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INTUITIONISTIC H-FUZZY REFLEXIVE RELATIONS

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Abstract: We introduce the subcategory $\mathbf{IRel_R}(H)$ of $\mathbf{IRel}(H)$ consisting of intuitionistic H-fuzzy reflexive relational space on sets and we study structures of $\mathbf{IRel_R}(H)$ in a viewpoint of the topological universe introduce by Nel. We show that $\mathbf{IRel_R}(H)$ is a topological universe over **Set**. Moreover, we show that exponential objects in $\mathbf{IRel_R}(H)$ are quite different from those in $\mathbf{IRel}(H)$.

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1. Introduction

In 1965, Zadeh [26] introduced a concept of a fuzzy set as the generalization of a crisp set. Also, he introduce a concept of a fuzzy relation as the generalization of a crisp relation in [27]. In 1986, Atanassov [1] introduced a notion of an intuitioninstic fuzzy set as the generalization of a fuzzy set. After that time, Banerjee and Basnet [2], Biswas [3], and Hur and his colloeagues applied

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the concept of intuitioninstic fuzzy sets to group theory. Çoker [6], Hur and his colleagues [16], and Lee and Lee [22] applied one to topology. Also, Hur and his colleagues [15] applied the notion of intuitioninstic fuzzy sets to topological group. In particular, Hur and his colleagues [18, 19] studied categorical structures of the category $\mathbf{ISet}(H)$ consisting of intuitioninstic H-fuzzy sets and the category $\mathbf{IRel}(H)$ consisting of intuitinistic H-fuzzy relational spaces in a viewpoint of topological universe, defined by Nel [23].

In this paper, we study categorical structures of the subcategory $\mathbf{IRel_R}(H)$ of $\mathbf{IRel}(H)$ consisting of intuitionistic H-fuzzy reflexive relational spaces on sets in a viewpoint of a topological universe. In particular, it is very interesting that exponential objects in $\mathbf{IRel_R}(H)$ are shown to be quite different from those in $\mathbf{IRel}(H)$ (see [19]).

For general background for lattice theory, we refer to [3, 20] and for general categorical background to [8, 9, 21, 23].

2. Preliminaries

We will introduce some well-known definitions and results [9,21] which are needed in a later sections.

Definition 1.1. (see [9]) A category $\mathbf{A}$ is said to be *well-powered* if each $\mathbf{A}$-object has a representative class of subobjects that is a set.

Dual Notion. co-(well-powered) [i.e., each object has a representative class of quotient objects which is a set].

Definition 1.2. (see [21]) Let $\mathbf{A}$ be a concrete category.

(1) The $\mathbf{A}$-*fibre* of a set X is the class of all $\mathbf{A}$-structures on X.

(2) $\mathbf{A}$ is called *properly fibred over* **Set** provided that the following conditions hold :

(i) (Fibre-Smallness) For each set X, the $\mathbf{A}$-fibre of X is a set.

(ii) (Terminal Separator Property) For each singleton set X, the $\mathbf{A}$-fibre of X has precisely one element.

(iii) If ξ and η are $\mathbf{A}$-structures on a set X such that $1_X : (X, \xi) \to (X, \eta)$ and $1_X : (X, \eta) \to (X, \xi)$ are $\mathbf{A}$-morphisms, then $\xi = \eta$.

Result 1.A. (see [21, Theorem 2.4; 9, Proposition 36.10 and 36.11]) *Let* $\mathbf{A}$ *be a well-powered and co-(well-powered) topological category and let* $\mathbf{B}$ *be a subcategory of* $\mathbf{A}$. *Then the following are equivalent:*

(1) $\mathbf{B}$ *is epireflective in* $\mathbf{A}$.

(2) $\mathbf{B}$ *is closed under the formation of initial monosources.*

(3) $\mathbf{B}$ *is closed under the formation of products and pullbacks in* $\mathbf{A}$.

Result 1.B. (see [21, Theorem 2.5]) *Let* **A** *be a well-powered and co-(well-powered) topological category and let* **B** *be a subcategory of* **A**. *Then the following are equivalent:*

(1) **B** *is bireflective in* **A**.

