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STRONG CONVERGENCE THEOREMS FOR k-STRICTLY ASYMPTOTICALLY PSEUDOCONTRACTIVE AND HEMICONTRACTIVE OPERATORS IN ARBITRARY REAL NORMED LINEAR SPACES

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Abstract: In this work, we establish strong convergence of the revised modified three-step iteration methods for k-strictly asymptotically pseudocontractive and hemicontractive operators in arbitrary real normed linear spaces. Our results extend the previous results for asymptotically nonexpansive mappings and those of Liu Qihou [10], which itself is an extension of those of Schu [11], for asymptotically pseudocontractive mappings to the modified three-step iteration methods.

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1. Introduction

Let K be a nonempty subset of a normed space B. A mapping $T : K \to K$ is said to be k-strictly asymptotically pseudocontractive if there exists a real sequence $\{k_n\}$ satisfying, $\lim\limits_{n\to\infty} k_n = 1$ and for some constant $k, 0 \le k < 1$,

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$$\|T^n x - T^n y\|^2 \le k_n^2 \|x-y\|^2 + k\|(x - T^n x) - (y - T^n y)\|^2 \tag{1.1}$$

for all $n \in N$ and all $x, y \in K$. When $k = 0$, then T is asymptotically nonexpansive. T is said to be asymptotically hemicontractive, if there exists a real sequence $\{k_n\}$, satisfying, $\lim_{n\to\infty} k_n = 1$ and for some constant $k, 0 \le k < 1$, $F(T)$ - the fixed point set of T - is nonempty and

$$\|T^n x - p\|^2 \le k_n^2 \|x-p\|^2 + k\|x - T^n x\|^2 \tag{1.2}$$

holds for all $x \in K$, $p \in F(T)$.

Schu [11] introduced the modified Ishikawa iteration method for arbitrary $x_1 \in K$ by

$$\left.\begin{array}{l} x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n, \\ y_n = a_n^{'} x_n + b_n^{'} T^n x_n + c_n^{'} v_n, \end{array}\right\}, \quad n \ge 1\,, \tag{1.3}$$

where K is a convex subset of a given Banach space, $\{u_n\}$, $\{v_n\}$ are sequences in K and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n^{'}\}$, $\{b_n^{'}\}$, $\{c_n^{'}\}$ are sequences in $[0,1]$ satisfying

$$a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = 1 \;\; \forall n \ge 0\,.$$

When $b_n^{'} = c_n^{'} = 0$ for all $n \ge 0$, then (1.3) reduces to the modified Mann iteration scheme with errors.

These iteration methods have been investigated by several authors including Schu [11], Owojori and Imoru [9], Osilike and Igbokwe [8], and others for fixed points of asymptotically nonexpansive and asymptotically demicontractive mappings in Banach spaces.

Recently Owojori and Imoru [9] introduced improved three-step schemes with errors as a generalization of the modified Mann and Ishikawa iteration methods with errors which contains that of Xu and Noor [5] as a special case. Convergence of the improved iteration method was established for pseudocontractive and accretive operators in arbitrary Banach spaces.

An improved modified three-step iteration scheme with errors which includes previous modified iteration methods as special cases is given by the following definition.

Definition 1.1. Let K be a nonempty closed bounded convex subset of a uniformly smooth Banach space and suppose T, S are uniformly continuous asymptotically nonexpansive selfmappings of K. Define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by

$$\left.\begin{array}{l} x_{n+1} = a_n x_n + b_n T^n y_n + c_n S^n x_n \\ y_n = a_n^{'} x_n + b_n^{'} S^n z_n + c_n^{'} v_n \\ z_n = a_n^{''} x_n + b_n^{''} T^n x_n + c_n^{''} \omega_n \end{array}\right\}, \quad n \ge 1\,, \tag{1.4}$$

where $\{v_n\}$, $\{\omega_n\}$ are arbitrary sequences in K and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$, $\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$ are real sequences in $[0,1]$ satisfying the following conditions:

(i) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$.

(ii) $\sum b_n = \infty$.

