International Journal of Pure and Applied Mathematics

Volume 20 No. 1 2005, 53-67

GLOBAL STABILITY OF A CLASS OF DELAYED NEURAL NETWORK MODELS UNDER DYNAMICAL THRESHOLDS

Fei-Yu Zhang[1 §], Ren-Hu Wang[2], Xing-Xue Yan[3]

[1]Department of Mathematics
Hexi University Zhangye
Gansu, 734000, P.R. CHINA
[1]e-mail: zhfy@hxu.edu.cn

Abstract: In this paper, we study dynamical behavior of a class of cellular neural networks system with distributed delays under dynamical thresholds. By using homeomorphism map, M-matrix and Lyapunov functions, some new criteria ensuring the existence, uniqueness, global asymptotic stability and global exponential stability of equilibrium point are derived. In the results, we do not require the activation function to be bounded, differentiable, and monotonic nondecreasing. Moreover, the symmetry of the connection matrix is not also necessary. Our criteria generalize and improve some known results in the literature.

AMS Subject Classification: 34A40, 34K13, 34K14, 34K20, 92B20
Key Words: equilibrium, Homeomorphism, M-matrix, Lyapunov function, Globally asymptotically stable, Globally exponentially stable

1. Introduction

Hopfield [6] considered a model of artificial neural networks as following

$$C_i v_i'(t) = -\frac{v_i(t)}{R_i} + \sum_{j=1}^{n} w_{ij}\phi(v_i(t)) + I_i, \quad i = 1, 2, ..., n. \tag{1}$$

By the help of generalized energy functions and LaSalle invariance principle, he

Received: February 28, 2005

§Correspondence author

obtained that the state of neural networks (i.e. the flow of the dynamics model) eventually converge to its equilibria set, which is very important for application of neural networks. Moreover, Hopfield implemented the dynamics model (1) by applying aggregate of regularly spaced circuits.

During the last 20 years, Hopfield neural networks has been extensively studied and developed, including both continuous-time and discrete-time setting and applied to associative memory, model identification and optimization problems, etc. Many essential features of these networks, such as qualitative properties of stability, oscillation, and convergence issues have been investigated by many authors (see, Hopfield [7] , Zhang and Jin [11], Hirsch [5], Gopalsamy and He [3], Liu and Dickson [9], Huang and Cao [8], Zhang, Li and Huo [10] and cited therein). The main form of Hopfield neural networks extensively considered is following model

$$x_i'(t) = -d_i x_i(t) + \sum_{i=1}^{n} a_{ij} f_j(x_i(t)) + \sum_{i=1}^{n} b_{ij} f_j\left(x_i(t-\tau)\right) + I_i, \qquad i = 1, 2, ..., n. \tag{2}$$

The constant fixed delays in (2) provide of a good approximation in simple circuits. However, due to the presence of parallel pathways with a verity of axon sizes and lengths, there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays. Under these environment, the signal propagation is not instantaneous and cannot be described with discrete delays. A suitable way is to introduce continuously distributed delays determined by a delay kernel. Gopalsamy and Leung [4] considered the following delayed neural networks under thresholds

$$x'(t) = -x(t) + a\tanh\left[x(t) - b\int_0^{\infty} k(s)x(t-s)ds - c\right], \quad t \geq 0, \tag{3}$$

where $a > 0, b \geq 0, a(1-b) < 1$ and $a(1+b) < 1$, $x \in C(\mathbb{R}, \mathbb{R})$ and $k \in C(\mathbb{R}^+, \mathbb{R}^+)$ is delayed-ker-function with

$$\int_0^{\infty} k(s)ds = 1, \tag{4}$$

and

$$\int_0^{\infty} sk(s)ds < +\infty. \tag{5}$$

For their physical meaning of signs in (3), one can refer to Gopalsamy and Leung [4]. If the delayed ker-function satisfies (4) and (5), then, by using Lyapunov function, they established a sufficient condition ensuring global asymptotic stability of the unique equilibrium $x^* = 0$ of system (3) with $c = 0$.

Cui [1] further considered the system (3). By using differential inequality and variations of constants, he obtained new criteria for global asymptotic stability of the equilibrium $x^* = 0$ of system (3) with $c = 0$.