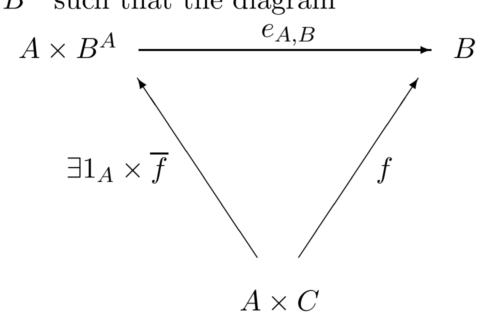
(2) **B** *is closed under the formation of initial sources.*

Result 1.C. (see [21, Theorem 2.6]) *If* **A** *is a (property fibred, resp.) topological category and* **B** *is a bireflective subcategory of* **A**, *then* **B** *is also a (property fibred, resp.) topological category. Moreover, every source in* **B** *which is initial in* **A** *is initial in* **B**.

Definition 1.3. (see [8]) A category **A** is called *cartesian closed* provided that the following conditions hold:

(1) For any **A**-objects A and B, there exists a product $A \times B$ in **A**.

(2) Exponential exist in **A**, i.e., for any **A**-object A, the functor $A \times - : \mathbf{A} \to \mathbf{A}$ has a right adjoint, i.e., for any **A**-object B, there exists an **A**-object B^A and a **A**-morphism $e_{A,B} : A \times B^A \to B$ (called the *evaluation*) such that for any **A**-object C and any **A**-morphism $f : A \times C \to B$, there exists a unique **A**-morphism $\overline{f} : C \to B^A$ such that the diagram



commutes.

Definition 1.4. (see [23]) A category **A** is called a *topological universe over* **Set** provided that the following conditions hold:

(1) **A** is well-structured over **Set**, i.e., (i) **A** is a concrete category ; (ii) **A** has the fibre-smallness condition; (iii) **A** has the terminal separator property.

(2) **A** is cotopological over **Set**.

(3) Final episinks in **A** are preserved by pullbacks, i.e., for any final episink $(g_\lambda : X \to Y)_\Lambda$ and any **A**-morphism $f : W \to Y$, the family $(e_\lambda : U_\lambda \to W)_\Lambda$, obtained by taking the pullback of f and g_λ for each λ, is again a final episink.

Definition 1.5. (see [25]) A category **A** is called a *topos* provided that the following conditions hold :

(1) There is a terminal object U in **A**, i.e., for each **A**-object A, there exists one and only one **A**-morphism from A to U.

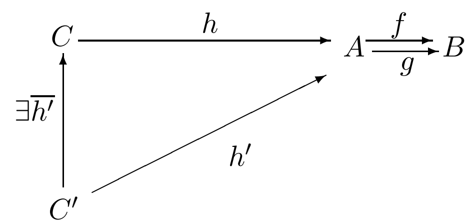
(2) **A** has equalizers i.e., for any **A**-objects A and B and **A**-morphisms

$$A \overset{f}{\underset{g}{\rightrightarrows}} B,$$

there exist an **A**-object C and an **A**-morphism $h : C \to A$ such that:

(a) $f \circ h = g \circ h$,

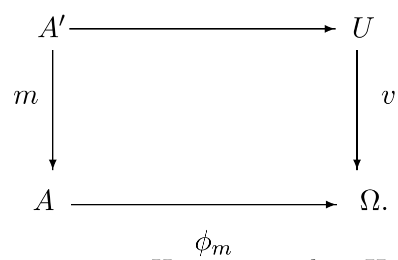
(b) for each **A**-object C' and **A**-morphism $h' : C' \to A$ with $f \circ h' = g \circ h'$, there exists a unique **A**-morphism $\overline{h'} : C' \to C$ such that $h' = h \circ \overline{h'}$, i.e., the diagram



commutes.

(3) **A** is cartesian closed.

(4) There is a subobject classifier in **A**, i.e., there is an **A**-object Ω and **A**-morphism $v : U \to \Omega$ such that for each **A**-monomorphism $m : A' \to A$, there exists a unique **A**-morphism $\phi_m : A \to \Omega$ such that the following diagram is a pullback:



Throughout this paper, we use H as a complete Heyting algebra.