(iii) $\alpha_n := b_n + c_n, \beta_n := b_n{'} + c_n{'}, \gamma_n := b_n" + c_n"$.

Remark. A special case of the iteration scheme (1.4) is also given by

$$\left.\begin{array}{l} x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n \\ y_n = a_n^{'} x_n + b_n^{'} T^n z_n + c_n^{'} v_n \\ z_n = a_n^{''} x_n + b_n^{''} T^n x_n + c_n^{''} \omega_n \end{array}\right\}, \quad n \geq 1\,, \tag{1.5}$$

where $\{u_n\}$, $\{v_n\}$ and $\{\omega_n\}$ are bounded sequences in K and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$, $\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$, are real sequences in $[0,1]$ satisfying:

(i) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$.

(ii) $\sum b_n = \infty$.

We readily observe that the revised iteration methods (1.4) and (1.5) are extensions of the modified three-step iteration scheme of B. Xu and Noor [13], with $\alpha_n = b_n$, $\beta_n = b_n^{'}$, $\gamma_n = b_n^{''}$, as well as the modified Mann and Ishikawa iteration methods with (and without) errors in the sense of Liu [5] and Xu [15]. Liu Qihou [10] established the convergence of the modified Mann and Ishikawa iteration methods for asymptotically demicontractive and hemicontractive mappings. In this paper, convergence of the iteration methods (1.5) and (1.4) for k-strictly asymptotically pseudocontractive and asymptotically hemicontractve mappings respectively are established.

2. Some Preliminary Results

In the Sequel, we shall require the following results.

Lemma 2.1. (see Xu [14]) *Let B be a uniformly smooth Banach space. Then B has modulus of smoothness of power type $q > 1$ if and only if there exist $j_q x \in J_q x$ and a constant $c > 0$ such that*

$$\|x + y\|^q \leq \|x\|^q + q < y, j_q(x) > +c\|y\|^q \tag{2.1}$$

for all $x, y \in B$.

By replacing y with $(-y)$ in (2.1), we obtain

$$\|x - y\|^q \leq \|x\|^q - q < y, j(x) > +\|y\|^q \leq \|x\|^q + \|y\|^q\,, \tag{2.2}$$

for all $x, y \in B$.

Applying Lemma 2.1, Chidume and Osilike [1] estabilshed the following lemma.

Lemma 2.2. (see Chidume and Osilike [1]) *Let B be a uniformly smooth Banach space with modulus of smoothness of power type $q > 1$. Then for all $x, y, z \in B$ and $\lambda \in [0,1]$, the following inequality*

$$\begin{aligned} &\|\lambda x + (1-\lambda)y - z\|^q \\ &\qquad \leq [1-\lambda(q-1)]\|y-z\|^q + \lambda c\|x-z\|^q - \lambda[1-\lambda^{q-1}c]\|x-y\|^q\,, \end{aligned} \tag{2.3}$$

holds where c is a positive constant.

The following result is also useful in this study.

Lemma 2.3. (see Weng [12]) *Let $\{\Phi_n\}$ be a nonnegative sequence of real numbers satisfying*

$$\Phi_{n+1} \leq (1-\delta_n)\Phi_n + \sigma_n\,, \tag{2.4}$$

where $\delta_n \in [0,1]$, $\sum \delta_n = \infty$ and $\sigma_n = o(\delta_n)$. Then $\lim\limits_{n\to\infty} \Phi_n = 0$.

3. The Main Results

Our main results in this work are the following.