Recently, Zhang, Li and Huo [10] considered the following more general model

$$x'(t) = -x(t) + af\left[x(t) - b\int_0^\infty k(s)x(t-s)ds - c\right], \quad t \geq 0, \tag{6}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a global Lipschtiz function. By using Brower's Theorem and Lyapunov function, they established some sufficient conditions for global asymptotic stability and global exponential stability of equilibrium x^* for the cases $c = 0$ and $c \neq 0$.

In this paper, our aim is to consider the multi-neurons model with delayed-ker-functions under dynamics thresholds. That is to say, we will consider the following more general multi-neurons model with delayed-ker-functions under dynamics thresholds

$$x_i'(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left[x_j(t) - b_j\int_0^\infty k_j(s)x_j(t-s)ds - c_j\right], \quad t \geq 0, \tag{7}$$

where $i = 1, 2, ..., n$, n denotes the numbers of units in the neural networks (7), $x_i(t)$ represents the states of the i-th neuron at time t, a_{ij} and d_j are positive constants, b_j and c_j are nonnegative constants, a_{ij} denotes the strength of the j-th neuron on the i-th neuron, b_j denotes a measure of the inhibitory influence of the past history of the j-th neuron, c_j denotes the neural threshold of the j-th neuron, d_j denotes the rate with which the j-th neuron will rest its potential to the resting state in isolation when disconnected from the networks and external inputs. $k_j : [0, +\infty) \to [0, +\infty)$ is a continuous delayed-ker-function satisfying (4) and (5), f_j denotes the output of the i-th neuron at time t and satisfies the following hypotheses:

(H) For each $j \in \{1, 2, ..., n\}$, $f_j : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz with Lipschitz constant $L_j > 0$, i.e.

$$|f_j(u) - f_j(v)| \leq L_j\,|u - v| \text{ for all } u, v \in \mathbb{R}.$$

By constructing suitable Lyapunov functions, and utilizing the property of M–matrix, some sufficient conditions are established for global asymptotic stability and global exponential stability of the equilibrium of (7). To prove existence and uniqueness of equilibrium, we will use homeomorphism map, which appeared in Forti and Tesi [2]. Compared with Brouwer's Theorem used in Zhang, Li and Huo [10], the advantage of homeomorphism map lies in that it guarantee not only existence but also uniqueness of equilibrium. In these results, we do not require the activation function to be bounded, differentiable, and monotonic nondecreasing. Moreover, the symmetry of the connection matrix is not also necessary. Therefore, our results include some known criteria of Gopalsamy and Leung [4], and Zhang, Li and Huo [10].

The initial condition associated with (7) is of the form

$$x_{0_i}(t) = \phi_i(t), \quad t \in (-\infty, 0], \quad i = 1, 2, ..., n,$$

where $\phi_i \in C((-\infty, 0], \mathbb{R})$, $\phi_i(t)$ is bounded on $(-\infty, 0]$, and the norm of $C((-\infty, 0], \mathbb{R})$ denote by

$$\|\phi(t)\| = \sup_{t\in(-\infty,0]} \sum_{i=1}^{n} |\phi_i(t)|,$$

where $\phi(t) = (\phi_1(t), ..., \phi_n(t))$.

For convenience, we introduce some notations. Let $x = (x_1, ..., x_n)^T$ denote a column vector (the symbol "T" denotes transpose of x) and $|x|$ denote the absolute-value vector given by $|x| = (|x_1|, ..., |x_n|)^T$. For matrix $A = (a_{ij})_{n\times n}$, A^T denotes the transpose of A, A^{-1} denotes the inverse of A, $[A]^s$ is defined as $[A]^s = \frac{(A^T+A)}{2}$. By $\|x\|_2$, we denotes a vector norm defined by $\|x\|_2 = (x_1^2 + ... + x_n^2)^{\frac{1}{2}}$, while $\|A\|$ denotes a matrix norm defined by $\|A\| = \left(\max\{\lambda : \lambda \text{ is an eigenvalue of } A^T A\}\right)^{\frac{1}{2}}$, E_n denotes the $n \times n$ identity matrix.

2. Existence and Uniqueness of the Equilibrium

In this section, we will consider existence and uniqueness of the equilibrium of system (7). Before starting our main results, we first give thefollowing definition and lemma.

Definition 1. (see [2]) A map $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a homeomorphism of $\mathbb{R}^n$ onto itself if $H \in C^0$, H is one-to-one, H is onto and the inverse map $H^{-1} \in C^0$.

