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SUPPORTS OF IDEMPOTENTS AND THE LIMITS OF AVERAGED CONVOLUTION SEQUENCES

A.K. Gaur
Department of Mathematics
Duquesne University
600 Forbes Avenue, Pittsburgh, PA 15282, USA
e-mail: gaur@mathcs.duq.edu

Abstract: In this note the supports of idempotents and the limits of averaged convolution sequences in the set of all Banach-valued probability measures on a compact semitopological semigroup are discussed.

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1. Introduction

A semitopological group is a group G endowed with a topology such that, for each a in G, the translations $x \to ax$ and $x \to xa$ are continuous on G, and such that the symmetry $x \to x^{-1}$ is continuous on G. A semitopological semigroup is a semitopological group without the continuity condition on symmetry $x \to x^{-1}$. For details on semitopological groups, see Bourbaki [1]. Let G be a compact semitopological semigroup. We denote by $P(G)$ the set of nonnegative and normalized Borel measures on G. If $P(G)$ is endowed with the weak star topology then it is a compact semitopological semigroup, where the multiplication is defined by convolution.

Let A be a unital Banach algebra. We write $C(G, A)$ as the algebra of all continuous functions from G to A. Let $C(G)$ denote the algebra of continuous functions from G to the set of complex numbers.

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For every x in G and for all functions f in $C(G)$, we define $f^*(x) = (f(x))^*$. A linear operator $T : C(G) \to A$ is positive if for all f in $C(G)$, $T(ff^*)$ is in A. Let $\sigma(G)$ be the σ-algebra of all Borel subsets of G. A partition of G in a finite (infinite) collection of pairwise disjoint clopen subsets of G, which cover G. We denote by $\pi(G)$, the partition of G. An operator T is weakly compact if it maps bounded sets into weakly sequentially compact sets. If T is weakly compact then the representing measure m of T has the value in A. The set of all weakly compact measures $m : \sigma(G) \to A$ is written as W_A. The positivity of an operator T guarantees that m is positive.

Definition 1.1. For B in $\sigma(G)$ we define the variation of m as follows: $v(m)B = \sup\{\Sigma||m(\pi_i)|| : \{\pi_i\}$ is in $\pi(B)\}$. Let μ and η be weakly compact measures. We define the convolution $\mu * \eta$ as follows:

$$\int f\,d\mu * \eta = \int d\mu(x) \int f(xy)\,d\eta(y), \quad f \in C(G), \quad x, y \in G$$

Definition 1.2. Let W_A be a subset of $L(C(G), A)$ endowed with the weak operator topology. The support of a measure μ is defined as the complement of $\bigcup\{U : U$ is open and $\mu(U) = 0\}$ and is denoted by supp μ, see Conway [3].

A measure $m : \sigma(G) \to A$ in W_A is called a Banach-valued probability measure on G, if $m > 0$ and $V(m)(G) = 1$. If Ω denotes the set of all Banach-valued probability measures, then it is a convex set in W_A. Further, it is plain to see if A is the set of complex numbers then W_A is the algebra of all bounded regular Boral measures. For more information on these measures, refer to Gaur [5].

Definition 1.3. Let Ω_0 be a subsemigroup of Ω. Then the supp Ω_0 is the closure of $\bigcup\{\text{supp}\mu : \mu \in \Omega_0\}$. It should be noted that if $\Omega_0(\mu) = \{\mu, \mu^2, \mu^3, \ldots\}$ then supp Ω_0 is the closed semigroup generated by supp μ.

Theorem 1.1. *For m in W_A there exists $m_\Omega \in \Omega$ with $m_\Omega \to m$ in the strong operator topology such that if m_Ω is positive then m is positive.*

Proof. From Corollary 5, p. 477 of Dunford [4], the set Ω has the same closure in the weak operator topology as it does in the strong operator topology.

