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ON INDICATOR OF NORMALS IN MINKOWSKI 3-SPACE

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Abstract: In this work, the curvature and torsion are studied of the indicators of Frenet frames in the Minkowski 3-space.

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1. Introduction

In this section, we collect some fundamental definitions and theorems.

Definition 1.1.. $\mathbb{R}^n$ with the metric tensor

$$\langle v_p, w_p \rangle = -\sum_{i=1}^{\nu} v_i w_i + \sum_{j=1}^{n} v_j w_j \tag{1}$$

is called semi-Euclidean space and is denoted by $\mathbb{R}^n_\nu$, where ν is called the index of the metric, see [2].

Definition 1.2. (see [2]) A tangent vector v to M is

$$\begin{aligned} &\text{if } \langle v, v \rangle > 0 \text{ or } v = 0, \text{ spacelike,} \\ &\text{if } \langle v, v \rangle < 0 \text{ or } v \neq 0, \text{ timelike.} \end{aligned}$$

Definition 1.3. A curve α in M semi-Riemannian manifold is spacelike if all of its velocity vectors $\alpha'(s)$ are spacelike similarly for timelike, see [2].

Definition 1.4. (see [3]) Let L be a 3-dimensional Lorentzian space. If (x_1, x_2, x_3) and (y_1, y_2, y_3) are the components of X and Y with respect to an

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allowable coordinate system, Lorentz exterior product $X \times Y$ is given by

$$X \times Y = (-x_2y_3 + x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1)\,.$$

Definition 1.5. Let α be spacelike curve. Lorentz exterior product for t, n are unit spacelike vectors and b is timelike vector as follows:

$$t \times n = b, \quad n \times b = -t, \quad t \times b = n. \tag{2}$$

Similarly, Lorentz exterior product for t, b are unit spacelike vectors and n is timelike vectors

$$t \times n = -b, \quad n \times b = -t, \quad b \times t = n \tag{3}$$

and for n, b are unit spacelike vectors and t is timelike vector

$$t \times n = -b, \quad n \times b = t, \quad t \times n = -b. \tag{4}$$

Theorem 1.1. *Let be $\beta : I \longrightarrow \mathbb{R}^3_1$ be a unit speed curve, κ is the curvature function τ is torsion of β. The Frenet derivative formulas of orthonormal vector field system $\{t, n, b\}$ is given by*

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\kappa & 0 \\ -\varepsilon_1\kappa & 0 & \varepsilon_3\tau \\ 0 & -\varepsilon_2\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{5}$$

where $< t, t >= \varepsilon_1$, $< n, n >= \varepsilon_2$, $< b, b >= \varepsilon_3$.

Definition 1.6. A curve α in M semi-Riemannian manifold is spacelike if all of its velocity vectors $\alpha'(s)$ are spacelike similarly for timelike, see [2].

Definition 1.7. Let $\alpha : I \longrightarrow \mathbb{R}^3_1$ be a unit speed curve. Then $\sigma(s) = T(s)$ is called tangent indiator of $\alpha(s)$.

2. The Indicator of Essential Normals in $\mathbb{R}^3_1$

Let $\alpha : I \longrightarrow \mathbb{R}^3_1$ be a unit speed curve. We take the curve $\eta(s) = N(s)$ as the indictor of essential normals of $\alpha(s)$.

Theorem 2.1. *Let $\alpha(s)$ be the unit speed, $\kappa > 0$ be a constant, and $\zeta(s)$ be an indicator of essential normals of $\alpha(s)$ in $\mathbb{R}^3_1$. In this circumstance, $\zeta(s)$ is plains and its first curvature $\kappa^* = 1$.*

Proof. Assume that κ is the first curvature and τ is second curvature of $\alpha : I \longrightarrow \mathbb{R}_1^3$ and κ^* is a first curvature and τ^* is a second curvature of ζ. Thus

$$n(s) = \frac{\varepsilon_2}{\kappa(s)} t'(s) = \zeta(s)\,, \tag{6}$$

where $T = \alpha'$.