Definition 1.6. (see [23]) A category **A** is called a *topological universe over* **Set** provided that the following conditions hold:

(1) **A** is well-structured over **Set**, i.e., (i) **A** is a concrete category; (ii) **A** has the fibre-smallness condition; (iii) **A** has the terminal separator property.

(2) **A** is cotopological over **Set**.

(3) Final episinks in **A** are preserved by pullbacks, i.e., for any final episink $(g_\lambda : X \to Y)_\Lambda$ and any **A**-morphism $f : W \to Y$, the family $(e_\lambda : U_\lambda \to W)_\Lambda$, obtained by taking the pullback of f and g_λ for each λ, is again a final episink.

Definition 1.7. (see [19]) Let X be a set. A pair $R = (\mu_R, \nu_R)$ is called an *intuitionistic H-fuzzy relation* (in shot, *IHFR*) on X if it satisfies the following conditions:

(i) $\mu_R : X \times X \to H$ and $\nu_R : X \times X \to H$ are mappings, where μ_R and ν_R denote the *degree of membership* (namely $\mu_R(x, y)$) and the *degree of nonmembership* (namely $\nu_R(x, y)$) of each $(x, y) \in X \times X$ to R.

(ii) $\mu_R \leq N(\nu_R)$, i.e., $\mu_R(x, y) \leq N(\nu_R(x, y))$ for each $(x, y) \in X \times X$.

In this case, (X, R) or (X, μ_R, ν_R) is called an *intuitionistic H-fuzzy relatinal space* (in short, *IHFRS*).

Definition 1.8. (see [19]) *Let (X, R_X) and (Y, R_Y) be an IHFRSs. A mapping $f : X \to Y$ is called a relation preserving mapping if $\mu_{R_X} \leq \mu_{R_Y} \circ f^2$ and $\nu_{R_X} \geq \nu_{R_Y} \circ f^2$, where $f^2 = f \times f$.*

From Definition 1.7 and Definition 1.8, we can form a concrete category **IRel**(H) consisting of all relational spaces and relation preserving mappings between them. Every **IRel**(H)-mapping will be called an **IRel***(H)-mapping*.

3. The Category $\mathbf{IRel}_{\boldsymbol{R}}(\boldsymbol{H})$

In this section, we obtain a subcategory $\mathbf{IRel}_{\mathbf{R}}(H)$ of **IRel**(H) which is a topological universe over **Set**. It is very interesting that exponential objects in $\mathbf{IRel}_{\mathbf{R}}(H)$ are shown to be quite different from those in **IRel**(H) constructed in [19].

Definition 2.1. An IHFR R on a set X is said to be *reflexive* if $\mu_R(x, x) = 1$ and $\nu_R(x, x) = 0$ for each $x \in X$.

The class of all intuitionistic H-fuzzy reflexive relational spaces and **IRel**(H)-mappings between them forms a subcategory of **IRel**(H) and denoted by $\mathbf{IRel}_{\mathbf{R}}(H)$.

It is clear that $\mathbf{IRel}_{\mathbf{R}}(H)$ is a full and isomorphism-closed subcategory of **IRel**(H).

We can easily obtain the following.

Proposition 2.2. $\mathbf{IRel}_{\mathbf{R}}(H)$ *is properly fibred over* **Set**.

Lemma 2.3. $\mathbf{IRel}_{\mathbf{R}}(H)$ *is closed under the formation of initial sources in* **IRel**(H).

Proof. Let $(f_\alpha : (X, R) \to (X_\alpha, R_\alpha))_\Gamma$ be any initial source in **IRel**(H) such that $(X_\alpha, R_\alpha) \in \mathbf{IRel}_{\mathbf{R}}(H)$ for each $\alpha \in \Gamma$. Let $x \in X$. Since R_α is reflexive for each $\alpha \in \Gamma$, $\mu_{R_\alpha} \circ f_\alpha^2(x, x) = 1$ and $\nu_{R_\alpha} \circ f_\alpha^2(x, x) = 0$. Thus $\mu_R(x, x) = \bigwedge_\Gamma \mu_{R_\alpha} \circ f_\alpha^2(x, x) = 1$ and $\nu_R(x, x) = \bigvee_\Gamma \nu_{R_\alpha} \circ f_\alpha^2(x, x) = 0$. So R is reflexive. Hence $(X, R) \in \mathbf{IRel}_{\mathbf{R}}(H)$. This completes the proof. □

From Result 1.B, Result 1.C and Lemma 2.3, we obtain the following result.

