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LOCALLY ALGEBRAIC REDUCED SCHEMES AND VECTOR BUNDLES: UNIQUE FACTORIZATION THEOREMS

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Abstract: Let X be a connected and reduced locally algebraic scheme such that all its irreducible components are projective and E a locally free sheaf on X with finite rank. Here we prove that E has a unique (up to permutations and isomorphisms of the factor) decomposition into irreducible locally free indecomposabole factors. Furthermore, the following conditions are equivalent:

(1) E is indecomposable;

(2) there is a connected positive-dimensional reduced closed subscheme Y such that $E|Y$ is indecomposable;

(3) $X = \cup_{\beta \in J} X_\beta$ with each X_β union of finitely many irreducible components of X and $E|X_\beta$ indecomposable for all $\beta \in J$.

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1. Locally Algebraic Reduced Schemes and Vector Bundles

Let $\mathbb{K}$ be an algebraically closed field. In this paper all locally algebraic schemes will be defined over $\mathbb{K}$. Here we prove the following results.

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Theorem 1. *Let X be a connected and reduced locally algebraic scheme such that all its irreducible components are projective and E a locally free sheaf on X with finite rank. The following conditions are equivalent:*

(i) E is indecomposable;

(ii) there is a connected positive-dimensional reduced closed subscheme Y such that $E|Y$ is indecomposable;

(iii) $X = \cup_{\beta \in J} X_\beta$ with each X_β connected union of finitely many irreducible components of X and $E|X_\beta$ indecomposable for all $\beta \in J$.

Theorem 2. *Let X be a connected and reduced locally algebraic scheme such that all its irreducible components are projective and E a locally free sheaf on X with finite rank. Then E has a unique (up to permutations and isomorphisms of the factor) decomposition into irreducible locally free indecomposabole factors.*

The following Cancellation Theorem is an immediate consequence of Theorem 2

Corollary 1. *Let X be a connected and reduced locally algebraic scheme such that all its irreducible components are projective and E, F and G locally free sheaves on X with finite rank. If $E \oplus F \cong E \oplus G$, then $F \cong G$.*

The following result is an immediate consequence of Theorem 1 and of a paper in preparation.

Theorem 3. *Let X be a connected and reduced locally algebraic scheme such that all its irreducible components are projective varieties. Every vector bundle on X is isomorphic to a direct sum of line bundles if and only if the following conditions are satisfied:*

(a) Every irreducible component of X is isomorphic to $\mathbf{P}^1$;

(b) X has only nodal singularities;

(c) every irreducible component of X contains at most two singular points of X.

Proof of Theorem 1. Since (ii) trivially implies (iii) and (iii) trivially implies (ii), it is sufficient to show that (i) implies (ii). Assume that (ii) is not true. Any non-trivial factor of a vector bundle on a locally algebraic scheme is locally free by Nakayama's Lemma. E is decomposable if and only if there is $f \in H^0(X, \mathrm{Hom}(E, E))$ such that $f \circ f = f$ (i.e. such that $f \circ (Id_E - f) = 0$) with $f \neq 0$ and $f \neq Id$: for any such endomorphism f we have $E \cong \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$. The same statement is true for $E|T$, where T is any union of some of the irreducible components of X. Lt I be the set of all irreducible components of X. Since the theorem is trivial if X is algebraic, we may assume that I is infinite. First, we assume that I is countable. Hence $X = \cup_{n\geq 1}X_n$ with X_n connected, X_n union of n irreducible components of X and $X_n \subset X_{n+1}$ for all n. Let $r_n \geq 1$ the minimal rank of factor of $E|X_n$. Let a_n be the rank of the maximal direct factor, A_n of $E|X_n$ which is a direct product of rank r_n direct factors. Hence a_n/r_n is a positive integer and $a_n \leq \mathrm{rank}(E)$. By assumption we have $1 \leq r_n < \mathrm{rank}(E)$. It is easy to check that $r_n \geq r_{n+1}$ for all n. Set $r := \min\{r_n\}_{n\geq 1}$. Hence $1 \leq r < \mathrm{rank}(E)$ and $r = r_n$ for all $n \gg 0$, say for all $n \geq n_0$. For all $n \geq n_0$ we have $a_n \geq a_{n+1}$ and hence there is an integer $n_1 \geq n_0$ such that $a_n = a_{n+1}$ for all integers $n \geq n_1$. Set $a := a_{n_1}$. First assume $a < \mathrm{rank}(E)$. We will show that E has a rank a factor; the same proof would show that this factor is a direct sum of a/r rank r indecomposable factors. Fix an integer $n \geq n_1$. Since $a_{n+1} = a_n$, it is easy to check that $A_{n+1}|X_n = A_n$ (equality as subsheaves of $E|X_n$, not only isomorphism as abstract locally free sheaves. Hence $A := \bigcup_{n\geq n_1} A_n$ is a well-defined subsheaf of E. It is easy to see using the projectors associated to all factors A_n of $E|_n$ that A is a rank a direct factor of E. Now assume $a = \mathrm{rank}(E)$. The same proof gives that E is a direct factor of a/r indecomposable rank r factors. Now assume that I is uncountable. Let U be the set of all $S \subseteq I$ such that $\bigcup_{i\in S} X_i$ is connected and $W := \{S \in U :$ Theorem 1 is true for the locally algebraic set $\bigcup_{i\in S} X_i\}$. The union of a chain of elements in W belongs to W. Hence by Zorn's Lemma U has a maximal element S with respect to the partial ordering induced by the inclusion of sets. Assume $S \neq I$. There is $j \in I\backslash S$ suc that $S_j \cup \bigcup_{i\in S} X_i$ is connected. The proof of the countable case shows that $\{i\} \cup S \in U$, contradiction. □

Proof of Theorem 2. Since the theorem is trivial if X is algebraic, we may assume that X has infinitely many irreducible components. First, we assume that X has countably many irreducible components. Hence $X = \cup_{n\geq 1}X_n$ with X_n connected, X_n union of n irreducible components of X and $X_n \subset X_{n+1}$ for all n. The proof of Theorem 1 shows that a Krull-Schmidt decomposition of $E|X_n$, $n \gg 0$, into direct factors without isomorphic common factors induces a decomposition of E and that each of these factors of E is (as for a corresponding factor of $E|X_n$ a direct sum of indecomposable and isomorphic factors. Even when X has uncountably many irreducible components the last part of the proof of Theorem 1 proves Theorem 2. □

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