INTUITIONISTIC H-FUZZY REFLEXIVE RELATIONS

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Abstract: We introduce the subcategory $\text{IRel}_R(H)$ of $\text{IRel}(H)$ consisting of intuitionistic H-fuzzy reflexive relational space on sets and we study structures of $\text{IRel}_R(H)$ in a viewpoint of the topological universe introduce by Nel. We show that $\text{IRel}_R(H)$ is a topological universe over $\text{Set}$. Moreover, we show that exponential objects in $\text{IRel}_R(H)$ are quite different from those in $\text{IRel}(H)$.

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1. Introduction

In 1965, Zadeh \cite{26} introduced a concept of a fuzzy set as the generalization of a crisp set. Also, he introduce a concept of a fuzzy relation as the generalization of a crisp relation in \cite{27}. In 1986, Atanassov \cite{1} introduced a notion of an intuitionistic fuzzy set as the generalization of a fuzzy set. After that time, Banerjee and Basnet \cite{2}, Biswas \cite{3}, and Hur and his colleagues applied
the concept of intuitionistic fuzzy sets to group theory. Çoker [6], Hur and his colleagues [16], and Lee and Lee [22] applied one to topology. Also, Hur and his colleagues [15] applied the notion of intuitionistic fuzzy sets to topological group. In particular, Hur and his colleagues [18, 19] studied categorical structures of the category $\text{ISet}(H)$ consisting of intuitionistic $H$-fuzzy sets and the category $\text{IRel}(H)$ consisting of intuitionistic $H$-fuzzy relational spaces in a viewpoint of topological universe, defined by Nel [23].

In this paper, we study categorical structures of the subcategory $\text{IRel}_R(H)$ of $\text{IRel}(H)$ consisting of intuitionistic $H$-fuzzy reflexive relational spaces on sets in a viewpoint of a topological universe. In particular, it is very interesting that exponential objects in $\text{IRel}_R(H)$ are shown to be quite different from those in $\text{IRel}(H)$ (see [19]).

For general background for lattice theory, we refer to [3, 20] and for general categorical background to [8, 9, 21, 23].

2. Preliminaries

We will introduce some well-known definitions and results [9,21] which are needed in a later sections.

**Definition 1.1.** (see [9]) A category $A$ is said to be well-powered if each $A$-object has a representative class of subobjects that is a set.

**Dual Notion.** co-(well-powered) [i.e., each object has a representative class of quotient objects which is a set].

**Definition 1.2.** (see [21]) Let $A$ be a concrete category.

1. The $A$-fibre of a set $X$ is the class of all $A$-structures on $X$.
2. $A$ is called properly fibred over $\text{Set}$ provided that the following conditions hold:
   1. (Fibre-Smallness) For each set $X$, the $A$-fibre of $X$ is a set.
   2. (Terminal Separator Property) For each singleton set $X$, the $A$-fibre of $X$ has precisely one element.
   3. If $\xi$ and $\eta$ are $A$-structures on a set $X$ such that $1_X : (X, \xi) \to (X, \eta)$ and $1_X : (X, \eta) \to (X, \xi)$ are $A$-morphisms, then $\xi = \eta$.

**Result 1.A.** (see [21, Theorem 2.4; 9, Proposition 36.10 and 36.11]) Let $A$ be a well-powered and co-(well-powered) topological category and let $B$ be a subcategory of $A$. Then the following are equivalent:

1. $B$ is epireflective in $A$.
2. $B$ is closed under the formation of initial monosources.
3. $B$ is closed under the formation of products and pullbacks in $A$. 
**Result 1.B.** (see [21, Theorem 2.5]) Let $A$ be a well-powered and co-(well-powered) topological category and let $B$ be a subcategory of $A$. Then the following are equivalent:

1. $B$ is bireflective in $A$.
2. $B$ is closed under the formation of initial sources.