Theorem 2.4. *(1)* $\mathbf{IRel_R}(H)$ *is a bireflective subcategory of* **IRel** (H).

(2) $\mathbf{IRel_R}(H)$ *is topological over* **Set**.

We show that $\mathbf{IRel_R}(H)$ is cotopological over **Set**, directly.

Theorem 2.5. $\mathbf{IRel_R}(H)$ *has final structures over* **Set**.

Proof. Let X be any set and let $((X_\alpha, R_\alpha))_\Gamma$ any family of intuitionistic H-fuzzy reflexive relational spaces indexed by a class Γ. Let $(f_\alpha : X_\alpha \to X)_\Gamma$ be any sink of mapping. We define two mappings $\mu_R : X \times X \to H$ and $\nu_R : X \times X \to H$ as follows: for each $(x, y) \in X \times X$,

$$\mu_R(x,y) = \begin{cases} \bigvee_\Gamma \bigvee_{(x_\alpha, y_\alpha) \in f_\alpha^{-1^2}(x,y)} \mu_{R_\alpha}(x_\alpha, y_\alpha) & \text{if } (x,y) \in (X \times X - \Delta_X), \\ 1 & \text{if } (x,y) \in \Delta_X, \end{cases}$$

and

$$\nu_R(x,y) = \begin{cases} \bigwedge_\Gamma \bigwedge_{(x_\alpha, y_\alpha) \in f_\alpha^{-1^2}(x,y)} \mu_{R_\alpha}(x_\alpha, y_\alpha) & \text{if } (x,y) \in (X \times X - \Delta_X), \\ 0 & \text{if } (x,y) \in \Delta_X, \end{cases}$$

where $\Delta_X = \{(x,x) : x \in X\}$ and $f_\alpha^{-1^2} = f_\alpha^{-1} \times f_\alpha^{-1}$. Then clearly $(X, R) \in \mathbf{IRel_R}(H)$. Moreover, we can easily check that $(f_\alpha : (X_\alpha, R_\alpha) \to (X, R))_\Gamma$ is a final sink in $\mathbf{IRel_R}(H)$. □

Theorem 2.6. *Final episinks in* $\mathbf{IRel_R}(H)$ *are preserved by pullbacks.*

Proof. Let $(g_\alpha : (X_\alpha, R_\alpha) \to (Y, R_Y))_\Gamma$ be any final episink in $\mathbf{IRel_R}(H)$ and let $f : (W, R_w) \to (Y, R_Y)$ any $\mathbf{IRel}(H)$-mapping, where $(W, R_W) \in \mathbf{IRel_R}(H)$. For each $\alpha \in \Gamma$, let us take $U_\alpha, R_{U_\alpha}, e_\alpha$ and p_α as in the process of the proof of Theorem 2.7 in [19]. By Theorem 2.4(1) and Result 1.A, $\mathbf{IRel_R}(H)$ is closed under the formation of pullbaks in $\mathbf{IRel}(H)$. Thus it is enough to show that $(e_\alpha : (U_\alpha, R_{U_\alpha}) \to (W, R_w))_\Gamma$ is final in $\mathbf{IRel_R}(H)$.

Suppose R is the final IHFR on W with respect to $(e_\alpha)_\Gamma$. By the process of the proof of Theorem 2.6 in [10], $\mu_{R_W} = \mu_R$. Let $(w, w') \in (W \times W - \Delta_W)$.