Lemma 1. (see [2]) *If $H(x) \in C^0$ satisfy the following conditions:*

(i) $H(x)$ is injective on $\mathbb{R}^n$, and

(ii) $\|H(x)\| \to \infty$ *as* $\|x\| \to \infty$.

Then $H(x)$ *is a homeomorphism of* $\mathbb{R}^n$.

In the following, we will consider the existence and uniqueness of the equilibrium of system (7). From (4), obviously, $x^* = (x_1^*, ..., x_n^*)$ is an equilibrium of system (7) if and only if the following condition holds

$$-d_i x_i^* + \sum_{j=1}^{n} a_{ij} f_j \left[(1-b_j)x_j^* - c_j\right] = 0, \quad i = 1, 2, ...n. \tag{8}$$

Defining a map $F(x)$ as following

$$F(x) = (F_1(x), ..., F_n(x)),$$

where

$$F_i(x) = -d_i x_i + \sum_{j=1}^{n} a_{ij} f_j \left[(1-b_j)x_j - c_j\right], \quad i = 1, 2, ..., n. \tag{9}$$

Theorem 1. *Assume that (H) and (4) hold and that*

$$DL^{-1} - A\left|E_n - B\right| \text{ is an M-matrix.} \tag{10}$$

Then system (7) has an unique equilibrium x^*.

Proof. Obviously, it is only needed to prove that the map $F(x)$ is homeomorphism on $\mathbb{R}^n$. Firstly, we will prove that the map $F(x)$ is injective on $\mathbb{R}^n$. Suppose that there exist x, $y \in \mathbb{R}^n$ with $x \neq y$ such that $F(x) = F(y)$, then $F_i(x) = F_i(y)$. From (9), we have

$$d_i(x_i - y_i) = \sum_{j=1}^{n} a_{ij} \left[(f_j(1-b_j)x_j - c_j) - (f_j(1-b_j)y_j - c_j)\right]. \tag{11}$$

From (11) and (H), we have

$$d_i \left|x_i - y_i\right| \le \sum_{j=1}^{n} L_j a_{ij} \left|1 - b_j\right| \left|x_j - y_j\right|, \tag{12}$$

and so

$$\left(DL^{-1} - A\left|E_n - B\right|\right)\left|x - y\right| \le 0. \tag{13}$$

Since $DL^{-1} - A\left|E_n - B\right|$ is an M-matrix, this implies that $(DL^{-1} - A \left|E_n - B\right|)^{-1}$ is a nonnegative matrix. In view of (13), we have $|x - y| \le 0$, which implies $x = y$. Hence, the map $F(x)$ is injective on $\mathbb{R}^n$.

In the following, we will prove that $\|F(x)\|_2 \to \infty$ as $\|x\|_2 \to \infty$ for two cases.

Case 1. If $f_j(u)$ is bounded on $\mathbb{R}$, then exist $N_j > 0$ such that

$$|f_j(u)| \leq N_j \text{ for all } u \in \mathbb{R}.$$

From (9), we obtain

$$d_i |x_i| \leq |F_i(x)| + \sum_{j=1}^{n} a_{ij} N_j.$$

Then

$$\|x\|_2 \leq \frac{1}{\min_{1\leq i\leq n} d_i} \left[2\|F(x)\|_2 + \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} N_j \right)^2 \right],$$

which implies that $\|F(x)\|_2 \to \infty$ as $\|x\|_2 \to \infty$.

Case 2. $f_j(u)$ is unbounded on $\mathbb{R}$. Let

$$\overline{F}(x) = F(x) - F(0). \tag{14}$$

To show $\|F(x)\|_2 \to \infty$ as $\|x\|_2 \to \infty$, it is sufficient to show $\left\|\overline{F}(x)\right\|_2 \to \infty$ as $\|x\|_2 \to \infty$. From (H), we get

$$f_j\left[(1-b_j)x_j - c_j\right] - f_j(-c_j) = k_j(\xi_j)(1-b_j)x_j, \quad j = 1, 2, ..., n,$$

where ξ_j lies between $(1-b_j)x_j - c_j$ and $-c_j$, and $|k_j(\xi_j)| \leq L_j$. Then

