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A COMMON FIXED POINT THEOREM AND ITS APPLICATIONS

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Abstract: In this article, a common fixed point theorem for a sequence of mappings in the fuzzy metric space is proved. This result offers an extension of Vasuki's Theorem. Moreover, Grabiec's Theorem is obtained.

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1. Introduction

Grabiec [2] proved fuzzy Banach contraction theorem and Vasuki [8] generalized Grabiec's result as a common fixed point theorem for a sequence of mappings in the fuzzy metric space. We present a new fixed point theorem and some results in the fuzzy metric space. This theorem implies an extension of Vasuki's Theorem. In order to do this, we recall some definitions from [2], [3], [4], [5] and [8].

Definition 1.1. A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a *triangular norm* (abbreviated, t-norm) if the following conditions are satisfied:

(I) $T(a,1) = a$ for every $a \in [0,1]$,

(II) $T(a,b) = T(b,a)$ for every $a, b \in [0,1]$,

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(III) $a \geq b,\ c \geq d \to T(a,c) \geq T(b,d)\ (a,b,c,d \in [0,1])$,

(IV) $T(a,T(b,c)) = T(T(a,b),c)\ (a,b,c \in [0,1])$.

Example 1.2. The *minimum* t-norm, T_M, is defined by $T_M(x,y) = \min(x,y)$, the *product* t-norm, T_P, is defined by $T_P(x,y) = x.y$, the *Lukasiewicz* t-norm, T_L, is defined by $T_L(x,y) = \max(x+y-1,0)$ and finally, the weakest t-norm, the *drastic product*, T_D, is defined by

$$T_D(x,y) = \begin{cases} \min(x,y), & \text{if} \quad \max(\text{x,y}) = 1, \\ 0, & \quad\quad \text{otherwise.} \end{cases}$$

Definition 1.3. The 3-tuple $(X,M,*)$ is said to be a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous $t-$norm and M is a fuzzy set on $X^2 \times [0,\infty[$ satisfying the following conditions: for all $x,y,z \in X$ and $t,s > 0$,

(I) $M(x,y,0) = 0$,

(II) $M(x,y,t) = 1$ for all $t > 0$, iff $x = y$,

(III) $M(x,y,t) = M(y,x,t)$,

(IV) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,

(V) $M(x,y,.) :\ [0,\infty[\to [0,1]$ is left-continuous.

Lemma 1.4. (see [2]) *$M(x,y,.)$ is nondecreasing for all $x,y \in X$.*

Definition 1.5. A sequence (x_n) in a fuzzy metric space $(X,M,*)$ is *G-Cauchy*, if $\lim_{n\to\infty} M(x_{n+p},x_n,t) \to 1$, for each $t > 0$ and $p \in N$. A sequence $\{x_n\}$ in a fuzzy metric space $(X,M,*)$ converges to x iff $\lim_{n\to\infty} M(x_n,x,t) = 1$ for all $t > 0$.

A fuzzy metric space in which every G-Cauchy sequence is convergent is called a G-complete fuzzy metric space.

Definition 1.6. A mapping $F : R \to R^+$ is called a *distribution function* if it is nondecreasing and left-continuous and it has the following properties:

(I) $\inf_{t\in R} F(t) = 0$,

(II) $\sup_{t\in R} F(t) = 1$.

Let D^+ be the set of all distribution functions F such that $F(0) = 0$ (F is a nondecreasing, left-continuous mapping from R into $[0,1]$ such that $\sup_{x\in R} F(x) = 1$).

Definition 1.7. A *probabilistic metric space* (briefly, PM-space) is an ordered pair (S,F), where S is a nonempty set and $F : S \times S \to D^+ \Big(F(p,q)$ is denoted by $F_{p,q}$ for every $(p,q) \in S \times S \Big)$ satisfies the following conditions:

1. $F_{u,v}(x) = 1$ for all $x > 0$ if and only if $u = v\ (u,v \in S)$.

2. $F_{u,v}(x) = F_{v,u}(x)$ for all $u, v \in S$ and $x \in R$.

3. If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$ for $u, v, w \in S$ and $x, y \in R^+$.

Definition 1.8. A *Menger probabilistic metric space* (briefly, *Menger PM-space*) (see [5]) is a triple (S, F, T), where (S, F) is a PM-space, T is a t-norm and the following inequality holds:

$$F_{u,v}(x + y) \geq T(F_{u,w}(x), F_{w,v}(y)),$$

for all $u, v, w \in S$ and every $x > 0,\ y > 0$. A Menger PM-space (S, F, T) with $F(S \times S) \subset D^+$ is called a *Menger space*.

