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STABLE COHERENT SYSTEMS (E, V) WITH $\mathrm{rank}(E) = \dim(V)$ ON PROJECTIVE VARIETIES

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Fix an integer $n \geq 2$, an integral projective variety X, a rank n torsion free sheaf E on X and an ample line bundle H on X. Here we prove the existence of an integer t_0 and a positive real number α_0 (depending only from X, E and H) such that for all integers $t \geq t_0$ there is an n-dimensional linear subspace $W \subseteq H^0(X, E(tH))$ such that the corent pair $(E(tH), W)$ is not α-stable (with respect to the polarization H) for all real numbers $\alpha \geq \alpha_0$.

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1. Introduction

For the general theory of coherent systems, see [2], [5] and [6]. Here we work over an algebraically closed field $\mathbb{K}$. In Section 2 we will prove the following result, which is an extension of [1], Theorem 1, to the case $k = n$.

Theorem 1. *Fix an integer $n \geq 2$, an integral projective variety X, a rank n vector bundle E on X and an ample line bundle H on X. There is an integer t_0 (depending only from n, X, H, E) such that for all integers $t \geq t_0$ a general n-dimensional linear subspace V of $H^0(X, E(tH))$ spans $E(tH)$ and the coherent system $(E(tH), V)$ is α-stable with respect to the polarization H for every $\alpha \gg 0$.*

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To state the aim of the last part of Introduction we introduce the following definition.

Definition 1. Let X be an integral projective variety and (E, V) a coherent system on X. A partial subsystem of (E, V) is a coherent system (E, W) on X with W a linear subspace of V.

Question 1. Let X be an integral projective variety and (E, V) a coherent system on X. Study the α-stability (with respect to a certain polarization H on X and for certain $\alpha \in \mathbb{R}$) of the partial subsystems (E, W) of (E, V) when W varies in the Grassmannian $Gr(x, V)$ of all x-dimensional linear subspaces of V.

To study Question 1 in one case we will use the following result, i.e. [1], Theorem 2.

Proposition 1. *Fix integers $k > n \geq 2$, an integral projective variety X, a rank n vector bundle E on X and an ample line bundle H on X. There is an integer t_0 (depending only from k, n, X, H, E) such that for all integers $t \geq t_0$ a general k-dimensional linear subspace V of $E(tH)$ spans $E(tH)$ and the natural map $u_{(E,V)} : \bigwedge^n(V) \to H^0(X, \det(E)(ntH))$ is injective.*

As an immediate corollary of Proposition 1 and of the proof of [1], Theorem 1, we get the following result.

Corollary 1. *Fix integers $k \geq n + 2 \geq 2$, an integral projective variety X, a rank n vector bundle E on X and an ample line bundle H on X. There is an integer t_0 and $\alpha_0 \in \mathbb{R}$ (depending only from n, X, H, E) such that for all integers $t \geq t_0$ a general k dimensional linear subspace $V \subseteq H^0(X, E(tH))$ has the following property: for all integers x such that $n < x \leq k$, all real numbers $\alpha \geq \alpha_0$ and all x-dimensional linear subspace W of V the coherent system $(E(tH), W)$ is α-stable with respect to the polarization H.*

2. Proof of Theorem 1

Proof of Theorem 1. We divide the proof into two parts, the first one being subdivided several times.

(i) Here we assume $\dim(X) = 1$. There is $\beta \in \mathbb{R}$, $\beta > 0$ such that $\mu(A) \leq \mu(E) + \beta$ for all non-zero subsheaves of H; here we compute the slope μ with respect to the polarization H. For this choice of the polarization we have $\mu(B) \leq \mu(E(zH)) + \beta$ for every $z \in Z$ and every non-zero subsheaf B of $E(zH)$. Let m be any integer such that mH is very ample. Since $\deg(H) > 0$, it is easy to see that we may take $m \geq 2p_a(X)+1$, and hence m depends only from X, not from the choice of H. Since H is ample, there is an integer t_1 (depending only from $p_a(X)$ and the constant β just introduced) such that for all integers $t \geq t_1$ the sheaf $E(tH)$ is spanned by its global sections. Fix any integer $t \geq t_1$. Since $E(tH)$ is spanned, the evaluation map $i_{t,W} : \mathcal{O}_X \otimes W \to E(tH)$ is an injective map of sheaves when W is a general n-dimensional linear subspace of $H^0(X, E(tH))$. Since $\dim(X) = 1$, we obtain an exact sequence