$$\begin{aligned}
\overline{F}(x) &= F(x) - F(0) \\
&= \Bigg(-d_1 x_1 + \sum_{j=1}^{n} a_{1j} f_j\left[(1-b_j)x_j - c_j\right] - \sum_{j=1}^{n} a_{1j} f_j(-c_j), ..., \\
&\qquad -d_n x_n + \sum_{j=1}^{n} a_{nj} f_j\left[(1-b_j)x_j - c_j\right] - \sum_{j=1}^{n} a_{nj} f_j(-c_j) \Bigg) \\
&= \left(-d_1 x_1 + \sum_{j=1}^{n} a_{1j} k_j(\xi_j)(1-b_j)x_j, ..., -d_n x_n + \sum_{j=1}^{n} a_{nj} k_j(\xi_j)(1-b_j)x_j \right) \\
&= \left[AK(\xi)(E_n - B) - D\right] x,
\end{aligned}$$

where $K(\xi) = \mathrm{diag}(k_1(\xi_1), ..., k_n(\xi_n))$.

Since $DL^{-1}-A\left|E_n-B\right|$ is an M-matrix, it is easy to see that $D-LA\left|E_n-B\right|$ is an M-matrix. Hence, there a positive definite diagonal matrix $C=$diag$(c_1,...,c_n)$ such that

$$\left[C\left(AL\left|E_n-B\right|-D\right)\right]^s\leq-\varepsilon E_n,$$

for sufficiently small $\varepsilon>0$. Since

$$\begin{aligned}[Cx]^T\overline{F}(x)&=[Cx]^T\left[AK(\xi)(E_n-B)-D\right]x\\&\leq|x|^T C\left(AL\left|E_n-B\right|-D\right)|x|\\&=|x|^T\left[C\left(AL\left|E_n-B\right|-D\right)\right]^s|x|\leq-\varepsilon\left\|x\right\|_2^2,\end{aligned}\quad(15)$$

then from (15), we obtain

$$\varepsilon\left\|x\right\|_2^2\leq\left\|C\right\|\left\|x\right\|_2\left\|\overline{F}(x)\right\|_2,$$

and so

$$\frac{\varepsilon\left\|x\right\|_2}{\left\|C\right\|}\leq\left\|\overline{F}(x)\right\|_2,$$

which implies that $\left\|\overline{F}(x)\right\|_2\to\infty$ as $\left\|x\right\|_2\to\infty$. From (14) we know that $\left\|F(x)\right\|_2\to\infty$.

By the above proof and Lemma 1, it is to see that the map $F(x)$ is homeomorphism on $\mathbb{R}^n$. Hence, there exists an unique point x^* such that $F(x^*)=0$, i.e., system (7) has an unique equilibrium x^*. The proof is complete. □

If x^* is an unique equilibrium of system (7), we set

$$y(t)=x(t)-x^*,$$

then for $i=1,2,...,n$,

$$\begin{aligned}y_i'(t)&=-d_iy_i(t)\\&\quad+\sum_{j=1}^n a_{ij}f_j\left[y_j(t)+x_j^*-b_j\int_0^\infty k_j(s)\left(y_j(t-s)+x_j^*\right)ds-c_j\right]\\&\quad-d_ix_i^*,\ t\geq0.\end{aligned}\quad(16)$$

By (H), (4) and (8), we have

$$\begin{aligned}y_i'(t)&=-d_iy_i(t)\\&\quad+\sum_{j=1}^n a_{ij}f_j\left[y_j(t)+x_j^*-b_j\int_0^\infty k_j(s)\left(y_j(t-s)+x_j^*\right)ds-c_j\right]\end{aligned}$$

$$-\sum_{j=1}^{n} a_{ij} f_j \left[(1-b_j)x_j^* - c_j\right]$$
$$\leq -d_i y_i(t) + \sum_{j=1}^{n} a_{ij} L_j \left| y_j(t) - b_j \int_0^{\infty} k_j(s) y_j(t-s) ds \right|. \quad (17)$$

Obviously, the equilibrium x^* of system (7) are globally asymptotically stable and globally exponentially stable if and only if the trivial solution $y(t) = 0$ of the system (16) are globally asymptotically stable and globally exponentially stable.

We remark that if $n = 1$, then the condition (10) reduces to

$$\frac{L}{d} a(1-b) < 1. \quad (18)$$

Therefore the following is true.

Corollary 1. *Assume that (H) ($n = 1$) and (18) hold. Then system (6) has an unique equilibrium x^*.*

Clearly, our Corollary 1 concludes Theorem 1 of Zhang, Li and Huo [10] for $d = 1$.