Theorem 3.1. *Let X be an arbitrary real normed linear space and K a nonempty closed bounded and convex subset of X. Suppose $T : K \to K$ is a completely continuous k-strictly asymptotically pseudoconcontractive mapping with real sequence $\{k_n\}$ satisfying, $k_n \geq 1$, for all n, $\lim\limits_{n\to\infty} k_n = 1$ and $k_n^p + 1 \leq p$,. For a given $x_1 \in K$, define sequence $\{x_n\}$ generated iteratively by*

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n \\ y_n &= a_n^{'} x_n + b_n^{'} T^n z_n + c_n^{'} v_n \\ z_n &= a_n^{''} x_n + b_n^{''} T^n x_n + c_n^{''} \omega_n \end{aligned}\right\}, \quad n \geq 1\,,$$

where $\{u_n\}$, $\{v_n\}$, $\{\omega_n\}$ are arbitrary sequences in K and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$, $\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$ are real sequences in $[0,1]$ satisfying the following conditions:

(i) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$,

(ii) $\sum b_n = \infty$; $\alpha_n := b_n + c_n, \beta_n := b_n{'} + c_n{'}, \gamma_n := b_n" + c_n"$,

(iii) $k_n^2(1-\beta_n) + \beta_n k_n^4(1-\gamma_n) + \beta_n\gamma_n k_n^2 \leq 1$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since T is demicontractive, then T has a fixed point in K. Let $p \in K$ be a fixed point of T. From our hypothesis, we have the following estimates

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|a_n x_n + b_n T^n y_n + c_n u_n - p\|^2 \\ &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n y_n - p) - c_n(T^n y_n - u_n)\|^2 \\ &\leq [1-\alpha_n]\|(x_n - p) - c_n(T^n y_n - u_n)\|^2 \\ &+\alpha_n\|(T^n y_n - p) - c_n(T^n y_n - u_n)\|^2 \\ &-\alpha_n(1-\alpha_n)\|(Ty_n - p) - (x_n - p)\|^2 .\end{aligned}$$

Expanding further and observing that $\alpha_n(1-\alpha_n) \geq 0$ for all $n > 0$, we have:

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1-\alpha_n)[\|x_n - p\|^2 + c_n\|T^n y_n - u_n\|^2] \\ &\quad + \alpha_n\{\|T^n y_n - p\|^2 + c_n\|T^n y_n - u_n\|^2\} \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T^n y_n - p\|^2 + \alpha_n\|T^n y_n - u_n\|^2] \,. \quad (3.1)\end{aligned}$$

But T is asymptotically demicontractive, therefore

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n[{k_n}^2\|y_n - p\|^2 \\ &\quad + k\|y_n - T^n y_n\|^2] + \alpha_n\|T^n y_n - u_n\|^2 \\ &= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n {k_n}^2\|y_n - p\|^2 + \alpha_n k\|y_n - T^n y_n\|^2 + \alpha_n\|T^n y_n - u_n\|^2 .\end{aligned}$$

Since T is uniformly continuous on the bounded set K, there exists a real number $M_1 < \infty$, such that

$$\|y_n - T^n y_n\|^2 \leq M_1, \quad \text{and} \quad \|T^n y_n - u_n\|^2 \leq M_1 .$$

Thus, we have

$$\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n[{k_n}^2\|y_n - p\|^2 + \alpha_n(1+k)M_1. \quad (3.2)$$

We also have similar estimates as follows:

$$\begin{aligned}\|y_n - p\|^2 &= \|a_n' x_n + b_n' T^n z_n + c_n' v_n - p\|^2 \\ &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n z_n - p) - c_n'(T^n z_n - v_n)\|^2 \\ &\leq (1-\beta_n)\|(x_n - p) - c_n'(T^n z_n - v_n)\|^2 \\ &+ \beta_n\|(T^n z_n - p) - c_n'(T^n z_n - v_n)\|^p - \beta_n(1-\beta_n)\|(T^n z_n - p) - (x_n - p)\|^2 .\end{aligned}$$

Expanding further and considering the fact that $c_n{}' \leq \beta_n$ and $\beta_n(1-\beta_n) \geq 0$, we have

$$\begin{aligned}\|y_n - p\|^2 &\leq (1-\beta_n)[\|x_n - p\|^2 + \beta_n\|T^n z_n - v_n\|^2] \\ &+\beta_n[\|T^n z_n - p\|^2 + \beta_n\|T^n z_n - v_n\|^2 \\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|T^n z_n - p\|^2 + \beta_n\|T^n z_n - v_n\|^2 \\ &\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n k_n^2\|z_n - p\|^2 \\ &+\beta_n k\|z_n - T^n z_n\|^2 + \beta_n\|T^n z_n - v_n\|^2 .\end{aligned}$$