2. A Common Fixed Point Theorem

Due to prove the common fixed point theorem, we need a lemma form [1] as follows:

Lemma 2.1. *Let the function $\phi(t) : [0, \infty) \to [0, \infty)$ satisfying the following condition: (ϕ_1) $\phi(t)$ is strictly increasing, $\phi(0) = 0$ and $\lim_{n\to\infty} \phi^n(t) = \infty$ for all $t > 0$, where $\phi^n(t)$ denotes the n-th iterative function of $\phi(t)$. Then $\phi(t) > t$ and $\phi^n(t) > \phi^{n-1}(t)$, for all $t > 0, n = 1, 2, \cdots$.*

Now, we prove a common fixed point theorem.

Theorem 2.2. *Let $\{T_n\}_n$ be a sequence of mappings of a G-complete fuzzy metric space $(X, M, *)$ into itself, where $*$ is a continuous t-norm and there exists a mapping $m : X \to N$ such that for any two mappings T_i and T_j and any $x, y \in X$*

$$M(T_i^{m(x)}x, T_j^{m(y)}y, t) \geq M(x, y, \phi_{i,j}(t)) \tag{2.1}$$

and

$$M(T_i^{m(x)}x, T_i^{m(x)}y, t) \geq M(x, y, \phi_{i,i}(t)), \tag{2.2}$$

where $\phi_{i,j}(t) : [0, \infty) \to [0, \infty)$ is a function such that $\phi_{i,j}(t) > \phi(t)$, for all $t > 0,\ i, j = 1, 2, \cdots$ and the function $\phi(t) : [0, \infty) \to [0, \infty)$ satisfies the condition (ϕ_1). In addition, suppose $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. Then the sequence $\{T_n\}_n$ has a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X and define

$$x_1 = T_1^{m(x_0)}x_0, x_2 = T_2^{m(x_1)}x_1, \cdots, x_n = T_n^{m(x_{n-1})}x_{n-1}, \cdots .$$

For convenience, put $m_i = m(x_i), i = 0, 1, 2, \cdots$. Then for all $t > 0$, by Lemma 1.4 and relation (2.1) we have

$$M(x_1, x_2, t) = M(T_1^{m_0} x_0, T_2^{m_1} x_1, t) \geq M(x_0, x_1, \phi_{1,2}(t)) \geq M(x_0, x_1, \phi(t))$$

and

$$M(x_2, x_3, t) = M(T_2^{m_1} x_1, T_3^{m_2} x_2, t) \geq M(x_1, x_2, \phi_{2,3}(t)) \geq M(x_0, x_1, \phi(\phi_{2,3}(t))) \geq M(x_0, x_1, \phi^2(t)),$$

and so on. By induction we have

$$M(x_n, x_{n+1}, t) \geq M(x_0, x_1, \phi^n(t)).$$

Let $p \in N$ be fixed, then

$$\begin{aligned} M(x_n, x_{n+p}, t) &\geq M(x_n, x_{n+1}, \tfrac{t}{p}) * \cdots * M(x_{n+p-1}, x_{n+p}, \tfrac{t}{p}) \\ &\geq M(x_0, x_1, \phi^n(\tfrac{t}{p})) * \cdots * M(x_0, x_1, \phi^{n+p-1}(\tfrac{t}{p})) \\ &\geq M(x_0, x_1, \phi^n(\tfrac{t}{p})) * \cdots * M(x_0, x_1, \phi^n(\tfrac{t}{p})) \\ &\to 1 * \cdots * 1 = 1 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ is a G-Cauchy sequence in X and hence it converges to some x in X. Now we prove that x is a common fixed point of $\{T_n\}_n$. For $t > 0$ we have

$$\begin{aligned} M(x, T_i^{m(x)} x, t) &\geq M(x, x_n, \tfrac{t}{2}) * M(T_n^{m_{n-1}} x_{n-1}, T_i^{m(x)} x, \tfrac{t}{2}) \\ &\geq M(x, x_n, \tfrac{t}{2}) * M(x_{n-1}, x, \phi_{n,i}(\tfrac{t}{2})) \\ &\geq M(x, x_n, \tfrac{t}{2}) * M(x_{n-1}, x, \phi(\tfrac{t}{2})) \\ &\to 1 * 1 = 1 \text{ as } n \to \infty. \end{aligned}$$