3. Global Stability Analysis

In this section, we will consider global asymptotic stability and global exponential stability of the unique equilibrium of system (7).

Theorem 2. *Assume that (H), (4) and (5) hold and that $DL^{-1} - A(E_n + B)$ is an M-matrix. Then the trivial solution of the system (16) is globally asymptotically stable.*

Proof. Since $DL^{-1} - A(E_n + B)$ is an M-matrix, hence, there exist $\xi_i > 0 (i = 1, 2, ..., n)$ such that

$$-\xi_i \frac{d_i}{L_i} + \sum_{j=1}^{n} \xi_j a_{ji} (1 + b_i) < 0,$$

then

$$-\xi_i d_i + \sum_{j=1}^{n} \xi_j L_i a_{ji} |1 - b_i| \leq -\xi_i d_i + \sum_{j=1}^{n} \xi_j L_i a_{ji} (1 + b_i) < 0. \quad (19)$$

Hence $DL^{-1} - A\,|E_n - B|$ is an M-matrix. By Theorem 1, we know that system (7) has an unique equilibrium x^*, and hence, system (16) has the trivial solution $y(t) = 0$.

Consider the Lyapunov function defined by

$$V_1(t) = \sum_{i=1}^{n} \xi_i \left\{ |y_i(t)| + \sum_{j=1}^{n} L_j a_{ij} b_j \int_0^{\infty} k_j(s) \left(\int_{t-s}^{t} |y_j(\tau)|\, d\tau \right) ds \right\}.$$

Calculating the upper right derivative $D^+V_1(t)$ along the solution of system (16), by (5), we get

$$D^+V_1(t)|_{(16)} = \sum_{i=1}^{n} \xi_i \times \left\{ (\mathrm{sgn} y_i(t))\, y_i'(t) + \sum_{j=1}^{n} L_j a_{ij} b_j \int_0^{\infty} k_j(s) \left[|y_j(t)| - |y_j(t-s)| \right] ds \right\}.$$

From (4) and (17), we have

$$\begin{aligned} D^+V_1(t)|_{(16)} &\le \sum_{i=1}^{n} \xi_i \left\{ -d_i |y_i(t)| + \sum_{j=1}^{n} L_j a_{ij} \left| y_j(t) - b_j \int_0^{\infty} k_j(s) y_j(t-s) ds \right| \right. \\ &\quad \left. + \sum_{j=1}^{n} L_j a_{ij} b_j \int_0^{\infty} k_j(s) \left[|y_j(t)| - |y_j(t-s)| \right] ds \right\} \\ &\le \sum_{i=1}^{n} \xi_i \left\{ -d_i |y_i(t)| + \sum_{j=1}^{n} L_j a_{ij} (1 + b_j) |y_j(t)| \right\} \\ &= \sum_{i=1}^{n} \left\{ -\xi_i d_i + \sum_{j=1}^{n} \xi_j L_i a_{ji} (1 + b_i) \right\} |y_i(t)| \le \alpha \sum_{i=1}^{n} |y_i(t)|, \end{aligned} \tag{20}$$

where

$$\alpha = \min_{1 \le i \le n} \left\{ -\xi_i d_i + \sum_{j=1}^{n} \xi_j L_i a_{ji} (1 + b_i) \right\}$$

and $\alpha < 0$ by (19). Therefore (20) means that the trivial solution of the system (16) is globally asymptotically stable, and hence, the equilibrium x^* of the system (7) is globally asymptotically stable. The proof is complete. □

If

$$\frac{L_i\left(1+b_i\right)}{d_i}\sum_{j=1}^{n} a_{ji} < 1,$$

and taking $\xi_i = 1$ in (19), then

$$-d_i + \sum_{j=1}^{n} L_i a_{ji}\left(1+b_i\right) < 0.$$

Hence this implies that $DL^{-1} - A\left(E_n + B\right)$ is an M-matrix. In view of Theorem 1, we have the following result.

Corollary 2. *Assume that (H), (4) and (5) hold and that*

$$\max_{1\le i\le n} \frac{L_i\left(1+b_i\right)}{d_i}\sum_{j=1}^{n} a_{ji} < 1.$$

Then system (7) has an unique equilibrium x^* which is globally asymptotically stable.

For $n = 1$, we have the following result.

Corollary 3. *Assume that (H) ($n = 1$), (4) and (5) hold and that*

$$La(1+b) < 1. \tag{21}$$

Then the unique equilibrium x^* of the system (6) is globally asymptotically stable.