$$0 \to \mathcal{O}_X \otimes W \to E(tH) \to G \to 0\,, \tag{1}$$

in which G is a skyscraper sheaf. Since $\dim(X) = 1$ and $\dim(\mathrm{Sing}(X)) = 0$, we may also assume that $i_{t,W}$ is surjective at each point of $\mathrm{Sing}(X)$ at which E is locally free and that $i_{t,W}$ has rank at least $n-1$ at each point of X_{reg}, i.e. for every $P \in X_{reg} \cap \mathrm{Supp}(G)$ there is a one dimensional linear subspace J_P of W such that J_P is the kernel of the composition of the map $\mathcal{O}_X \otimes W \to E(tH)$ with the restriction map $H^0(X, E(tH)) \to H^0(\{P\}, E(tH)|\{P\})$. Hence we obtain a finite family $\{J_P\}_{P \in X_{reg} \cap \mathrm{Supp}(G)}$ of one-dimensional linear subspaces. If $t \geq t_1 + 2m$, we may even assume sthat $H^0(X, E(tH))$ spans the jet sheaf of E at each point $P \in X_{reg}$. Hence for general W we may also assume $h^0(X, A) = 1$ for every non-zero subsheaf A of G supported by a smooth point of X. Let G_s the part of G supported by the singular points of X. Set $\gamma(t) := h^0(X, G_s)$ for a general n-dimensional linear subspace of $H^0(X, E(tH))$. Wjusr saw that $\gamma(t') = \gamma(t)$ for all $t' \geq t \geq t_1 + 2m$ and that the non-negative integer $\gamma(t)$ is a measure of the non-local freeness of the torsion free sheaf E. Set $\gamma := \gamma(t)$ for any $t \geq t_1 + 2m$. Varying W among the sufficiently general n-dimensional linear subspaces of $H^0(X, E(tH))$ we obtain a group $J(t)$ of permutations of the set $\Sigma := X_{reg} \cap \mathrm{Supp}(G)$. There is an integer $t_2 \geq t_1 + 2m$ (depending only from $p_a(X)$ and E) such that $h^0(X, E(tH)) \geq 6n^2$ and $\deg(E(tH)) \geq 6 + \gamma$ for every $t \geq t_2$. For every $Q \in \mathrm{Sing}(X)$ let $\Delta_Q(E)$ denote the minimal length of the cokernel of an injective map ${\mathcal{O}_{X,Q}}^{\oplus n} \to E_Q$ of $\mathcal{O}_{X,Q}$-modules, where E_Q denotes the germ of E at Q ([3], Definition 2.2.3; for a related invariant, see [4]). Hence $\Delta_Q(E) = 0$ if and only if E is locally free at Q and $\Delta_Q(E(tH)) = \Delta_Q(E)$ for all $t \in \mathbb{Z}$. Set $\Delta(E) := \sum_{Q \in \mathrm{Sing}(X)} \Delta_Q(E)$. Hence $\Delta(E) = \Delta(E(tH))$ for all $t \in \mathbb{Z}$.

First Claim. *For $t \geq t_2 + 12m$ the permutation group $J(t)$ is either the full symmetric group of Σ or the alternating group of Σ.*

Proof of the First Claim. By the classification of 6-transitive finite permutation groups ([7], Theorem 2.4), it is sufficient to show that $J(t)$ is 6-transitive. For any $P \in X_{reg}$ set $I(t,P) := \{f \in H^0(X, E(tH)) : f|\{P\} = 0\}$. Since

