utility value of one to the most preferred strategy.

We will extend u_i to a function defined for all events $S' \subset S$ as follows $u_i(S') = \sum_{s \in S'} u_i(s) \left(\prod_{j=1}^{n} \sigma_j(s_j|S') \right)$. We will normalize the the utility functions such that $u_i(s) \geq 0$ and $u_i(S) = 1$, the the set of functions defined by $f(s) := u_i(s)\left(\prod_{j=1}^{n} \sigma_j(s_j)\right)$ is a probability measure, i.e. $f : S \to [0, 1)$.

Example 4.2. Let us consider a positive utility function linear transformation as a utility function. A positive utility function's linear transformation is a function $u(w) = aw + b$, where a is positive and b is the equilibrium point. A linear transformation $u(w) = aw + b$ means that there is an isomorphism between u and w, which maps points from u to corresponding points of w, and vice versa.

The utility functions specified up to a positive linear transformation are called interval scales. Our definition of a utility functions require an interval scale to be invariant only under positive linear transformations.

5. Equilibrium Point and Stability

Consider the game $G = \left\langle N, (\Gamma_i)_{i\in N}, (u_i)_{i\in N} \right\rangle$. Denote for each player $i \in N$ and each profile $\sigma_{-i} \in \Gamma_{-i}$ of actions of his opponent the set of *best replies*, i.e. the actions that player i cannot improve upon $B_i(\sigma_{-i})$:

$$B_i(\sigma_{-i}) := \{\sigma_i \in \Gamma_i | \forall \sigma_i' \in \Gamma_i : u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})\}.$$

An action $s_i \in S_i$ is weakly dominated by a strategy $\sigma_i \in \Gamma_i$ if

$$\forall s_{-i} \in S_{-i} : u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i}),$$

with strict inequality for at least one s_{-i} and *strictly dominated* if all inequalities are strict. A *strictly dominated* action is clearly never a best reply.

Definition 5.1. Let $G = \left\langle N, (\Gamma_i)_{i\in N}, (u_i)_{i\in N} \right\rangle$ be a local game. A strategy profile $\sigma \in \Gamma$ is an equilibrium point in the Lyapunov sense if for every player $i \in N$ and for every $\sigma_i \in \Gamma_i$ the utility functions $u_i(\sigma_{-i}^*, \sigma_i^*) = 0$.

Theorem 5.1. *Every local game $G = \left\langle N, (\Gamma_i)_{i\in N}, (u_i)_{i\in N} \right\rangle$ has a local strategy Nash/Lyapunov equilibrium point.*

Proof. Nash equilibrium point exists in a non-empty, compact, convex subset of a Euclidian space by Kakutani's Fixed-Point Theorem ([8]). On the other hand, by the definition of local Lyapunov function u_i we have that $\exists s^* \in \prod_{i\in N} S_i$ such that $u_i((s^*) = 0) = 0$. □

Remark 5.1. The potentiality of the previous theorem remains in its formal simplicity proof for the existence of an equilibrium point.

Theorem 5.2. *The local equilibrium point in the Lyapunov sense of a game* $G = \left\langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$ *coincides with the Nash equilibrium.*

Proof. ($\Longrightarrow$) The equilibrium point in the Nash sense of a game $G = \left\langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$ is satisfied by a Lyapunov function $v = \sigma^T A \sigma$ ([11]), where the payoff matrix is described by A, the increment of v is negative definte, considering that Δv satisfies the following condition $A^T P A - P = -Q$, where P and Q are negative definte and the eingenvalues of A, $\{\lambda_i\}$, are bounded in [0,1).

($\Longleftarrow$) The equilibrium point in the Lyapunov sense of a game $G = \left\langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$ is satisfied by $v = \sigma^{*T} A \sigma^*$. In agreement with Lyapunov $|\sigma^{*T} A \sigma^* - \sigma^T A \sigma^*| > \varepsilon$, it is equivalent to $\sigma^{*T} A \sigma^* > \varepsilon + \sigma^T A \sigma^*$. When σ becomes σ^* we have that $\varepsilon \to 0$, then $\sigma^{*T} A \sigma^* \geq \sigma^T A \sigma^*$ and this is the Nash equilibrium condition. □

Corolary 5.1. *The local game* $G = \left\langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$ *is asymptotically stable in the Lyapunov sense.*

Proof. Since $|u_i(\sigma') - u_i(\sigma)| > \varepsilon$ for all $\sigma, \sigma' \in \Gamma$, where $\sigma, \sigma' \neq \sigma^*$, the game is asymptotically stable (see [12]). □

Definition 5.2. Let $G = \left\langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$ be a robust game. A strategy profile $\sigma \in \Gamma$ is a robust equilibrium point in the Lyapunov sense if for every player $i \in N$ and for every $\sigma_i \in \Gamma_i$ the utility functions $u_i(\sigma^*_{-i}, \sigma^*_i) = 0$.