Remark 2. If $n = 1$, then our Theorem 2 reduces to Theorem 2 of Zhang, Li and Huo [10] and improves Proposition 2.4 of Gopalsamy and Leung [4] by dropping the condition $a\left(1-b\right) < 1$.

Theorem 3. *Assume that (H) and (4) hold and that:*

(i) $\int_0^\infty sk_j(s)e^s ds = M_j < +\infty$, and

(ii) $DL^{-1} - A\left(E_n + BM\right)$ is an M-matrix, where M =diag$(M_1, ..., M_n)$ is an positive definite diagonal matrix.

Then the trivial solution of the system (16) is globally exponentially stable.

Proof. Since $1 = \int_0^\infty k_j(s)ds < \int_0^\infty sk_j(s)e^s ds$, so $M_j \ge 1$. By (ii), there exist $\xi_i > 0$ $(i = 1, 2, ..., n)$ such that

$$-\xi_i \frac{d_i}{L_i} + \sum_{j=1}^{n} \xi_j a_{ji}\left(1 + b_i M_i\right) < 0.$$

Then

$$-\xi_i d_i + \sum_{j=1}^{n} \xi_j a_{ji} |1 - b_i| L_i \leq -\xi_i d_i + \sum_{j=1}^{n} \xi_j a_{ji} (1 + b_i M_i) L_i < 0. \tag{22}$$

By (22), we know that $DL^{-1} - A|E_n - B|$ is an M-matrix. By Theorem 1, we see that system (7) has an unique equilibrium x^*, and hence, system (16) has the trivial solution $y(t) = 0$.

By (22), there exists $\varepsilon > 0$ such that

$$\xi_i(\varepsilon - d_i) + \sum_{j=1}^{n} \xi_j a_{ji} (1 + b_i M_i) L_i < 0. \tag{23}$$

Consider the Lyapunov function defined by

$$\begin{aligned} V_2(t) = {} & \sum_{i=1}^{n} \xi_i \\ & \times \left\{ e^{\varepsilon t} |y_i(t)| + \sum_{j=1}^{n} L_j a_{ij} b_j \int_0^\infty k_j(s) \left(\int_{t-s}^{t} |y_j(\tau)| e^{\varepsilon(\tau+s)} d\tau \right) ds \right\}. \end{aligned}$$

Calculating the upper right derivative $D^+V_2(t)$ along the solution of system (16), by the condition (i) of Theorem 3, we have

$$\begin{aligned} D^+V_2(t)|_{(16)} = {} & \sum_{i=1}^{n} \xi_i \left\{ \varepsilon e^{\varepsilon t} |y_i(t)| + e^{\varepsilon t} (\mathrm{sgn} y_i(t)) y_i'(t) \right. \\ & \left. + \sum_{j=1}^{n} L_j a_{ij} b_j \int_0^\infty k_j(s) \left[|y_j(t)| e^{\varepsilon(t+s)} - |y_j(t-s)| e^{\varepsilon t} \right] ds \right\}. \end{aligned}$$

Further, by (4), (17) and (i), we obtain

$$\begin{aligned} D^+V_2(t)|_{(16)} = {} & e^{\varepsilon t} \sum_{i=1}^{n} \xi_i \left\{ (\varepsilon - d_i) |y_i(t)| \right. \\ & + \sum_{j=1}^{n} L_j a_{ij} \left| y_j(t) - b_j \int_0^\infty k_j(s) y_j(t-s) ds \right| \\ & \left. + \sum_{j=1}^{n} L_j a_{ij} b_j \int_0^\infty k_j(s) \left[|y_j(t)| e^{\varepsilon s} - |y_j(t-s)| \right] ds \right\} \end{aligned}$$

$$\le e^{\varepsilon t}\sum_{i=1}^{n}\xi_i\Bigg\{(\varepsilon-d_i)\,|y_i(t)| + \sum_{j=1}^{n}L_j a_{ij}\left[|y_j(t)| + b_j\,|y_j(t)|\int_0^{\infty}k_j(s)e^{\varepsilon s}ds\right]\Bigg\}$$

$$< e^{\varepsilon t}\sum_{i=1}^{n}\xi_i\left\{(\varepsilon-d_i)\,|y_i(t)| + \sum_{j=1}^{n}L_j a_{ij}\,(1+b_jM_j)\,|y_j(t)|\right\}$$