$E(tH))$ is spanned, $I(t,P)$ is a codimension n linear subspace of $H^0(X,E(tH))$. We fix an integer b such that $0 \le b \le b+1$ and assume that Σ is b-transitive. To prove First Claim in six steps it is sufficient to prove that $J(t)$ is $(b+1)$-transitive. Fix $b+1$ general points $P_1,\dots,P_b,Q$ of X_{reg}. Since mH is very ample and $E((t-6m)H)$ is spanned, we see that for general W the $(b+1)$ codimensiona n linear spaces $I(t,P_1)\cap W,\dots,I(t,P_b)\cap W, I(t,Q)\cap W$ are general in W. Move around Q in X_{reg} and use that X is irreducible. □

Second Claim. *Use all the notation introduced in the proof of First Claim and fix a general n-dimensional linear subspace W of $H^0(X,E(tH))$ and take the associated $\Sigma := X_{reg}\cap \mathrm{Supp}(G)$. Then for any subset S of Σ the linear span of $\cup_{P\in S}(I(t,P)\cap W)$ has dimension $\min\{n,\sharp(S)\}$.*

Proof of the Second Claim. The last part of the proof of First Claim proved Second Claim when $\sharp(S)\le 6$. To get the general case it is sufficient to prove Second Claim for all S such that $\sharp(S)\le n$. To get the general case use that $J(t)$ is at least $(\sharp(\Sigma)-2)$-transitive. □

Third Claim. *Fix any integer $t\ge t_2+12m$ and any general n-dimensional linear subspace W of $H^0(X,E(tH))$. Then for any integer x such that $1\le x<n$ and any rank x subsheaf F of $E(tH)$ we have $\dim(W\cap H^0(X,F))\le x$.*

Proof of Third Claim. Just use the injectivity of the map of sheaves $\mathcal{O}_X\otimes W\to E(tH)$. □

Fourth Claim. *Fix any integer $t\ge t_2+12m$ and any general n-dimensional linear subspace W of $H^0(X,E(tH))$. Then for any integer x such that $1\le x<n$ and any x-dimensional linear subspace M of W, the saturation in $E(tH)$ of the image $A(M)$ of the evaluation map $u:\mathcal{O}_X\otimes M\to E(tH)$ has saturation with degree at most $x+\Delta(E)$.*

Proof of Fourth Claim. By Second Claim there are at most x points $P\in X_{reg}\cap\mathrm{Supp}(G)$ such that $J(t,P)\cap W\subseteq M$. For any $Q\in\mathrm{Sing}(X)$ the map u has local degree at most $\Delta_Q(E)$. □

Fix any integer $t\ge t_2+12m$, any integer x such that $1\le x<n$ and any general n-dimensional linear subspace W of $H^0(X,E(tH))$. Let (F,M) be a coherent subsystem of $(E(tH),W)$ such that $\mathrm{rank}(F)=x$. By Fourth Claim either $\dim(M)\le x-1$ or $\deg(F)\le x+\gamma$. First, assume $\dim(M)\le x-1$. Since $\mu(F)\le\mu(E(tH))+\beta$, we have $\mu_\alpha(F,M)<\mu_\alpha(E(tH))$ if $\alpha>(x-1)\beta/x$. Notice that $(x-1)\beta/x$ depends only from E, not from t. Now assume $\deg(F)\le x+\gamma+\Delta(E)$. Since $\mu(E(tH))=\mu(E)+t\deg(H)$, there is an integer $t_0\ge t_2+12m$, such that $\mu(E(tH))>1+\gamma+\Delta(E)$. By Third Claim we have $\mu_\alpha(F,M)<\mu_\alpha(E(tH))$ for every $\alpha\ge 0$, concluding the proof when X is a curve. □

(ii) Here we assume $\dim(X) > 1$. By part (i) we may assume that the result is true for all varieties X' such that $1 \leq \dim(X') < \dim(X)$. Since H is ample, there is an integer m such that mH is very ample. Hence X is covered by a family of hypersurfaces $\{Y_c\}$ with $Y_c \in |mH|$. Furthermore, for $t \gg 0$ we may also assume $h^1(X, E(t-m)) = 0$. Hence the restriction map $H^0(X, E(tH)) \to H^0(Y_c, (E|Y_c)(t(H|Y_c)))$ is surjective for all Y_c. Take any subcoherent system (F, M) of $(E(tH), W)$ and apply the inductive assumption to its restriction to Y_c with respect to the polarization $H|Y_c$. $\square$

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