Therefore $x = T_i^{m(x)} x$. Now we prove x is a unique common fixed point of the sequence $\{T_i^{m(x)}\}$. Due to this, suppose $y \in X$ be a fixed point of $T_i^{m(x)}$, then for $t > 0$ from (2.2) we have

$$M(x, y, t) = M(T_i^{m(x)} x, T_i^{m(x)} y, t) \geq M(x, y, \phi_{i,i}(t)) \geq M(x, y, \phi(t)).$$

On the other hand, Lemma 1.4 and Lemma 2.1 imply that

$$M(x, y, t) \leq M(x, y, \phi(t)).$$

Thus the two last inequalities implies that $M(x, y, t) = M(x, y, \phi(t))$ for all $t > 0$. Therefore, by induction

$$M(x, y, t) = M(x, y, \phi(t)) = \cdots = M(x, y, \phi^n(t)), \qquad \text{for all } t > 0, n = 1, 2, \cdots .$$

Then

$$M(x, y, t) = \lim_{n\to\infty} M(x, y, \phi^n(t)) = 1, \text{ for all } t > 0.$$

Hence $x = y$, i.e. x is a unique common fixed point of $\{T_i^{m(x)}\}$. Also,

$$T_i x = T_i T_i^{m(x)} x = T_i^{m(x)} T_i x,$$

and this means that, $T_i x$ is also a common fixed point of $\{T_i^{m(x)}\}$. Therefore, $x = T_i x$. Hence x is a common fixed point of $\{T_i\}_i$. If $y \in X$ be a common fixed point of $\{T_i\}_i$, then $T_i^{m(x)} y = y$. So $y = x$. Therefore x is a unique common fixed point of the sequence $\{T_i\}_i$. □

Now, as an application of the above theorem we prove the following Vauski's Theorem [8].

Corollary 2.3. *Let $\{T_n\}_n$ be a sequence of a G-complete fuzzy metric space $(X, M, *)$ into itself, where $*$ is a continuous t-norm such that for any two mapping T_i and T_j we have $M(T_i^m x, T_j^m y, \alpha_{i,j} t) \geq M(x, y, t)$ for some m and $0 < \alpha_{i,j} < k < 1$, $i, j = 1, 2, \cdots$, $x, y \in X$. In addition, suppose $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. Then the sequence $\{T_n\}_n$ has a unique common fixed point.*

Proof. In Theorem 2.2 take, $\phi_{i,j}(t) = \frac{t}{\alpha_{i,j}}$, $i, j = 1, 2, \cdots$ and $\phi(t) = \frac{t}{k}$ and $m(x) = m$ for all $x \in X$. □

Corollary 2.4. *Let T be a mapping of a G-complete fuzzy metric space $(X, M, *)$ into itself, where $*$ is a continuous t-norm and there exists a mapping $m : X \to N$ such that for all $x, y \in X$*

$$M(T^{m(x)}x, T^{m(x)}y, t) \geq M(T^{m(x)}x, T^{m(y)}y, t) \geq M(x, y, \phi(t)),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a function satisfies the condition (ϕ_1). In addition, suppose $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. Then T has a unique fixed point (call x_0) and the iterative sequence $\{T^n x\}$ is converging to x_0 for all $x \in X$.

Proof. By applying Theorem 2.2, T has a unique fixed point x_0. For $n > m(x_0)$, we have $n = km(x_0) + s$, for some $0 \leq s < m(x_0)$. Thus

$$\begin{aligned} M(x_0, T^n x, t) &= M(T^{m(x_0)}x_0, T^{km(x_0)+s}x, t) \\ &\geq M(x_0, T^{(k-1)m(x_0)+s}x, \phi(t)) \geq \cdots \geq M(x_0, T^s x, \phi^k(t)) \to 1 \text{ as } n \to \infty. \end{aligned}$$

Hence $\lim_{n\to\infty} T^n x = x_0$ for all $x \in X$. □

Remark 2.5. Let (X, F, T) be a Menger space, taking $\phi(t) = \frac{t}{k}$, $0 < k < 1$, $m(x) = 1$ for all $x \in X$ and suppose either T be Minimum t-norm, or $T(a, a) \geq a$ for all $a \in [0, 1]$. Then the main results of [6] are special cases of Corollary 2.4.

Corollary 2.6. *Corollary 2.4 implies the Grabiec's Theorem.*
Proof. Taking $\phi(t) = \frac{t}{k}$, $0 < k < 1$ and $m(x) = 1$ for all $x \in X$. □

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