Uniform continuity of T on K also implies that there exists a real number $M_2 < \infty$ such that

$$\|T^n z_n - v_n\|^2 \leq M_2 \text{ and } \|z_n - T^n z_n\|^2 \leq M_2.$$

Therefore,

$$\|y_n - p\|^2 \leq (1-\beta_n)\|x_n - p\|^2 + \beta_n k_n^2\|z_n - p\|^2 + \beta_n(1+k)M_2 \tag{3.3}$$

Substitute (3.3) into (3.2), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n[{k_n}^2\|x_n - p\|^2 \\ + \alpha_n\beta_n k_n^4\|z_n - p\|^2 + \alpha_n\beta_n k_n^2(1+k)M_2 + \alpha_n(1+k)M_1 .\end{aligned} \tag{3.4}$$

By similar procedure as above, we also have the following

$$\begin{aligned}\|z_n - p\|^2 &= \|a_n^{''}x_n + b_n^{''}T^n x_n + c_n^{''}\omega_n - p\|^2 \\ &= \|(1-\gamma_n)(x_n - p) + \gamma_n(T^n x_n - p) - c_n^{''}(T^n x_n - \omega_n)\|^2 \\ &\leq (1-\gamma_n)\|(x_n - p) - c_n^{''}(T^n x_n - \omega_n)\|^2 + \gamma_n\|(T^n x_n - p) - c_n^{''}(T^n x_n - \omega_n)\|^2 .\end{aligned}$$

Since $c_n^{''} \leq \gamma_n$ and T is asymptotically demicontractive, we have

$$\begin{aligned}\|z_n - p\|^2 &\leq (1-\gamma_n)[\|x_n - p\|^2 + \gamma_n\|T^n x_n - \omega_n\|^2] \\ &+\gamma_n[\|T^n x_n - p\|^2 + \gamma_n\|T^n x_n - \omega_n\|^2 \\ &= (1-\gamma_n)\|x_n - p\|^2 + \gamma_n[\|T^n x_n - p\|^2 + \gamma_n\|T^n x_n - \omega_n\|^2 \\ &\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n k_n^2\|x_n - p\|^2 \\ &+\gamma k\|x_n - T^n x_n\|^2 + \gamma_n\|T^n x_n - \omega_n\|^2 .\end{aligned}$$

Continuity of T on the bounded set K implies that there exists a real number $M_3 < \infty$ such that

$$\|x_n - T^n x_n\|^2 \leq M_3 \text{ and } \|T^n x_n - \omega_n\|^2 \leq M_3 .$$

Therefore,

$$\|z_n - p\|^2 \le [1 - \gamma_n(1 - k_n^2)]\|x_n - p\|^2 + \gamma_n(1 + k)M_3 \,. \tag{3.5}$$

Substituting (3.5) into (3.4) gives

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n[{k_n}^2\|x_n - p\|^2 \\
&\quad + \alpha_n\beta_n k_n^4\{[1 - \gamma_n(1 - k_n^2)]\|x_n - p\|^2 + \gamma_n(1 + k)M_3\} \\
&\quad + \alpha_n\beta_n k_n^2(1 + k)M_2 + \alpha_n(1 + k)M_1 \\
&= \{(1 - \alpha_n) + \alpha_n {k_n}^2(1 - \beta_n) + \alpha_n\beta_n k_n^4[1 - \gamma_n(1 - k_n^2)]\}\|x_n - p\|^2 \\
&\quad + \alpha_n\beta_n\gamma_n k_n^4(1 + k)M_3 + \alpha_n\beta_n k_n^2(1 + k)M_2 + \alpha_n(1 + k)M_1 \,.
\end{aligned}$$

