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A CLASSIFICATION OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: In this paper, a solution of the ordinary second-order differential equation of the form $\ddot{x} + (q_1(t) + q_2(t))x = 0$ is found, in terms of a solution of the simpler ordinary differential equation $\ddot{x} + q_1(t)x = 0$. Several forms of the first equation are generated through a suggested procedure. It is observed that their solutions have some value in mathematical physics and engineering applications see Takayama [3], Arfken [1], and Laham [2].

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1. Introduction

Takayama [1] discussed the problem of finding the solution of the ordinary differential equation

$$\ddot{y} + (q_1(t) + q_2(t))y = 0 \tag{1.1}$$

in terms of the the solution of the simpler ordinary differential equation:

$$\ddot{y} + q_1(t)y = 0\,. \tag{1.2}$$

Under the condition that $\ddot{\phi} - \frac{2\dot{\psi}\dot{\phi}}{\psi} = 0$ and $q_2(t) = \dot{\phi}^2$, Takayama obtained a

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particular class of equation (1.1) in the form:

$$\ddot{y} + (q_1(t) + \frac{1}{\psi^4})y = 0, \tag{1.3}$$

whose solution $y = \psi e^{\pm i \int \frac{dt}{\psi^2}}$ is given in terms of the solution ψ of equation (1.2); $\ddot{\psi} + q_1(t)\psi(t) = 0$.

In this paper, we present different classes of ordinary differential equations and their solutions. Also, we will concerned only with a particular solution of equation (1.1) as it is well known that a second independent solution could always be found. To demonstrate the work we give examples that provide the solution of equations that appear in various applications.

2. Classifications

In this section, new classes of differential equations are obtained. Takayama [1] obtained the solution of his particular class of differential equation in an exponential form. To obtain more classes, we assume the solution of (1.1) in the form:

$$y = \psi e^{\phi}, \tag{2.1}$$

where ψ is a solution of equation (1.2) and $\phi(t)$ is a twice differentiable function in the domain of interest. Thus,

$$\dot{y} = \dot{\psi} e^{\phi} + \psi \dot{\phi} e^{\phi},$$

and

$$\ddot{y} = \ddot{\psi} e^{\phi} + 2\dot{\psi}\dot{\phi} e^{\phi} + \ddot{\phi}\psi e^{\phi} + (\dot{\phi})^2 \psi e^{\phi}. \tag{2.2}$$

Using equations (2.1) and (1.2), equation (2.2) can be written as

$$\ddot{y} + (q_1(t) - \ddot{\phi} - \dot{\phi}^2 - \frac{2\dot{\psi}\dot{\phi}}{\psi})y = 0, \tag{2.3}$$

which is in the form of (1.1). Hence $y = \psi e^{\phi}$ is a solution of (1.1) with

$$q_2(t) = -\ddot{\phi} - \dot{\phi}^2 - \frac{2\dot{\psi}\dot{\phi}}{\psi}. \tag{2.4}$$

According to equation (2.4), we have many choices. Each choice generate a new class of differential equations as in the following cases. Takayama [1] assumes that $\ddot{\phi} - \frac{2\dot{\psi}\dot{\phi}}{\psi} = 0$ and $q_2(t) = -\dot{\phi}^2$ see (Case 6).

Case 1. Choose $\ddot{\phi} = -\dot{\phi}^2$, that is $\phi(t)$ =log$t$. Hence equation (2.4) gives

$$q_2(t) = \frac{-2\dot{\psi}}{t\psi}. \tag{2.5}$$

Therefore, by (2.3) the class of generated equations has the form

$$\ddot{y} + (q_1(t) - \frac{2\dot{\psi}}{t\psi})y = 0, \tag{2.6}$$

which is by (2.1) has a solution of the form:

$$y = \psi e^{\phi} = t\psi\,. \tag{2.7}$$

Example 3.1. The equation $\ddot{\psi} - \alpha^2\psi = 0$ has $q_1(t) = -\alpha^2$ and $\psi(t) = e^{\alpha t}$, hence equations (2.5) and (2.6) give $q_2(t) = \frac{-2\alpha}{t}$ and the class of generated equations is:

$$\ddot{y} - (\alpha^2 + \frac{2\alpha}{t})y = 0,$$

which is by (2.7) has a solution of the form $y = te^{\alpha t}$. In general, the solution of the following class of generated equations:

$$\ddot{y} - (\alpha^2 + \frac{2n\alpha}{t} + \frac{n(n-1)}{t^2})y = 0,$$

is given by $y = t^n e^{\alpha t}$.

Case 2. Choose $\frac{2\dot{\phi}\dot{\psi}}{\psi} = -\dot{\phi}^2$, that is $\phi = log(\frac{1}{\psi^2})$. Then equation (2.4) gives that:

$$q_2(t) = -\ddot{\phi} = \frac{2\ddot{\psi}}{\psi} - 2(\frac{\dot{\psi}}{\psi})^2. \tag{2.8}$$

Thus by equation (2.3), we generate the class of equations

$$\ddot{y} + (q_1(t) + \frac{2\ddot{\psi}}{\psi} - (\frac{\dot{\psi}}{\psi})^2)y = 0, \tag{2.9}$$

which is by (2.1) has the solution of the form:

$$y = \psi e^{\phi} = \frac{1}{\psi}\,. \tag{2.10}$$

Example 3.2. The equation $\ddot{\psi} + \alpha^2\psi = 0$ has $q_1(t) = \alpha^2$ and $\psi(t) = \cos(\alpha t)$, therefore equation (2.8) gives $q_2(t) = -2\alpha^2\sec^2(\alpha t)$, and by (2.9) the class of generated equations is:

$$\ddot{y} + \alpha^2(1 - 2\sec^2(\alpha t))y = 0,$$

which is by (2.10) has the solution $y = \sec(\alpha t)$.