$$= e^{\varepsilon t}\sum_{i=1}^{n}\left\{\xi_i(\varepsilon-d_i) + \sum_{j=1}^{n}\xi_j L_i a_{ji}\,(1+b_iM_i)\right\}|y_i(t)| \le \beta e^{\varepsilon t}\sum_{i=1}^{n}|y_i(t)|\,,$$

where

$$\beta = \min_{1\le i\le n}\left\{\xi_i(\varepsilon-d_i) + \sum_{j=1}^{n}\xi_j L_i a_{ji}(1+b_iM_i)\right\},$$

and $\beta < 0$ by (23). So we have $V_2(t) < V_2(0)$ for $t \ge 0$. Since

$$e^{\varepsilon t}\min_{1\le i\le n}\xi_i\sum_{i=1}^{n}|y_i(t)| \le V_2(t), \quad t\ge 0,$$

and

$$V_2(0) = \sum_{i=1}^{n}\xi_i\left\{|y_i(0)| + \sum_{j=1}^{n}L_j a_{ij} b_j\int_0^{\infty}k_j(s)\left(\int_{-s}^{0}|y_j(\tau)|\,e^{\varepsilon(\tau+s)}d\tau\right)ds\right\}$$

$$\le \left\{\max_{1\le i\le n}\xi_i + \sum_{i=1}^{n}\xi_i\left(\sum_{j=1}^{n}L_j b_j a_{ij}\int_0^{\infty}k_j(s)e^{\varepsilon s}\left(\int_{-s}^{0}e^{\varepsilon\tau}d\tau\right)ds\right)\right\}$$

$$\times\,\|y(0)\| \le \left\{\max_{1\le i\le n}\xi_i + \sum_{i=1}^{n}\xi_i\left(\sum_{j=1}^{n}\varepsilon^{-1}L_j b_j a_{ij} M_j\right)\right\}\|y(0)\|\,,$$

where $y(0) = x^* - \phi$, then

$$e^{\varepsilon t}\min_{1\le i\le n}\xi_i\sum_{i=1}^{n}|y_i(t)|$$

$$\leq \left\{ \max_{1\leq i\leq n} \xi_i + \sum_{i=1}^{n} \xi_i \left(\sum_{j=1}^{n} \varepsilon^{-1} L_j b_j a_{ij} M_j \right) \right\} \|x^* - \phi\| ,$$

and so

$$\sum_{i=1}^{n} |y_i(t)| \leq \gamma \|x^* - \phi\| e^{-\varepsilon t}, \tag{24}$$

where

$$\gamma = \frac{1}{\min_{1\leq i\leq n} \xi_i} \left\{ \max_{1\leq i\leq n} \xi_i + \sum_{i=1}^{n} \xi_i \left(\sum_{j=1}^{n} \varepsilon^{-1} L_j b_j a_{ij} M_j \right) \right\} \geq 1$$

is a constant. From (24), we see that the trivial solution of the system (16) is globally exponentially stable, and hence, the equilibrium x^* of the system (7) is globally exponentially stable. The proof is complete. □

Corollary 4. *Assume that (H) ($n = 1$) and (4) hold and that:*

(i) $\int_0^\infty sk(s)e^s ds = M < +\infty$, *and*

(ii) $La(1 + bM) < d$.

Then the unique equilibrium x^* of the system (16) is globally exponentially stable.

4. Two Examples

In this section, we will give two examples to illustrate our results.

Example 1. Consider the following model

$$\left\{ \begin{array}{l} x_1'(t) = -x_1(t) + \frac{1}{3}\tanh\left[x_1(t) - \frac{1}{4}\int_0^\infty e^{-s}x_1(t-s)ds - 1\right] \\ \qquad +\frac{1}{2}\tanh\left[x_2(t) - \frac{1}{3}\int_0^\infty e^{-s}x_2(t-s)ds - 2\right], \\ x_2'(t) = -\frac{3}{4}x_2(t) + \frac{1}{6}\tanh\left[x_1(t) - \frac{1}{4}\int_0^\infty e^{-s}x_1(t-s)ds - 1\right] \\ \qquad +\frac{1}{4}\tanh\left[x_2(t) - \frac{1}{3}\int_0^\infty e^{-s}x_2(t-s)ds - 2\right]. \end{array} \right. \tag{25}$$

Functions $f_j(x) = \tanh x$ satisfies hypotheses (H) with $L_j = 1$ and $k_j(s) = e^{-s}$ satisfy (4) and (5) for $j = 1, 2$. Then

$$DL^{-1} - A(E_2 + B) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{5}{4} & 0 \\ 0 & \frac{4}{3} \end{bmatrix} = \begin{bmatrix} \frac{7}{12} & -\frac{2}{3} \\ -\frac{5}{24} & \frac{5}{12} \end{bmatrix}.$$

It is easy to verify that the matrix $DL^{-1} - A(E_2 + B)$ is an M-matrix. Therefore, by Theorem 2, system (25) has a globally asymptotically stable equilibrium.