Let $M = \max[M_1, M_2, M_3]$, then we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \{(1 - \alpha_n) + \alpha_n k_n^2(1 - \beta_n) + \alpha_n\beta_n k_n^4[1 - \gamma_n(1 - k_n^2)]\}\|x_n - p\|^2 \\
&\quad + \alpha_n(1 + k)(1 + \beta_n k_n^2 + \beta_n\gamma_n k_n^4)M = [1 - t_n]\|x_n - p\|^2 + \sigma_n \,,
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
t_n &= \alpha_n - \alpha_n {k_n}^2(1 - \beta_n) - \alpha_n\beta_n k_n^4[1 - \gamma_n(1 - k_n^2)] \\
&= \alpha_n - \alpha_n k_n^2 + \alpha_n\beta_n k_n^2 - \alpha_n\beta_n k_n^4 + \alpha_n\beta_n\gamma_n k_n^4 - \alpha_n\beta_n\gamma_n k_n^6 \\
&= \alpha_n[1 - {k_n}^2(1 - \beta_n) - \beta_n k_n^4(1 - \gamma_n) - \beta_n\gamma_n k_n^6] \\
&\le 1 - k_n^2(1 - \beta_n) - \beta_n k_n^4(1 - \gamma_n) - \beta_n\gamma_n k_n^6 \le 1 \,.
\end{aligned}$$

Observe that $t_n \ge 0$ by our hypothesis. Thus $0 \le t_n \le 1$. Since $\sum \alpha_n = \infty$, then $\sum t_n = \infty$. Also, let

$$\sigma_n = \alpha_n(1 + k)(1 + \beta_n k_n^2 + \beta_n\gamma_n k_n^4)M.$$

Clearly, $\sigma_n = o(t_n)$. In (3.6), put

$$\|x_n - p\|^2 = \rho_n,$$

then we have

$$\rho_{n+1} = (1 - t_n)\rho_n + \sigma_n.$$

Hence, by Lemma (2.3), we have

$$\lim_{n\to\infty} \rho_n = 0.$$

This implies that sequence $\{x_n\}$ converges strongly to p. The proof is complete. $\square$

Remark. Theorem 3.1 is clearly an extension of the results of Yeol Je Cho, Haiyun Zhou and Shin Min Kang [16] as well as Owojori and Imoru [9], to the revised three-step iteration procedure and asymptotically demicontractive operators respectively. By similar procedure as in the proof of the theorem above, it can be shown that the iteration method given by

$$\left.\begin{array}{l} x_{n+1} = a_n x_n + b_n T^n y_n + c_n T^n x_n \\ y_n = a_n^{'} x_n + b_n^{'} T^n z_n + c_n^{'} v_n \\ z_n = a_n^{''} x_n + b_n^{''} T^n x_n + c_n^{''} \omega_n \end{array}\right\}, \quad n \geq 1\,,$$

where $\{v_n\}$, $\{\omega_n\}$ are arbitrary sequences in K and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$, $\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$ are real sequences in $[0,1]$ satisfying certain conditions, also converges strongly to the fixed point of asymptotically demicontractive operator T.

We now investigate the convergence of the revised iteration method given by (1.5) for common fixed points of asymptotically demicontractive operators in uniformly smooth Banach spaces. Our result is the following theorem.

Theorem 3.2. *Let X be an arbitrary real normed linear space and K a nonempty closed bounded and convex subset of X. Suppose S, T are uniformly continuous and asymptotically demicontractive selfmapping of K with real sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\lim k_n = 1$. Define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by*

$$\left.\begin{array}{l} x_{n+1} = a_n x_n + b_n T^n y_n + c_n S^n x_n \\ y_n = a_n^{'} x_n + b_n^{'} S^n z_n + c_n^{'} v_n \\ z_n = a_n^{''} x_n + b_n^{''} T^n x_n + c_n^{''} \omega_n \end{array}\right\}, \quad n \geq 1\,,$$

where $\{v_n\}$, $\{\omega_n\}$ are arbitrary sequences in K and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$, $\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$ are real sequences in $[0,1]$ satisfying the following conditions:

(i) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$,

(ii) $\sum b_n = \infty$; $\alpha_n := b_n + c_n$,$\beta_n := b_n{'} + c_n{'}$, $\gamma_n := b_n" + c_n"$,

(iii) $0 \leq 2\beta_n(1+\gamma_n) \leq \dfrac{3k_n^2 - 1}{k_n^2(1-k_n^2}$.