**Result 1.C.** (see [21, Theorem 2.6]) If $A$ is a (property fibred, resp.) topological category and $B$ is a bireflective subcategory of $A$, then $B$ is also a (property fibred, resp.) topological category. Moreover, every source in $B$ which is initial in $A$ is initial in $B$.

**Definition 1.3.** (see [8]) A category $A$ is called cartesian closed provided that the following conditions hold:

1. For any $A$-objects $A$ and $B$, there exists a product $A \times B$ in $A$.
2. Exponential exist in $A$, i.e., for any $A$-object $A$, the functor $A \times - : A \to A$ has a right adjoint, i.e., for any $A$-object $B$, there exists an $A$-object $B^A$ and a $A$-morphism $e_{A,B} : A \times B^A \to B$ (called the evaluation) such that for any $A$-object $C$ and any $A$-morphism $f : A \times C \to B$, there exists a unique $A$-morphism $f : C \to B^A$ such that the diagram

$$
\begin{array}{ccc}
A \times B^A & \xrightarrow{e_{A,B}} & B \\
\downarrow \exists 1_A \times f & & \downarrow f \\
A \times C & & \\
\end{array}
$$

commutes.

**Definition 1.4.** (see [23]) A category $A$ is called a topological universe over $\textbf{Set}$ provided that the following conditions hold:

1. $A$ is well-structured over $\textbf{Set}$, i.e., (i) $A$ is a concrete category; (ii) $A$ has the fibre-smallness condition; (iii) $A$ has the terminal separator property.
2. $A$ is cotopological over $\textbf{Set}$.
3. Final episinks in $A$ are preserved by pullbacks, i.e., for any final episink $(g_\lambda : X \to Y)_\Lambda$ and any $A$-morphism $f : W \to Y$, the family $(e_\lambda : U_\lambda \to W)_\Lambda$, obtained by taking the pullback of $f$ and $g_\lambda$ for each $\lambda$, is again a final episink.

**Definition 1.5.** (see [25]) A category $A$ is called a topos provided that the following conditions hold:

1. There is a terminal object $U$ in $A$, i.e., for each $A$-object $A$, there exists one and only one $A$-morphism from $A$ to $U$. 
(2) \( \mathcal{A} \) has equalizers i.e., for any \( \mathcal{A} \)-objects \( A \) and \( B \) and \( \mathcal{A} \)-morphisms
\[
A \xrightarrow{f} B,
\]
there exist an \( \mathcal{A} \)-object \( C \) and an \( \mathcal{A} \)-morphism \( h : C \to A \) such that:
(a) \( f \circ h = g \circ h \),
(b) for each \( \mathcal{A} \)-object \( C' \) and \( \mathcal{A} \)-morphism \( h' : C' \to A \) with \( f \circ h' = g \circ h' \), there exists a unique \( \mathcal{A} \)-morphism \( \overline{h} : C' \to C \) such that \( h' = h \circ \overline{h} \), i.e., the diagram
\[
\begin{array}{ccc}
C' & \xrightarrow{h'} & C \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
\]
commutes.

(3) \( \mathcal{A} \) is cartesian closed.

(4) There is a subobject classifier in \( \mathcal{A} \), i.e., there is an \( \mathcal{A} \)-object \( \Omega \) and \( \mathcal{A} \)-morphism \( v : U \to \Omega \) such that for each \( \mathcal{A} \)-monomorphism \( m : A' \to A \), there exists a unique \( \mathcal{A} \)-morphism \( \phi_m : A \to \Omega \) such that the following diagram is a pullback:
\[
\begin{array}{ccc}
A' & \xrightarrow{m} & U \\
\downarrow & & \downarrow \\
A & \xrightarrow{\phi_m} & \Omega
\end{array}
\]

Throughout this paper, we use \( H \) as a complete Heyting algebra.

