A CONSTRUCTION OF LIMITS AND COLIMITS OF TOPOLOGICAL STRUCTURES

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Abstract: A construction of limits and colimits in the category of topological structures over a first order language is given. The construction readily specializes to the full subcategory of models of an equational theory.

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1. Introduction

Let $L$ be a first order language, which for simplicity will be assumed not to have relation symbols. Let $\text{Struct}_L$ be the category of structures in $L$. An object is a set $S$, with each $n$-ary function of $L$ interpreted as an $n$-ary function on $S$. A morphism is a homomorphism of structures (i.e., “preserves the functions”).

Let $\text{TopS}_L$ denote the category where:

1. $S$ is equipped with a topology,
2. the functions of the structure are continuous (the structure and the topology are “compatible”), and
3. morphisms are continuous as well as structure preserving.

It is well-known that a diagram in $\text{Struct}_L$ has a limit or colimit. We will show that this can be equipped with a topology, such that the resulting topological structure, together with the limit or colimit cone in $\text{Struct}_L$, comprise a limit or colimit in $\text{TopS}_L$.

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Let $T$ be a set of (implicitly universally quantified) equations in $L$. Let $\text{Mdl}_T$ denote the full subcategory of $\text{Struct}_L$ consisting of the models of $T$. Let $\text{TopM}_T$ denote the full subcategory of $\text{TopS}_L$. The construction of limits and colimits in $\text{Struct}_L$ can be adapted to $\text{Mdl}_T$. We will show that in a similar way the construction of limits and colimits in $\text{TopS}_L$ can be adapted to $\text{TopM}_T$.

In particular a simple description results of the amalgamated product in the category of topological groups, which is stated to be lacking in Khan and Morris [1].

2. Substructures

**Lemma 1.** Given $X$ in $\text{TopS}_L$ and a substructure $W \subseteq X$, the subspace topology is compatible with the substructure.

**Proof.** Suppose $V \subseteq W$ is open, and $n$ is an $n$-ary function; let $f_r$ denote the restriction to $W^n$. Then $w \in f_r^{-1}[V]$ iff $w \in W^n$ and $w \in f^{-1}[V]$. That is, $f_r^{-1}[V] = W^n \cap f^{-1}[V]$, so $f_r^{-1}[V]$ is open in $W^n$. □

3. Products

**Lemma 2.** Given objects $X_i$ in $\text{TopS}_L$, the product topology is compatible with the product structure. The product topology is that induced by the projections. The product structure equipped with the product topology is the product in $\text{TopS}_L$.

**Proof.** Let $U$ be the subbasic open set which is $U_j$ in component $j$ and $X_i$ for $i \neq j$; suppose $f$ is $n$-ary. Then $f((x_{1i}), \ldots, (x_{ni})) \in U$ iff $f(x_{1j}, \ldots, x_{nj}) \in U_j$. So $f^{-1}[U]$ is in fact the subbasic open set which is $f^{-1}[U_j]$ in component $j$ and $X_i^n$ for $i \neq j$, and in particular is open. The product topology is that induced by the projections since this is true of the topological spaces. Thus, given a cone from the $X_i$ to any $Y \in \text{TopS}_L$, the induced map from $Y$ to $\times_i X_i$ (the product in $\text{Struct}_L$) is continuous. □

4. Limits

**Theorem 3.** The limit of a diagram in $\text{TopS}_L$ is the limit in $\text{Struct}_L$, equipped with the topology induced by the limit cone.

**Proof.** The equalizer in $\text{Set}$ is the equalizer in $\text{TopS}_L$. □
5. Quotients

**Lemma 4.** Given $X$ in $\text{TopS}_L$ and a congruence relation $\equiv$ on $X$, the quotient topology is compatible with the quotient structure.