Theorem 5.3. *Every robust game* $G = \left\langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$ *has a robust Nash equilibrium point.*

Proof. Since we are asking f in (1) to be Lipschitz, then by Picard (see [6]) there is a solution satisfying the initial conditions of (1). Then the set $\mathcal{I} \subseteq \mathbb{R}^n$ is not empty. □

Theorem 5.4. *Every robust game* $G = \left\langle N, (\Gamma_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$ *has a robust Lyapunov equilibrium point.*

Proof. For definition of robust Lyapunov function u_i the set $\mathcal{I} \subseteq \mathbb{R}^n$ is nonempty. □

If $\mathcal{I} \subseteq \mathbb{R}^n$ consists of just an equilibrium point x_0, this is the local notion of global asymptotic stability for the solution $x(t) \equiv x_0$.

Remark 5.2. The potential of the previous theorem remains in its formal proof simplicity for the existence of an equilibrium point.

Theorem 5.5. *The equilibrium point in the robust Lyapunov sense of a game $G = \left\langle N, (\Gamma_i)_{i\in N}, (u_i)_{i\in N} \right\rangle$ coincides with a robust Nash equilibrium point.*

Proof. ($\Longrightarrow$) The equilibrium point, in the Nash sense, of a game $G = \left\langle N, (\Gamma_i)_{i\in N}, (u_i)_{i\in N} \right\rangle$ is satisfied by a Lyapunov function $v = \sigma^T A \sigma$ ([11]), where the payoff matrix is described by $A = \sum_{i=1}^{n} \omega_i A_i$, the increment of v is negative, considering that Δv satisfies the following condition $A^T P A - P = -Q$, where P and Q are positive definite and the eingenvalues of A, $\{\lambda_{ij}\}$, are bounded in [0,1).

($\Longleftarrow$) The equilibrium point in the Lyapunov sense of a game $G = \left\langle N, (\Gamma_i)_{i\in N}, (u_i)_{i\in N} \right\rangle$ is satisfied by $v = \sigma^{*T} A \sigma^*$. In agreement with Lyapunov $|\sigma^{*T} A \sigma^* - \sigma^T A \sigma^*| > \varepsilon$, it is equivalent to $\sigma^{*T} A \sigma^* > \varepsilon + \sigma^T A \sigma^*$. When σ becomes σ^* we have that $\varepsilon \to 0$, then $\sigma^{*T} A \sigma^* \geq \sigma^T A \sigma^*$ and this is the Nash equilibrium condition. □

Corolary 5.2. *The robust game $G = \left\langle N, (\Gamma_i)_{i\in N}, (u_i)_{i\in N} \right\rangle$ is robust uniformly globally asymptotically stable in the Lyapunov sense.*

Proof. Since $|u_i(\sigma') - u_i(\sigma)| > \varepsilon$ for all $\sigma, \sigma' \in \Gamma$, where $\sigma, \sigma' \neq \sigma^*$, the game is robust uniformly globally asymptotically stable (see [12]). □

Example 5.1. The entropy in a game $G = \left\langle N, (\Gamma_i)_{i\in N}, (u_i)_{i\in N} \right\rangle$ maximizes the utility only in Nash's equilibrium or in Lyapunov equilibrium point. The utility function satisfies that in the equilibrium point the standard deviation

$$E_i = \sum_{i=1}^{n} (u_i(\sigma_i, \sigma_{-i}) - u_i(\sigma_i', \sigma_{-i}))(\prod_{j=i}^{n} \sigma_j(s_j)) = 0.$$

Outside the equilibrium point we find the standard deviation $E_i \neq 0$.

6. Conclusions and Future Works

The Lyapunov method induces a new equilibrium and stability concept in noncooperative games. We proved that the equilibrium concept in local and robust Lyapunov sense coincides with the equilibrium concept of Nash, representing an alternative way to calculate the equilibrium and stability of the game. It is the most important contribution of this paper. We show that an utility function isomorphism can be attained.

We also have shown conditions under which the game is stable in local and robust Lyapunov sense. The Lyapunov method is based on the determination of a function able to show stability and instability of a system. We believe that there are many applications in game theory where robust stability properties are of interest.

However, the main disadvantage of Lyapunov's stability criterion is that it gives only the sufficient conditions for stability. Necessary conditions can attained under certain space restrictions, but we will leave the proof for a upcoming work. Furthermore, there are no unique methods of determining the function for a wide class of systems. As future work, it will also be of interest to formally tackle the time complexity of the game.

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