**Definition 1.6.** (see [23]) A category \( \mathcal{A} \) is called a topological universe over \( \text{Set} \) provided that the following conditions hold:

(1) \( \mathcal{A} \) is well-structured over \( \text{Set} \), i.e., (i) \( \mathcal{A} \) is a concrete category; (ii) \( \mathcal{A} \) has the fibre-smallness condition; (iii) \( \mathcal{A} \) has the terminal separator property.

(2) \( \mathcal{A} \) is cotopological over \( \text{Set} \).

(3) Final episinks in \( \mathcal{A} \) are preserved by pullbacks, i.e., for any final episink \( (g_\lambda : X \to Y)_\lambda \) and any \( \mathcal{A} \)-morphism \( f : W \to Y \), the family \( (e_\lambda : U_\lambda \to W)_\lambda \), obtained by taking the pullback of \( f \) and \( g_\lambda \) for each \( \lambda \), is again a final episink.

**Definition 1.7.** (see [19]) Let \( X \) be a set. A pair \( R = (\mu_R, \nu_R) \) is called an intuitionistic \( H \)-fuzzy relation (in shot, IHFR) on \( X \) if it satisfies the following
conditions:

(i) $\mu_R : X \times X \to H$ and $\nu_R : X \times X \to H$ are mappings, where $\mu_R$ and $\nu_R$ denote the degree of membership (namely $\mu_R(x, y)$) and the degree of nonmembership (namely $\nu_R(x, y)$) of each $(x, y) \in X \times X$ to $R$.

(ii) $\mu_R \leq N(\nu_R)$, i.e., $\mu_R(x, y) \leq N(\nu_R(x, y))$ for each $(x, y) \in X \times X$.

In this case, $(X, R)$ or $(X, \mu_R, \nu_R)$ is called an intuitionistic $H$-fuzzy relational space (in short, IHFRS).

**Definition 1.8.** (see [19]) Let $(X, R_X)$ and $(Y, R_Y)$ be an IHFRSs. A mapping $f : X \to Y$ is called a relation preserving mapping if $\mu_{R_X} \leq \mu_{R_Y} \circ f^2$ and $\nu_{R_X} \geq \nu_{R_Y} \circ f^2$, where $f^2 = f \times f$.

From Definition 1.7 and Definition 1.8, we can form a concrete category $\text{IRel}(H)$ consisting of all relational spaces and relation preserving mappings between them. Every $\text{IRel}(H)$-mapping will be called an $\text{IRel}(H)$-mapping.

### 3. The Category $\text{IRel}_R(H)$

In this section, we obtain a subcategory $\text{IRel}_R(H)$ of $\text{IRel}(H)$ which is a topological universe over $\text{Set}$. It is very interesting that exponential objects in $\text{IRel}_R(H)$ are shown to be quite different from those in $\text{IRel}(H)$ constructed in [19].

**Definition 2.1.** An IHFR $R$ on a set $X$ is said to be reflexive if $\mu_R(x, x) = 1$ and $\nu_R(x, x) = 0$ for each $x \in X$.

The class of all intuitionistic $H$-fuzzy reflexive relational spaces and $\text{IRel}(H)$-mappings between them forms a subcategory of $\text{IRel}(H)$ and denoted by $\text{IRel}_R(H)$.

It is clear that $\text{IRel}_R(H)$ is a full and isomorphism-closed subcategory of $\text{IRel}(H)$.

We can easily obtain the following.

**Proposition 2.2.** $\text{IRel}_R(H)$ is properly fibred over $\text{Set}$.

**Lemma 2.3.** $\text{IRel}_R(H)$ is closed under the formation of initial sources in $\text{IRel}(H)$.

**Proof.** Let $(f_\alpha : (X, R) \to (X_\alpha, R_\alpha))_\Gamma$ be any initial source in $\text{IRel}(H)$ such that $(X_\alpha, R_\alpha) \in \text{IRel}_R(H)$ for each $\alpha \in \Gamma$. Let $x \in X$. Since $R_\alpha$ is reflexive for each $\alpha \in \Gamma$, $\mu_{R_\alpha} \circ f_\alpha^2(x, x) = 1$ and $\nu_{R_\alpha} \circ f_\alpha^2(x, x) = 0$. Thus $\mu_R(x, x) = \bigwedge_\Gamma \mu_{R_\alpha} \circ f_\alpha^2(x, x) = 1$ and $\nu_R(x, x) = \bigvee_\Gamma \nu_{R_\alpha} \circ f_\alpha^2(x, x) = 0$. So $R$ is reflexive. Hence $(X, R) \in \text{IRel}_R(H)$. This completes the proof. \hfill \Box