Then:

$$\begin{aligned}
\nu_{R_W}(w,w') &= \nu_{R_W}(w,w') \vee \nu_{R_W}(w,w') \geq \nu_{R_W}(w,w') \vee \nu_{R_Y} \circ f^2(w,w) \\
&\quad \text{(Since } f:(W,R_W)\to(Y,R_Y) \text{ is an } \mathbf{IRel}(H)\text{-mapping)} \\
&= \nu_{R_W}(w,w') \vee \nu_{R_Y}(f(w),f(w')) \\
&= \nu_{R_W}(w,w') \vee [\bigwedge_{\Gamma} \bigwedge_{(x_\alpha,x'_\alpha)\in {g_\alpha^{-1}}^2(f(w),f(w'))} \nu_{R_\alpha}(x_\alpha,x'_\alpha)] \\
&\quad \text{(Since } (g_\alpha)_\Gamma \text{ is final)} \\
&= \bigwedge_{\Gamma} \bigwedge_{(x_\alpha,x'_\alpha)\in {g_\alpha^{-1}}^2(f(w),f(w'))} [\nu_{R_W}(w,w') \vee \nu_{R_\alpha}(x_\alpha,x'_\alpha)] \\
&= \bigwedge_{\Gamma} \bigwedge_{((w,x_\alpha),(w',x'_\alpha))\in {e_\alpha^{-1}}^2(w,w')} [\nu_{R_W}(w,w') \vee \nu_{R_\alpha}(x_\alpha,x'_\alpha)] \\
&= \bigwedge_{\Gamma} \bigwedge_{((w,x_\alpha),(w',x'_\alpha))\in {e_\alpha^{-1}}^2(w,w')} \nu_{R_{U_\alpha}}((w,x_\alpha),(w',x'_\alpha)).
\end{aligned}$$

Thus $\nu_{R_W}(w,w') \geq \nu_R(w,w')$, i.e., $\nu_{R_W} \geq \nu_R$. On the other hand, by the similar argument as the process of the proof of Theorem 2.7 in [19], we have $\nu_R \geq \nu_{R_W}$ on $W \times W - \Delta_W$. So $\nu_R = \nu_{R_W}$ on $W \times W - \Delta_W$. Now let $w \in \Delta_W$. Then clearly $\nu_R(w,w) = 0 = \nu_{R_W}(w,w)$. Hence $\nu_R = \nu_{R_W}$ on $W \times W$. This completes the proof. □

Hence, by Proposition 2.2, Theorem 2.4(2) and Theorem 2.6, we obtain the following result.

Theorem 2.7. *$\mathbf{IRel_R}(H)$ is a topological universe over* **Set**. *Hence $\mathbf{IRel_R}(H)$ is a concrete quasitopos in the sense of E.J. Dubuc [7].*

Theorem 2.8. *$\mathbf{IRel_R}(H)$ has exponential objects. Hence $\mathbf{IRel_R}(H)$ is cartesian closed over* **Set**.

Proof. For any $\mathbf{X} = (X, R_X), \mathbf{Y} = (Y, R_Y) \in \mathbf{IRel_R}(H)$, let $Y^X = \hom_{\mathbf{IRel_R}(H)}(X,Y)$. We define two mappings $\mu_R : Y^X \times Y^X \to H$ and $\nu_R : Y^X \times Y^X \to H$ sa follows: for each $(f,g) \in Y^X \times Y^X$,

$$\mu_R(f,g) = \begin{cases} 1 & \text{if } D(f,g) = \emptyset, \\ \bigwedge_{(x,y)\in D(f,g)} \mu_{R_Y}(f(x),g(y)) & \text{if } D(f,g) \neq \emptyset, \end{cases}$$

and

$$\nu_R(f,g) = \begin{cases} 0 & \text{if } E(f,g) = \emptyset, \\ \bigvee_{(x,y)\in E(f,g)} \nu_{R_Y}(f(x),g(y)) & \text{if } E(f,g) \neq \emptyset, \end{cases}$$

where $D(f,g) = \{(x,y) \in X \times X : \mu_{R_X}(x,y) > \mu_{R_Y}(f(x),g(y))\}$ and $E(f,g) = \{(x,y) \in X \times X : \nu_{R_X}(x,y) < \nu_{R_Y}(f(x),g(y))\}$.

