SUPPORTS OF IDEMPOTENTS AND THE LIMITS OF AVERAGED CONVOLUTION SEQUENCES

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Abstract: In this note the supports of idempotents and the limits of averaged convolution sequences in the set of all Banach-valued probability measures on a compact semitopological semigroup are discussed.

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1. Introduction

A semitopological group is a group $G$ endowed with a topology such that, for each $a$ in $G$, the translations $x \to ax$ and $x \to xa$ are continuous on $G$, and such that the symmetry $x \to x^{-1}$ is continuous on $G$. A semitopological semigroup is a semitopological group without the continuity condition on symmetry $x \to x^{-1}$. For details on semitopological groups, see Bourbaki [1]. Let $G$ be a compact semitopological semigroup. We denote by $P(G)$ the set of nonnegative and normalized Borel measures on $G$. If $P(G)$ is endowed with the weak star topology then it is a compact semitopological semigroup, where the multiplication is defined by convolution.

Let $A$ be a unital Banach algebra. We write $C(G, A)$ as the algebra of all continuous functions from $G$ to $A$. Let $C(G)$ denote the algebra of continuous functions from $G$ to the set of complex numbers.

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For every \( x \) in \( G \) and for all functions \( f \) in \( C(G) \), we define \( f^*(x) = (f(x))^* \). A linear operator \( T : C(G) \to A \) is positive if for all \( f \) in \( C(G) \), \( T(f^*) \) is in \( A \). Let \( \sigma(G) \) be the \( \sigma \)-algebra of all Borel subsets of \( G \). A partition of \( G \) in a finite (infinite) collection of pairwise disjoint clopen subsets of \( G \), which cover \( G \). We denote by \( \pi(G) \), the partition of \( G \). An operator \( T \) is weakly compact if \( |
abla| \) maps bounded sets into weakly sequentially compact sets. If \( T \) is weakly compact then the representing measure \( m \) of \( T \) has the value in \( A \). The set of all weakly compact measures \( m : \sigma(G) \to A \) is written as \( W_A \). The positivity of an operator \( T \) guarantees that \( m \) is positive.

**Definition 1.1.** For \( B \) in \( \sigma(G) \) we define the variation of \( m \) as follows:
\[
\nu(m)B = \sup\{\Sigma||m(\pi_i)|| : \{\pi_i\} \text{ is in } \pi(B)\}.
\]
Let \( \mu \) and \( \eta \) be weakly compact measures. We define the convolution \( \mu \ast \eta \) as follows:
\[
\int f \, d\mu \ast \eta = \int d\mu(x) \int f(xy) \, d\eta(y), \quad f \in C(G), \quad x, y \in G
\]

**Definition 1.2.** Let \( W_A \) be a subset of \( L(C(G), A) \) endowed with the weak operator topology. The support of a measure \( \mu \) is defined as the complement of \( \bigcup\{U : U \text{ is open and } \mu(U) = 0\} \) and is denoted by \( \text{supp } \mu \), see Conway [3].

A measure \( m : \sigma(G) \to A \) in \( W_A \) is called a Banach-valued probability measure on \( G \), if \( m > 0 \) and \( V(m)(G) = 1 \). If \( \Omega \) denotes the set of all Banach-valued probability measures, then it is a convex set in \( W_A \). Further, it is plain to see if \( A \) is the set of complex numbers then \( W_A \) is the algebra of all bounded regular Borel measures. For more information on these measures, refer to Gaur [5].

**Definition 1.3.** Let \( \Omega_0 \) be a subsemigroup of \( \Omega \). Then the \( \text{supp } \Omega_0 \) is the closure of \( \bigcup\{\text{supp } \mu : \mu \in \Omega_0\} \). It should be noted that if \( \Omega_0(\mu) = \{\mu, \mu^2, \mu^3, \ldots\} \) then \( \text{supp } \Omega_0 \) is the closed semigroup generated by \( \text{supp } \mu \).

**Theorem 1.1.** For \( m \) in \( W_A \) there exists \( m_\Omega \in \Omega \) with \( m_\Omega \to m \) in the strong operator topology such that if \( m_\Omega \) is positive then \( m \) is positive.

**Proof.** From Corollary 5, p. 477 of Dunford [4], the set \( \Omega \) has the same closure in the weak operator topology as it does in the strong operator topology.