From Result 1.B, Result 1.C and Lemma 2.3, we obtain the following result.
Theorem 2.4. (1) $\mathrm{IRel}_R(H)$ is a bireflective subcategory of $\mathrm{IRel}(H)$.

(2) $\mathrm{IRel}_R(H)$ is topological over $\mathrm{Set}$.

We show that $\mathrm{IRel}_R(H)$ is cotopological over $\mathrm{Set}$, directly.

Theorem 2.5. $\mathrm{IRel}_R(H)$ has final structures over $\mathrm{Set}$.

Proof. Let $X$ be any set and let $((X_\alpha, R_\alpha))_{\Gamma}$ any family of intuitionistic $H$-fuzzy reflexive relational spaces indexed by a class $\Gamma$. Let $(f_\alpha : X_\alpha \to X)_{\Gamma}$ be any sink of mapping. We define two mappings $\mu_R : X \times X \to H$ and $\nu_R : X \times X \to H$ as follows: for each $(x, y) \in X \times X$,

$$
\mu_R(x, y) = \begin{cases} 
\bigvee_{(x_\alpha, y_\alpha) \in f_\alpha^{-1}(x, y)} \mu_{R_\alpha}(x_\alpha, y_\alpha) & \text{if } (x, y) \in (X \times X - \Delta_X), \\
1 & \text{if } (x, y) \in \Delta_X,
\end{cases}
$$

and

$$
\nu_R(x, y) = \begin{cases} 
\bigwedge_{(x_\alpha, y_\alpha) \in f_\alpha^{-1}(x, y)} \mu_{R_\alpha}(x_\alpha, y_\alpha) & \text{if } (x, y) \in (X \times X - \Delta_X), \\
0 & \text{if } (x, y) \in \Delta_X,
\end{cases}
$$

where $\Delta_X = \{(x, x) : x \in X\}$ and $f_\alpha^{-1} = f_\alpha^{-1} \circ f_\alpha^{-1}$. Then clearly $(X, R) \in \mathrm{IRel}_R(H)$. Moreover, we can easily check that $(f_\alpha : (X_\alpha, R_\alpha) \to (X, R))_{\Gamma}$ is a final sink in $\mathrm{IRel}_R(H)$.

Theorem 2.6. Final episinks in $\mathrm{IRel}_R(H)$ are preserved by pullbacks.

Proof. Let $(g_\alpha : (X_\alpha, R_\alpha) \to (Y, R_Y))_{\Gamma}$ be any final episink in $\mathrm{IRel}_R(H)$ and let $f : (W, R_W) \to (Y, R_Y)$ any $\mathrm{IRel}(H)$-mapping, where $(W, R_W) \in \mathrm{IRel}_R(H)$. For each $\alpha \in \Gamma$, let us take $U_\alpha, R_{U_\alpha}, e_\alpha$ and $p_\alpha$ as in the process of the proof of Theorem 2.7 in [19]. By Theorem 2.4(1) and Result 1.A, $\mathrm{IRel}_R(H)$ is closed under the formation of pullbacks in $\mathrm{IRel}(H)$. Thus it is enough to show that $(e_\alpha : (U_\alpha, R_{U_\alpha}) \to (W, R_W))_{\Gamma}$ is final in $\mathrm{IRel}_R(H)$.