*Proof.* For $x \in X$ let $\bar{x}$ denote the image under the canonical epimorphism, and extend the notation to subsets of $X$, and to the functions of the structure. Suppose $f(\bar{x}_1, \ldots, \bar{x}_n) \in \bar{V}$ where $\bar{V}$ is open in $\bar{X}$. Then $f(x_1, \ldots, x_n) \in V$ (where $V = \bigcup \bar{V}$ is the saturated open set whose image is $\bar{V}$). Thus there are open sets $U_1, \ldots, U_n$ such that if $x_k \in U_k$ for $1 \leq k \leq n$ then $f(x_1, \ldots, x_n) \in V$. Since $V$ is saturated and $\equiv$ is a congruence relation, $U_k$ may be replaced by its saturation. Then if $\bar{x}_k \in \bar{U}_k$ for $1 \leq k \leq n$ then $f(\bar{x}_1, \ldots, \bar{x}_n) \in \bar{V}$. □

6. Coproducts

So far everything has been determined by forgetful functors; this is no longer the case for the coproduct. The underlying structure is the coproduct structure, but it must be given a topology specific to the new category.

Let $t$ denote a term over $L$. We suppose that the distinct variables from left to right are $x_1, \ldots, x_r$. Let $H$ denote the set of closed terms $\hat{t}$ of the form $t(c_1, \ldots, c_r)$, where $c_i \in X_i$. This is a structure in an obvious and well-known manner. We define the topology $T_H$ on $H$ as that whose subbasic open sets are $U_t, U_1, \ldots, U_r$, where $U_t$ is an open set of $X_t$. The closed terms in this open set are those, where $c_i \in U_i$. From hereon $H$ is supposed to be equipped with this topology.

**Lemma 5.** If $f$ is an $n$-ary function of $L$ then $f$ is a continuous function on $H$.

*Proof.* A closed term $f(\hat{t}_1, \ldots, \hat{t}_n)$ is in a subbasic open set iff the term $t$ has the closed term as an instance, and the open sets contain the constants. An (subbasic) open set for $\hat{t}_k$ is obtained by taking $t_k$ as the term, and the open sets for its constants as for $t$. □

**Lemma 6.** The map $X_i \mapsto H$ mapping $c$ to itself is continuous.

*Proof.* The inverse image of $U_{c,W}$ is $W$.

Let $\equiv$ be the equivalence image on $H$, where two closed terms are equal iff all their constants are from the same structure and they are equal in the structure. This is easily seen to be a congruence relation in the structure $H$. It is not difficult to show that $H/\equiv$ is the coproduct in $\text{Struct}_L$ (see Dowd [1] for example).

**Theorem 7.** $H/\equiv$, equipped with the quotient topology, is the coproduct in $\text{TopS}_L$. 
Proof. Given $K \in \text{Top}_S$ and $\text{Top}_S$ morphisms $\nu_i : X_i \mapsto K$, there is a unique $\text{Struct}_L$ morphism $\phi : H \mapsto K$ such that $\phi(c) = \nu_i(c)$ for each $c \in X_i$. Suppose $V \subseteq K$ is open and $\phi(\hat{t}) \in V$. Write $\hat{t}$ as $t(c_1, \ldots, c_r)$ where $c_l \in X_{i_l}$. Let $t_K$ be the function on $K$ interpreting $t$. Since $t_K$ is continuous there are open sets $U_l \subseteq X_{i_l}$ such that for $\langle c'_1, \ldots, c'_r \rangle \in U_1 \times \cdots \times U_r$, $t_K(\nu_{i_1}(c'_1), \ldots, \nu_{i_r}(c'_r)) \in V$. Thus, $\phi[U_1 \times \cdots \times U_r] \subseteq V$, and $\phi$ has been shown to be continuous. If $d = f(c_1, \ldots, c_n)$ in $X_i$ then $\phi(d) = \phi(f(c_1, \ldots, c_n))$, so $\phi$ respects $\equiv$. Thus, $\phi$ factors through the canonical epimorphism in $\text{Top}$, and the theorem follows.

7. Colimits

**Theorem 8.** The colimit of a diagram in $\text{Top}_S$ is obtained from the coproduct and the coequalizer in the usual way.

Proof. Immediate.

8. Equational Theories

Theorem 3 holds in $\text{Top}_M_T$ because substructures and products of models are models. Theorem 7 must be modified, in a standard manner, namely, the relation $\equiv$ is modified to include the equivalences arising from $T$.

References