If S and T have a common fixed point in K, then the sequence $\{x_n\}$ converges strongly to the common fixed point of S and T.

Proof. Let $p \in K$ be the common fixed point of S and T. From the our hypothesis above and observing that, for all n, $\alpha_n(1-\alpha_n) \geq 0$, we have the following estimates

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|a_n x_n + b_n T^n y_n + c_n S^n x_n - p\|^2 \\ &= \|(1-\alpha_n)(x_n-p)+\alpha_n(T^n y_n - p) - c_n(T^n y_n - S^n x_n)\|^2 \leq \\ (1-\alpha_n)\|(x_n-p) &- c_n(T^n y_n - S^n x_n)\|^2 + \alpha_n\|(T^n y_n - p) - c_n(T^n y_n - S^n x_n)\|^2 \\ &\leq (1-\alpha_n)[\|x_n-p\|^2 + c_n\|T^n y_n - S^n x_n\|^2] \\ &\quad + \alpha_n[\|T^n y_n - p\|^2 + c_n\|T^n y_n - S^n x_n\|^2] \\ &= (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\|T^n y_n - p\|^2 + c_n\|T^n y_n - S^n x_n\|^2 \,.\end{aligned}$$

By Xu [14], we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\|T^n y_n - p\|^2 + \alpha_n\|T^n y_n - p\|^2 \\ + \alpha_n\|p - S^n y_n\|^2 &\leq (1-\alpha_n)\|x_n-p\|^2 + 2\alpha_n[k_n^2\|y_n-p\|^2 + k\|y_n - T^n y_n\|^2] \\ &\quad + \alpha_n[k_n^2\|x_n-p\|^2 + k\|x_n - S^n x_n\|^2] \,.\end{aligned}$$

Since T, S are asymptotically demicontractive, the above yields

$$\begin{aligned}\|x_{n+1}-p\|^2 &= [(1-\alpha_n)+\alpha_n k_n^2]\|x_n-p\|^2 + 2\alpha_n k_n^2\|y_n-p\|^2 \\ &\quad + 2\alpha_n k\|y_n - T^n y_n\|^2 + \alpha_n k\|x_n - S^n x_n\|^2 \,.\end{aligned}$$

Since S, T are uniformly continuous on the bounded set K, then there exists a real number $M_4 < \infty$ such that

$$\|y_n - T^n y_n\|^2 \leq M_4 \quad \text{and} \quad \|x_n - S^n x_n\|^2 \leq M_4 \,.$$

Therefore, we have

$$\|x_{n+1}-p\|^2 \leq [1-\alpha_n(1-k_n^2)]\|x_n-p\|^2 + 2\alpha_n k_n^2\|y_n-p\|^2 + 3\alpha_n k M_4. \quad (3.7)$$

By the same procedure as above, we also have the following estimates

$$\begin{aligned}\|y_n-p\|^2 \quad &\leq (1-\beta_n)[\|x_n-p\|^2 + c_n'\|S^n z_n - v_n\|^2] \\ &+\beta_n[\|S^n z_n - p\|^2 + c_n'\|S^n z_n - v_n\|^2] \\ &\leq (1-\beta_n)\|x_n-p\|^2 + \beta_n\|S^n z_n - p\|^2 + \beta_n\|S^n z_n - v_n\|^2 \\ &\leq (1-\beta_n)\|x_n-p\|^2 + \beta_n k_n^2\|z_n-p\|^2 \\ &+\beta_n k\|z_n - S^n z_n\|^2 + \beta_n\|S^n z_n - v_n\|^2 \,.\end{aligned}$$