Suppose $R$ is the final IHFR on $W$ with respect to $(e_\alpha)_{\Gamma}$. By the process of the proof of Theorem 2.6 in [10], $\mu_{R_W} = \mu_R$. Let $(w, w') \in (W \times W - \Delta_W)$. 

Then:

\[ \nu_{R'w}(w, w') = \nu_{Rw}(w, w') \cap \nu_{Rw}(w, w') \geq \nu_{Rw}(w, w') \cap \nu_{Ry} \circ f^2(w, w) \]

Since \( f : (W, R_w) \rightarrow (Y, R_Y) \) is an IRel(H)-mapping

\[ = \nu_{Rw}(w, w') \cap \nu_{Ry}(f(w), f(w')) \]

\[ = \nu_{Rw}(w, w') \vee \bigwedge_{\Gamma} \bigwedge_{(x, x') \in \mathbb{g}_a} \nu_{Ra}(x, x') \]

(Since \( \nu_{Rw} \) is a concrete quasitopos in the sense of E.J. Dubuc [7].)

\[ = \bigwedge_{\Gamma} \bigwedge_{(x, x') \in \mathbb{g}_a} \nu_{Ra}(x, x') \]

\[ = \bigwedge_{\Gamma} \nu_{R_{(w, w')}}((w, x), (w', x')) \].

Thus \( \nu_{Rw}(w, w') \geq \nu_R(w, w') \), i.e., \( \nu_{Rw} \geq \nu_R \). On the other hand, by the similar argument as the process of the proof of Theorem 2.7 in [19], we have \( \nu_R \geq \nu_{Rw} \) on \( W \times W - \Delta_a \). So \( \nu_R = \nu_{Rw} \) on \( W \times W - \Delta_a \). Now let \( w \in \Delta_a \). Then clearly \( \nu_R(w, w) = 0 = \nu_{Rw}(w, w) \). Hence \( \nu_R = \nu_{Rw} \) on \( W \times W \). This completes the proof. \qed

Hence, by Proposition 2.2, Theorem 2.4(2) and Theorem 2.6, we obtain the following result.

**Theorem 2.7.** IRel(H) is a topological universe over Set. Hence IRel(H) is a concrete quasitopos in the sense of E.J. Dubuc [7].

**Theorem 2.8.** IRel(H) has exponential objects. Hence IRel(H) is cartesian closed over Set.

**Proof.** For any \( X = (X, R_X), Y = (Y, R_Y) \in \text{IRel}(H) \), let \( Y^X = \text{hom}_{\text{IRel}(H)}(X, Y) \). We define two mappings \( \mu_R : Y^X \times Y^X \rightarrow H \) and \( \nu_R : Y^X \times Y^X \rightarrow H \) as follows: for each \( (f, g) \in Y^X \times Y^X \),

\[ \mu_R(f, g) = \begin{cases} 1 & \text{if } D(f, g) = \emptyset, \\ \bigwedge_{(x, y) \in D(f, g)} \mu_R(f(x), g(y)) & \text{if } D(f, g) \neq \emptyset, \end{cases} \]

and

\[ \nu_R(f, g) = \begin{cases} 0 & \text{if } E(f, g) = \emptyset, \\ \bigvee_{(x, y) \in E(f, g)} \nu_R(f(x), g(y)) & \text{if } E(f, g) \neq \emptyset, \end{cases} \]
where \(D(f, g) = \{(x, y) \in X \times X : \mu_{RX}(x, y) > \mu_{RY}(f(x), g(y))\}\) and \(E(f, g) = \{(x, y) \in X \times X : \nu_{RX}(x, y) < \nu_{RY}(f(x), g(y))\}\).

Then it is clear that \(E(f, g) \neq \emptyset\) if and only if \(D(f, g) \neq \emptyset\) for each \((f, g) \in Y^X \times Y^X\) and \(N(\mu_{R}(f, g) \geq \mu_{R}(f, g))\) for each \((f, g) \in Y^X \times Y^X\). Thus \((Y^X, R) \in \text{IRel}(H)\). Since \(f : X \to Y\) is an \(\text{IRel}(H)\)-mapping, \(D(f, f) = \emptyset = E(f, f)\).