Then it is clear that $E(f,g) \neq \emptyset$ if and only if $D(f,g) \neq \emptyset$ for each $(f,g) \in Y^X \times Y^X$ and $N(\nu_R(f,g) \geq \mu_R(f,g)$ for each $(f,g) \in Y^X \times Y^X$. Thus $(Y^X, R) \in \mathbf{IRel}(H)$. Since $f : \mathbf{X} \to \mathbf{Y}$ is an $\mathbf{IRel}(H)$-mapping, $D(f,f) = \emptyset = E(f,f)$. So $(Y^X, R) \in \mathbf{IRel_R}(H)$. Let $\mathbf{Y^X} = (Y^X, R)$. Now we define a mapping $e_{X,Y} : X \times Y^X \to Y$ by $e_{X,Y}(a,f) = f(a)$ for each $(a,f) \in X \times Y^X$. Let $((a,f),(b,g)) \in (X \times Y^X) \times (X \times Y^X)$. Then, by the process of the proof of Remark 2.8 in [10], $\mu_{R_X \times R} \leq \mu_{R_Y} \circ e^2_{X,Y}$. Suppose $E(f,g) = \emptyset$. Then:

$$\begin{aligned}\nu_{R_X \times R}((a,f),(b,g)) &= \nu_{R_X}(a,b) \vee \nu_R(f,g)\\ &= \nu_{R_X}(a,b) \geq \nu_{R_Y}(f(a),g(b)) = \nu_{R_Y}(e_{X,Y}(a,f), e_{X,Y}(b,g))\\ &= \nu_{R_Y} \circ e^2_{X,Y}((a,f),(b,g)).\end{aligned}$$

Suppose $E(f,g) \neq \emptyset$. Then:

$$\begin{aligned}\nu_{R_X \times R}((a,f),(b,g)) &= \nu_{R_X}(a,b) \vee \nu_R(f,g)\\ &= \nu_{R_X}(a,b) \vee [\bigvee_{(x,y)\in E(f,g)} \nu_{R_Y}(f(x),g(y))] \geq \nu_{R_Y}(f(a),g(b))\\ &= \nu_{R_Y} \circ e^2_{X,Y}((a,f),(b,g)).\end{aligned}$$

In all, $\nu_{R_X \times R} \geq \nu_{R_Y} \circ e^2_{X,Y}$ So $e_{X,Y} : \mathbf{X} \times \mathbf{Y^X} \to \mathbf{Y}$ is an $\mathbf{IRel}(H)$-mapping.

For any $\mathbf{Z} = (Z, R_Z) \in \mathbf{IRel_R}(H)$, let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y^X}$ be any $\mathbf{IRel}(H)$-mapping. Define $\overline{h} : Z \to Y^X$ by $[\overline{h}(c)](a) = h(a,c)$ for each $c \in Z$ and each $a \in X$. Let $c \in Z$ and let $a,b \in X$. Then, by the process of the proof of Remark 2.8 in [6], $\mu_{R_X} \leq \mu_{R_Y} \circ [\overline{h}(c)]^2$. On the other hand,

$$\begin{aligned}\nu_{R_Y} \circ [\overline{h}(c)]^2(a,b) &= \nu_{R_Y}([\overline{h}(c)](a), [\overline{h}(c)](b))\\ &= \nu_{R_Y}(h(a,c),h(b,c)) = \nu_{R_Y} \circ h^2((a,c),(b,c)) \leq \nu_{R_X \times R_Z}((a,c),(b,c))\\ &= \nu_{R_X}(a,b) \vee \nu_{R_Z}(c,c) = \nu_{R_X}(a,b).\end{aligned}$$

Thus $\nu_{R_X} \geq \nu_{R_Y} \circ [\overline{h}(c)]^2$. So $\overline{h}(c) : \mathbf{X} \to \mathbf{Y}$ is an $\mathbf{IRel}(H)$-mapping for each $c \in Z$ and thus $\overline{h}$ is well-defined. Now let $c, c' \in Z$. Then, by the process of the proof of Remark 2.8 in [6], $\mu_{R_Z} \leq \mu_R \circ \overline{h}^2$. We will show that $\nu_{R_Z} \geq \nu_R \circ \overline{h}^2$.

Suppose $E(\overline{h}(c), \overline{h}(c')) = \emptyset$. Then $\nu_R \circ \overline{h}^2(c,c') = \nu_R(\overline{h}(c), \overline{h}(c')) = 0 \leq \nu_{R_Z}(c,c')$.