The differentiation with respect to s of (6) gives

$$n'(s) = \zeta'(s). \tag{7}$$

From equation (5) and equation (7),

$$\zeta' = -\varepsilon_1 \kappa t + \varepsilon_3 \tau b. \tag{8}$$

Therefore

$$\zeta'' = -\varepsilon_2(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)n \tag{9}$$

and the differentiation of (9) gives

$$\zeta''' = \varepsilon_1\varepsilon_2\kappa(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)t - \varepsilon_2\varepsilon_3\tau(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)b. \tag{10}$$

From Lorentz exterior product of ζ' and ζ'',

$$\zeta' \times \zeta'' = \varepsilon_1\varepsilon_2\kappa(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)t \times n - \varepsilon_2\varepsilon_3\tau(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)b \times n. \tag{11}$$

and

$$\begin{aligned} \left|\left\langle \zeta' \times \zeta'', \zeta' \times \zeta'' \right\rangle\right|^{1/2} &= \left| -\left\langle \zeta', \zeta' \right\rangle \left\langle \zeta'', \zeta'' \right\rangle + \left\langle \zeta', \zeta'' \right\rangle^2 \right|^{1/2} \\ &= \left|\varepsilon_1\kappa^2 + \varepsilon_3\tau^2\right|^{3/2}. \end{aligned} \tag{12}$$

Thus

$$\kappa^* = 1\,, \tag{13}$$

since

$$\kappa = \frac{|\langle \alpha' \times \alpha'', \alpha' \times \alpha'' \rangle|^{1/2}}{|\langle \alpha', \alpha' \rangle|^{3/2}}.$$

We have

$$\begin{aligned} \left\langle \zeta' \times \zeta'', \zeta''' \right\rangle &= -\varepsilon_1\varepsilon_3\kappa\tau(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)^2 \langle t \times n, b \rangle \\ &\quad -\varepsilon_1\varepsilon_3\kappa\tau(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)^2 \langle b \times n, t \rangle\,. \end{aligned} \tag{14}$$

From equation (2), equation (3) and equation (4)

$$\left\langle \zeta' \times \zeta'', \zeta''' \right\rangle = 0 \tag{15}$$

and from (12) and (15) we have

$$\tau^* = \frac{\langle \zeta' \times \zeta'', \zeta''' \rangle}{\left|\langle \zeta' \times \zeta'', \zeta' \times \zeta'' \rangle\right|^{1/2}}. \qquad \square$$

3. The Frenet Frame of Indicator of Essential Normals in $\mathbb{R}_1^3$

One can obtain t^*, n^*, b^*, and new frame of the curve $\zeta(s)$ using Theorem 2.1 as follows:

$$t^* = \frac{\zeta'}{\left|\langle \zeta', \zeta' \rangle\right|^{1/2}},$$

and

$$t^* = \frac{-\varepsilon_1 \kappa t + \varepsilon_3 \tau b}{\left|\varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2\right|^{1/2}}. \tag{16}$$

If

$$b^* = \frac{\zeta' \times \zeta''}{\left|\langle \zeta' \times \zeta'', \zeta' \times \zeta'' \rangle\right|^{1/2}},$$

then

$$b^* = \frac{\varepsilon_1 \varepsilon_2 \kappa (\varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2) t \times n - \varepsilon_2 \varepsilon_3 \tau (\varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2) b \times n}{\left|\varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2\right|^{3/2}}. \tag{17}$$

If t is timelike, n and b are spacelike, from equation (4) follows that

$$n^* = t^* \times b^* = \frac{(-\kappa^2 + \tau^2)(\kappa^2 - \tau^2)}{\left|-\kappa^2 + \tau^2\right|^2} n\,.$$

If $\tau^2 < \kappa^2$ and $\tau^2 > \kappa^2$, respectively

$$\left|-\kappa^2 + \tau^2\right| = \kappa^2 - \tau^2$$

and

$$\left|-\kappa^2 + \tau^2\right| = -\kappa^2 + \tau^2\,,$$

$$n^* = -n\,.$$

If b is timelike, t and n are spacelike, from equation (2),

$$n^* = -n\,.$$

If n is timelike, t and b are spacelike, from equation (3)

$$n^* = n\,.$$

Corollary 2. *Let n be normal of unit speed curve $\alpha(s)$ and n^* also be normal of unit speed curve $\zeta(s)$ and n^* be normal of indicator of essential normals. There is a relation between n and n^* vectors as follows:*

$$n^* = \pm n. \tag{18}$$

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