So \((Y^X, R) \in \text{IRel}_R(H)\). Let \(Y^X = (Y^X, R)\). Now we define a mapping \(e_{X,Y} : X \times Y^X \to Y\) by \(e_{X,Y}(a, f) = f(a)\) for each \((a, f) \in X \times Y^X\). Let \(((a, f), (b, g)) \in (X \times Y^X) \times (X \times Y^X)\). Then, by the process of the proof of Remark 2.8 in [10], \(\mu_{RX \times R} \leq \mu_{RY} \circ e_{X,Y}^2\). Suppose \(E(f, g) = \emptyset\). Then:

\[
\nu_{RX \times R}((a, f), (b, g)) = \nu_{RX}(a, b) \lor \nu_{R}(f, g)
\]

\[
= \nu_{RX}(a, b) \lor \nu_{R}(f, g) \lor [\bigvee_{(x, y) \in E(f, g)} \nu_{R}(f(x), g(y))] \geq \nu_{R}(f(a), g(b))
\]

\[
= \nu_{R}(f(a), g(b)).
\]

Suppose \(E(f, g) \neq \emptyset\). Then:

\[
\nu_{RX \times R}((a, f), (b, g)) = \nu_{RX}(a, b) \lor \nu_{R}(f, g)
\]

\[
= \nu_{RX}(a, b) \lor \nu_{R}(f, g) \lor [\bigvee_{(x, y) \in E(f, g)} \nu_{R}(f(x), g(y))] \geq \nu_{R}(f(a), g(b))
\]

\[
= \nu_{R}(f(a), g(b)).
\]

In all, \(\nu_{RX \times R} \geq \nu_{R} \circ e_{X,Y}^2\). So \(e_{X,Y} : X \times Y^X \to Y\) is an \(\text{IRel}(H)\)-mapping.

For any \(Z = (Z, R_Z) \in \text{IRel}_R(H)\), let \(h : X \times Z \to Y^X\) be any \(\text{IRel}(H)\)-
mapping. Define \(\overline{h} : Z \to Y^X\) by \(\overline{h}(c)(a) = h(a, c)\) for each \(c \in Z\) and each \(a \in X\). Let \(c \in Z\) and let \(a, b \in X\). Then, by the process of the proof of Remark 2.8 in [6], \(\mu_{RX} \leq \mu_{RY} \circ |\overline{h}(c)|_2^2\). On the other hand,

\[
\nu_{RY} \circ |\overline{h}(c)|_2^2(a, b) = \nu_{RY}(|\overline{h}(c)|_2(a, b))
\]

\[
= \nu_{RY}(h(a, c), h(b, c)) \leq \nu_{RX \times R_Z}((a, c), (b, c))
\]

\[
= \nu_{RX}(a, b) \lor \nu_{R_Z}(c, c) = \nu_{RY}(a, b).
\]

Thus \(\nu_{RX} \geq \nu_{RY} \circ |\overline{h}(c)|_2^2\). So \(\overline{h}(c) : X \to Y\) is an \(\text{IRel}(H)\)-mapping for each \(c \in Z\) and thus \(\overline{h}\) is well-defined. Now let \(c, c' \in Z\). Then, by the process of the proof of Remark 2.8 in [6], \(\mu_{R_Z} \leq \mu_{R} \circ \overline{h}^2\). We will show that \(\nu_{R_Z} \geq \nu_{R} \circ \overline{h}^2\).