Let m be in W_A such that there exists m_Ω in Ω with $m_\Omega \to m$ in the strong operator topology. Let f be in $C(G)$. Then $m_\Omega(f) = m(f)$. In this case $v(m)(B) = 1 = ||m(B)||$ for B in $\sigma(G)$ (see Definition 1.1). Since $m_\Omega(ff^*) \to m(ff^*)$ for every f in $C(G)$ and $K(A)$ is closed ($K(A)$ is the positive cone of A). Hence, the positivity of m_Ω implies m is positive. $\square$

Remark 1.1. The above theorem basically establishes that the set Ω is closed in W_A endowed with weak operator topology.

Theorem 1.2. *The set Ω is a compact semitopological semigroup.*

Proof. From Proposition 1 in Gaur [5] the convolution of probability measures is continuous in Ω.

Let W_A be a subset of $W_{A^{**}}$. Then by Brooks [2] $W_{A^{**}}$ is a subset of $L[C(G, A^*), \mathbb{C}]$. We note that $\mathbb{C}$ is reflexive and hence the closed unit sphere of $L[C(G, A^*), \mathbb{C}]$ is compact in the weak operator topology. This in fact follows from p. 512 in Dunford [4]. ☐

2. The Limit Theorem

In the following limit theorem it is shown that the limit of an averaged convolution sequence in Ω is an idempotent in Ω.

Theorem 2.1. *If $x_n = \sum_{i=1}^{n} \frac{u^i}{n}$, where μ belongs to Ω, then the sequence $\{x_n\}$ converges to $e(\mu)$ such that $e(\mu)\mu = \mu e(\mu) = e(\mu)$ and $e(\mu)$ is an idempotent in Ω. Also,* $\operatorname{supp} e(\mu)$ *is a minimal ideal of* $\operatorname{supp} \Omega_0$.

Proof. The set Ω is compact and hence the sequence $\{x_n\}$ has a cluster point, say x by Lemma 9, p. 29 in Dunford [4]. First, we will show that the cluster point x of the sequence $\{x_n\}$ is unique and idempotent.

Consider $(\mu - 1)x_n$. Then

$$(\mu - 1)x_n = \frac{1}{n}[(\mu - 1)\sum_{i=1}^{n} \mu^i]$$
$$= \frac{1}{n}(\mu^2 + \mu^3 + \ldots + \mu^{n+1} - \mu - \mu^2 - \ldots \mu^n) = \frac{1}{n}(\mu^{n+1} - \mu)\,.$$

Let f be an element of $C(G)$ and a^* is in A^*. Then

$$|(\frac{a^*\mu^{n+1} - a^*\mu}{n})(f)| = |\frac{a^*}{n}(\mu^{n+1}(f) - \mu(f))| \leq \frac{2}{n}||a^*||\,.$$

This shows that $(\mu - 1)x_n = \frac{\mu^{n+1} - \mu}{n}$ converges to the zero measure in the weak operator topology. Therefore, $\mu x = x = x\mu$. In fact, we have $x_n x = x = x_n$ and $x^2 = x$. ☐

Now we prove the uniqueness of x. Let y be any other cluster point of $\{x_n\}$. Then $xy = x = yx$ and $yx = xy = y$. This shows that $x = y$. Hence $x_n \to x$. If we assume $x = e(\mu)$ then $\mu e(\mu) = e(\mu)\mu = e(\mu)$ since $e(\mu)$ belongs to the closed convex hull of $\Omega_0(\mu)$, it follows that supp $e(\mu)$ is a subset of supp Ω_0. We also remark that supp $\Omega_0 =$ supp $\mathrm{co}[\Omega_0(\mu)] =$ supp $\overline{\mathrm{co}}[\Omega_0(\mu)]$. Let supp $e(\mu) = P$ and supp $\mu = Q$. Then by Lemma 1 in Pym [6] and Theorem 1 in Gaur [5], we have $\overline{PQ^n} = \overline{Q^n P} = P$, for all n. This shows that P is an ideal of supp Ω_0 which is also minimal by Remark 2 in Gaur [5].

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