Suppose $E(\overline{h}(c), \overline{h}(c')) \neq \emptyset$. Then:

$$\begin{aligned}
\nu_R \circ \overline{h}^2(c, c') &= \nu_R(\overline{h}(c), \overline{h}(c')) \\
&= \bigvee_{(a,b)\in E(\overline{h}(c),\overline{h}(c'))} \nu_{R_Y}([\overline{h}(c)](a), [\overline{h}(c')](b)) \\
&= \bigvee_{(a,b)\in E(\overline{h}(c),\overline{h}(c'))} \nu_{R_Y}(h(a, c), h(b, c')) \\
&= \bigvee_{(a,b)\in E(\overline{h}(c),\overline{h}(c'))} \nu_{R_Y} \circ h^2((a, c), (b, c')) \\
&\leq \bigvee_{(a,b)\in E(\overline{h}(c),\overline{h}(c'))} \nu_{R_X \times R_Z}((a, c), (b, c')) \\
&= \bigvee_{(a,b)\in E(\overline{h}(c),\overline{h}(c'))} [\nu_{R_X}(a, b) \vee \nu_{R_Z}(c, c')].
\end{aligned}$$

On the other hand, let $(a, b) \in E(\overline{h}(c), \overline{h}(c'))$. Then:

$$\begin{aligned}
&\nu_{R_X}(a, b) < \nu_{R_Y}([\overline{h}(c)](a), [\overline{h}(c')](b)) = \nu_{R_Y}(h(a, c), h(b, c')) \\
&= \nu_{R_Y} \circ h^2((a, c), (b, c')) \leq \nu_{R_X \times R_Z}((a, c), (b, c')) = \nu_{R_X}(a, b) \vee \nu_{R_Z}(c, c').
\end{aligned}$$

Thus $\nu_{R_X}(a, b) < \nu_{R_Z}(c, c')$. So $\nu_R \circ \overline{h}^2(c, c') \leq \nu_{R_Z}(c, c')$. In all, $\nu_{R_Z} \geq \nu_R \circ \overline{h}^2$. Hence $\overline{h}$ is an **IRel**(H)-mapping. Moreover, $\overline{h}$ is unique and $e_{X,Y} \circ (1_X \times \overline{h}) = h$. This completes the proof. □

Remark 2.9. (1) In [24], Y. Noh obtained exponential objects in $\mathbf{Rel_R}(I)$, where $I = [0, 1]$. In Theorem 2.8, we showed that the construction of an exponential object in $\mathbf{Rel_R}(I)$ is applicable to the case of $\mathbf{IRel_R}(H)$.

(2) We note that exponential objects in $\mathbf{IRel_R}(H)$ are quite different from those in **IRel**(H) constructed in Theorem 2.9 in [19].

(3) $\mathbf{IRel_R}(H)$ has no subobject classifier.

Example 2.10. Let $H = \{0, 1\}$ be the two points chain and let $X = \{a, b\}$. Let R_1 and R_2 be the intuitionistic H-fuzzy reflexive relations on X given by:

$\mu_{R_1}(a, a) = \mu_{R_1}(b, b) = 1$, $\mu_{R_1}(a, b) = \mu_{R_1}(b, a) = 0$;

$\nu_{R_1}(a, a) = \nu_{R_1}(b, b) = 0$, $\nu_{R_1}(a, b) = \nu_{R_1}(b, a) = 1$;

$\mu_{R_2}(a, a) = \mu_{R_2}(b, b) = 1$, $\mu_{R_2}(a, b) = \mu_{R_2}(b, a) = 0$;

$\nu_{R_2}(a, a) = \nu_{R_2}(b, b) = 0$, $\nu_{R_2}(a, b) = \nu_{R_2}(b, a) = 1$.

Let $1_X : (X, R_1) \to (X, R_2)$ be the identity mapping. Then clearly 1_X is both a monomorphism and an epimorphism in $\mathbf{IRel_R}(H)$. But 1_X is not an isomorphism in $\mathbf{IRel_R}(H)$. Hence $\mathbf{IRel_R}(H)$ has no subobject classifier (see [5]).

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