Example 2. Consider the following model

$$\begin{cases} x_1'(t) = -x_1(t) + \frac{1}{3}\tanh\left[x_1(t) - \frac{1}{4}\int_0^\infty 2e^{-2s}x_1(t-s)ds - 1\right] \\ \qquad + \frac{1}{4}\tanh\left[x_2(t) - \frac{1}{3}\int_0^\infty 2e^{-2s}x_2(t-s)ds - \frac{3}{2}\right], \\ x_2'(t) = -\frac{5}{6}x_2(t) + \frac{1}{6}\tanh\left[x_1(t) - \frac{1}{4}\int_0^\infty 2e^{-2s}x_1(t-s)ds - \frac{3}{2}\right] \\ \qquad + \frac{1}{3}\tanh\left[x_2(t) - \frac{1}{3}\int_0^\infty 2e^{-2s}x_2(t-s)ds - \frac{3}{2}\right]. \end{cases} \tag{26}$$

Functions $f_j(x) = \tanh x$ satisfies hypotheses (H) with $L_j = 1$ and $k_j(s) = 2e^{-2s}$ satisfy (4) and (i) with $M_j = 2$ for $j = 1, 2$. Then

$$DL^{-1} - A(E_2 + BM) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{5}{6} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{5}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{12} \\ -\frac{1}{4} & \frac{5}{18} \end{bmatrix}.$$

It is easy to verify that the matrix $DL^{-1} - A(E_2 + BM)$ is an M-matrix. Therefore, condition (ii) of Theorem 3 holds. By Theorem 3, system (26) has a globally exponentially stable equilibrium.

Acknowledgements

This work is supported by the Foundation of the Education Department of Gansu Province (048-02).

References

[1] W.Y. Cui, Global stability of a class of neural networks model under dynamical thresholds with delays, *Journal of Biomathematics*, **15**, No. 4 (2000), 420-424, In Chinese.

[2] M. Forti, A. Tesi, New conditions for global stability of neural networks with application to linear and quadratic programming problems, *IEEE Transaction on Circuits and Systems: Fundamental Theory and Application*, **42**, No. 7 (1995), 354-366.

[3] K. Gopalsamy, X.Z. He, Delayed-independent stability in bidirectional associative memory networks, *IEEE Trans. on Neural Networks*, **6**, No. 5 (1994), 998-1002.

[4] K. Gopalsmy, Leung, K.C. Issic, Convergence under dynamical thresholds with delays, *IEEE Trans. on Neural Networks*, **8**, No. 2 (1997), 341-348.

[5] M.W. Hirsch, Convergent activation dynamics in continuous-time networks, *Neural Networks*, **2** (1989), 331-349.

[6] J.J. Hopfield, Neural networks and physical system with emergent collective computation abilities, *Proc. Nat. Acad. Sci. USA.*, **79** (1982), 2554-2558.

[7] J.J. Hopfield, Neurons with graded response have collective computational properties like those of two-state neurons, *Proc. Nat. Acad. Sci. USA.*, **81** (1984), 3088-3092.

[8] H. Huang, J.D. Cao, On global asymptotic stability of recurrent neural networks with time-varying delays, *Applied Mathematics and Computation*, **142** (2003), 143-154.

[9] X.Z. Liu, R. Dickson, Stability analysis of Hopfield neural network with uncertainty, *Mathematical and Computer Modelling*, **34** (2001), 353-363.

[10] F.Y. Zhang, W.T. Li, H.F. Huo, Global stability of a class of delayed cellular neural networks under dynamical thresholds, *International J. Appl. Math.* (2003), To Appear.

[11] J.Y. Zhang, X.Z. Jin, Global stability analysis in delayed Hopfield neural network models, *Neural Networks*, **13** (2000), 745-753.