Uniformly continuity of S, T on the bounded set K implies that there exists a real number $M_5 < \infty$ such that

$$\|z_n - S^n z_n\|^2 \le M_5 \quad \text{and} \quad \|S^n z_n - v_n\|^2 \le M_5\,.$$

Therefore, we have

$$\|y_n - p\|^2 \le (1-\beta_n)\|x_n - p\|^2 + \beta_n k_n^2 \|z_n - p\|^2 + \beta_n (1+k) M_5. \tag{3.8}$$

Substitute (3.8) into (3.7) yields

$$\begin{aligned}\|x_{n+1} - p\|^2 \le [1-\alpha_n(1-k_n^2)]\|x_n - p\|^2 + 2\alpha_n[k_n^2[(1-\beta_n)\|x_n - p\|^2 \\ + \beta_n k_n^2 \|z_n - p\|^2 + \beta_n(1+k)M_5] + 3\alpha_n k M_4 \\ = [1-\alpha_n(1-k_n^2) + 2\alpha_n k_n^2(1-\beta_n)]\|x_n - p\|^2 \\ + 2\alpha_n\beta_n k_n^4 \|z_n - p\|^2 + 2\alpha_n\beta_n k_n^2(1+k)M_5 + 3\alpha_n k M_4\,.\end{aligned} \tag{3.9}$$

Furthermore, equation (3.5) also holds true in this case, i.e.

$$\|z_n - p\|^2 \le [1-\gamma_n(1-k_n^2)]\|x_n - p\|^2 + \gamma_n(1+k)M_3\,. \tag{3.5}$$

Substitute (3.5) into (3.9), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 \le [1-\alpha_n(1-k_n^2) + 2\alpha_n k_n^2(1-\beta_n)]\|x_n - p\|^2 \\ + 2\alpha_n\beta_n k_n^4[1-\gamma_n(1-k_n^2)]\|x_n - p\|^2 + \\ + \alpha_n\beta_n k_n^4\gamma_n(1+k)M_3 + 2\alpha_n\beta_n k_n^2(1+k)M_5 + 3\alpha_n k M_4 \\ = \{1-\alpha_n + \alpha_n k_n^2(1-\beta_n) + 2\alpha_n\beta_n k_n^4[1-\gamma_n(1-k_n^2)]\} \\ \|x_n - p\|^2 + 2\alpha_n\beta_n k_n^4\gamma_n(1+k)M_3 \\ + 2\alpha_n\beta_n k_n^2(1+k)M_5 + 3\alpha_n k M_4 = [1-t_n]\|x_n - p\|^2 + \sigma_n\,,\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}t_n = \alpha_n - 3\alpha_n {k_n}^2 + 2\alpha_n\beta_n k_n^2 - 2\alpha_n\beta_n k_n^4 + 2\alpha_n\beta_n\gamma_n k_n^4 - 2\alpha_n\beta_n\gamma_n k_n^6 \\ = \alpha_n[1 - k_n^2(3-2\beta_n) - 2\beta_n k_n^4(1-\gamma_n) - 2\beta_n\gamma_n k_n^6] \le \alpha_n \le 1\,.\end{aligned}$$

By our hypothesis, it is clear that $t_n \ge 0$ and $\sum t_n = \infty$. Thus $0 \le t_n \le 1$.

Also, let

$$\sigma_n = 2\alpha_n\beta_n k_n^4\gamma_n(1+k)M_3 + 2\alpha_n\beta_n k_n^2(1+k)M_5 + 3\alpha_n k M_4.$$

Clearly, $\sigma_n = o(t_n)$. In (3.10), let

$$\|x_n - p\|^2 = \rho_n.$$

Then we have

$$\rho_{n+1} = (1 - t_n)\rho_n + \sigma_n.$$

Hence, by Lemma (2.3), we have

$$\lim_{n\to\infty} \rho_n = 0.$$

This implies that the sequence $\{x_n\}$ converges strongly to p. The proof is complete. $\square$

Remark. It is clear that Theorem 3.2 is an extension of Theorem 3.1 and other relevant previous fixed point results on asymptotically demicontractive mappings to common fixed point of two such operators.

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