Let \( m \) be in \( W_A \) such that there exists \( m_\Omega \) in \( \Omega \) with \( m_\Omega \to m \) in the strong operator topology. Let \( f \) be in \( C(G) \). Then \( m_\Omega(f) = m(f) \). In this case \( \nu(m)(B) = 1 = ||m(B)|| \) for \( B \) in \( \sigma(G) \) (see Definition 1.1). Since \( m_\Omega(ff^*) \to m(ff^*) \) for every \( f \) in \( C(G) \) and \( K(A) \) is closed (\( K(A) \) is the positive cone of \( A \)). Hence, the positivity of \( m_\Omega \) implies \( m \) is positive. \( \square \)
Remark 1.1. The above theorem basically establishes that the set $\Omega$ is closed in $W_A$ endowed with weak operator topology.

Theorem 1.2. The set $\Omega$ is a compact semitopological semigroup.

Proof. From Proposition 1 in Gaur [5] the convolution of probability measures is continuous in $\Omega$.

Let $W_A$ be a subset of $W_{A^*}$. Then by Brooks [2] $W_{A^*}$ is a subset of $L[C(G, A^*), \mathbb{C}]$. We note that $\mathbb{C}$ is reflexive and hence the closed unit sphere of $L[C(G, A^*), \mathbb{C}]$ is compact in the weak operator topology. This in fact follows from p. 512 in Dunford [4].

2. The Limit Theorem

In the following limit theorem it is shown that the limit of an averaged convolution sequence in $\Omega$ is an idempotent in $\Omega$.

Theorem 2.1. If $x_n = \sum_{i=1}^{n} \frac{\mu_i}{n}$, where $\mu$ belongs to $\Omega$, then the sequence $\{x_n\}$ converges to $e(\mu)$ such that $e(\mu)\mu = \mu e(\mu) = e(\mu)$ and $e(\mu)$ is an idempotent in $\Omega$. Also, $\text{supp} e(\mu)$ is a minimal ideal of $\text{supp} \Omega_0$.

Proof. The set $\Omega$ is compact and hence the sequence $\{x_n\}$ has a cluster point, say $x$ by Lemma 9, p. 29 in Dunford [4]. First, we will show that the cluster point $x$ of the sequence $\{x_n\}$ is unique and idempotent.

Consider $(\mu - 1)x_n$. Then

$$(\mu - 1)x_n = \frac{1}{n}[(\mu - 1) \sum_{i=1}^{n} \mu_i^i]$$

$$= \frac{1}{n}(\mu^1 + \mu^2 + \ldots + \mu^{n+1} - \mu^2 - \ldots \mu^n) = \frac{1}{n}(\mu^{n+1} - \mu).$$

Let $f$ be an element of $C(G)$ and $a^*$ is in $A^*$. Then

$$|(\frac{a^*\mu^{n+1} - a^*\mu}{n})(f)| = |\frac{a^*}{n}(\mu^{n+1}(f) - \mu(f))| \leq \frac{2}{n}|a^*||.$$

This shows that $(\mu - 1)x_n = \frac{\mu^{n+1} - \mu}{n}$ converges to the zero measure in the weak operator topology. Therefore, $\mu x = x = x \mu$. In fact, we have $x_n x = x = x_n$ and $x^2 = x$. 

\[\square\]
Now we prove the uniqueness of $x$. Let $y$ be any other cluster point of $\{x_n\}$. Then $xy = x = yx$ and $yx = xy = y$. This shows that $x = y$. Hence $x_n \to x$. If we assume $x = e(\mu)$ then $\mu e(\mu) = e(\mu)\mu = e(\mu)$ since $e(\mu)$ belongs to the closed convex hull of $\Omega_0(\mu)$, it follows that $\text{supp } e(\mu)$ is a subset of $\text{supp } \Omega_0$. We also remark that $\Omega_0 = \text{supp } \text{co}[\Omega_0(\mu)] = \text{supp } \overline{\text{co}}[\Omega_0(\mu)]$. Let $\mu \in \text{supp } e(\mu) = P$ and $\text{supp } \mu = Q$. Then by Lemma 1 in Pym [6] and Theorem 1 in Gaur [5], we have $PQ^n = Q^nP = P$, for all $n$. This shows that $P$ is an ideal of $\text{supp } \Omega_0$ which is also minimal by Remark 2 in Gaur [5].

References