Suppose \(E(\overline{h}(c), \overline{h}(c')) = \emptyset\). Then \(\nu_{R} \circ \overline{h}^2(c, c') = \nu_{R}(\overline{h}(c), \overline{h}(c')) = 0 \leq \nu_{R_Z}(c, c')\).
Suppose $E(\overline{h}(c), \overline{h}(c')) \neq \emptyset$. Then:

\[
\nu_R \circ \overline{h}^2 (c, c') = \nu_R (\overline{h}(c), \overline{h}(c'))
= \bigvee_{(a,b) \in E(\overline{h}(c), \overline{h}(c'))} \nu_{R_Y} ([\overline{h}(c)](a), [\overline{h}(c')](b))
= \bigvee_{(a,b) \in E(\overline{h}(c), \overline{h}(c'))} \nu_{R_Y} (h(a,c), h(b,c'))
= \bigvee_{(a,b) \in E(\overline{h}(c), \overline{h}(c'))} \nu_{R_Y} \circ \overline{h}^2 ((a,c), (b,c'))
\leq \bigvee_{(a,b) \in E(\overline{h}(c), \overline{h}(c'))} \nu_{RX \times R_Z} ((a,c), (b,c'))
= \bigvee_{(a,b) \in E(\overline{h}(c), \overline{h}(c'))} [\nu_{RX} (a) \vee \nu_{R_Z} (c, c')].
\]

On the other hand, let $(a,b) \in E(\overline{h}(c), \overline{h}(c'))$. Then:

\[
\nu_{RX} (a) < \nu_{R_Y} ([\overline{h}(c)](a), [\overline{h}(c')](b)) = \nu_{R_Y} (h(a,c), h(b,c'))
= \nu_{R_Y} \circ \overline{h}^2 ((a,c), (b,c')) \leq \nu_{RX \times R_Z} ((a,c), (b,c')) = \nu_{RX} (a) \vee \nu_{R_Z} (c, c').
\]

Thus $\nu_{RX} (a) < \nu_{R_Z} (c, c')$. So $\nu_R \circ \overline{h}^2 (c, c') \leq \nu_{R_Z} (c, c')$. In all, $\nu_{R_Z} \geq \nu_R \circ \overline{h}^2$. Hence $\overline{h}$ is an $I_{R}(H)$-mapping. Moreover, $\overline{h}$ is unique and $e_{X,Y} \circ (1_X \times \overline{h}) = h$. This completes the proof. \(\square\)

**Remark 2.9.** (1) In [24], Y. Noh obtained exponential objects in $I_{R}(I)$, where $I = [0,1]$. In Theorem 2.8, we showed that the construction of an exponential object in $I_{R}(I)$ is applicable to the case of $I_{R}(H)$.

(2) We note that exponential objects in $I_{R}(H)$ are quite different from those in $I_{R}(H)$ constructed in Theorem 2.9 in [19].

(3) $I_{R}(H)$ has no subobject classifier.

**Example 2.10.** Let $H = \{0,1\}$ be the two points chain and let $X = \{a, b\}$. Let $R_1$ and $R_2$ be the intuitionistic H-fuzzy reflexive relations on $X$ given by:

\[
\begin{align*}
\mu_{R_1}(a,a) &= \mu_{R_1}(b,b) = 1, \mu_{R_1}(a,b) = \mu_{R_1}(b,a) = 0; \\
\nu_{R_1}(a,a) &= \nu_{R_1}(b,b) = 0, \nu_{R_1}(a,b) = \nu_{R_1}(b,a) = 1; \\
\mu_{R_2}(a,a) &= \mu_{R_2}(b,b) = 1, \mu_{R_2}(a,b) = \mu_{R_2}(b,a) = 0; \\
\nu_{R_2}(a,a) &= \nu_{R_2}(b,b) = 0, \nu_{R_2}(a,b) = \nu_{R_2}(b,a) = 1.
\end{align*}
\]

Let $1_X : (X, R_1) \rightarrow (X, R_2)$ be the identity mapping. Then clearly $1_X$ is both a monomorphism and an epimorphism in $I_{R}(H)$. But $1_X$ is not an
isomorphism in $\text{IRel}_{\mathbb{R}}(H)$. Hence $\text{IRel}_{\mathbb{R}}(H)$ has no subobject classifier (see [5]).

References