Case 3. Let $\phi(t) = t^m$. Hence equation (2.4) gives that

$$q_2(t) = -m(m-1)t^{m-2} - m^2 t^{2m-2} - 2mt^{m-1}\frac{\dot{\psi}}{\psi}. \tag{2.11}$$

Therefore by (2.3), we have the class of generated equations of the form:

$$\ddot{y} + (q_1(t) - mt^{m-1}(\frac{m-1}{t} + mt^{m-1} + \frac{2\dot{\psi}}{\psi}))y = 0, \tag{2.12}$$

which is by (2.1) has the solution of the form:

$$y = \psi e^{t^m}. \tag{2.13}$$

Example 3.3. If $\ddot{\psi} = 0$ then $q_1(t) = 0$ and $\psi(t) = t$. Therefore, equation (2.12) gives the class of generated equations:

$$\ddot{y} - mt^{m-1}(\frac{m-1}{t} + mt^{m-1} + \frac{2}{t})y = 0\,,$$

which is by (2.13) has the solution $y = te^{t^m}$. When $m = 1$, we see that the solution of the equation:

$$\ddot{y} - (1 + \frac{2}{t})y = 0,$$

has the form $y = te^t$. When $m = -1$, the solution of the generated equation

$$\ddot{y} - \frac{1}{t^4}y = 0,$$

is given by $te^{\frac{1}{t}}$.

Case 4. Let $\phi = \cos(t^n)$. Then by (2.4),

$$q_2(t) = n^2 t^{2n-2}\cos t^n + n(n-1)t^{n-2}\sin t^n - n^2 t^{2n-2}\sin^2 t^n + 2n(\frac{\dot{\psi}}{\psi})t^{n-1}\sin t^n. \tag{2.14}$$

Therefore, (2.3) gives the class of generated equations:

$$\ddot{y} + (q_1(t) + n^2 t^{2n-2}(\cos t^n - \sin^2 t^n) + n(n-1)t^{n-2}\sin t^n + 2n(\frac{\dot{\psi}}{\psi})t^{n-1}\sin t^n)y = 0, \tag{2.15}$$

which is by (2.1) has the solution of the form:

$$y = \psi e^{\cos t^n}. \tag{2.16}$$

Example 3.4. The equation $\ddot{\psi} - \frac{m(m-1)}{t^2}\psi = 0$ has $q_1(t) = -\frac{m(m-1)}{t^2}$ and $\psi(t) = t^m$. Equation (2.15) gives the class of generated equations:

$$\ddot{y} + (-\frac{m(m-1)}{t^2} + n^2 t^{2n-2}(\cos t^n - \sin^2 t^n) + n(n+2m-1)t^{n-2}\sin t^n)y = 0,$$

which is by (2.16) has the solution $y = t^m e^{\cos t^n}$. When $n = 1$, we see that the class of equations:

$$\ddot{y} + (-\frac{m(m-1)}{t^2} + \cos(t) - \sin^2(t) + (\frac{2m}{t})\sin(t))y = 0$$

has the solution $y = t^m e^{\cos(t)}$.

Case 5. In this case let $\phi = n \log t$. Equation (2.4) gives

$$q_2(t) = -\frac{n(n-1)}{t^2} - \frac{2n}{t}(\frac{\dot{\psi}}{\psi}). \tag{2.17}$$

Thus by equation (2.3), we obtain the class of generated equations:

$$\ddot{y} + (q_1(t) - \frac{n(n-1)}{t^2} - \frac{2n}{t}(\frac{\dot{\psi}}{\psi}))y = 0, \tag{2.18}$$

which is by (2.1) has the solution

$$y = t^n \psi. \tag{2.19}$$

Example 3.5. The equation $\ddot{\psi} + (2m+1-t^2)\psi = 0$ has $q_1(t) = 2m+1-t^2$ and $\psi(t) = H_m(t)e^{-\frac{1}{2}t^2}$, where $H_m(t)$ is the Hermite polynomial of degree m. Equations (2.17)-(2.19) give $q_2(t) = 2n - \frac{n(n-1)}{t^2} - \frac{4nm}{t}\frac{H_{m-1}(t)}{H_m(t)}$ and the solution of the generated equations:

$$\ddot{y} + (1 + 2(n+m) - t^2 - \frac{n(n-1)}{t^2} - \frac{n(n-1)}{t^2} - \frac{4nm}{t}\frac{H_{m-1}(t)}{H_m(t)})y = 0,$$

is $y = t^n H_m e^{-\frac{1}{2}t^2}$, where we have used the well relationship $\frac{d}{dt}H_m(t) = 2mH_{m-1}(t)$. When $m = 1$, we see that the solution of the equation:

$$\ddot{y} + (2n + 3 - t^2 - \frac{n(n-5)}{t^2})y = 0,$$

is given by $y = 2t^{n+1}e^{-\frac{1}{2}t^2}$.

Case 6. In this case we will show that the special case given by (1.3) corresponding to the choice $\dot{\phi} = \frac{\pm i}{\psi^2}$. Then equation (2.4) gives $q_2(t) = \frac{1}{\psi^4}$. This satisfies the conditions $(\ddot{\phi} - \frac{\pm 2\dot{\phi}\dot{\psi}}{\psi} = 0, q_2 = -(\dot{\phi})^2)$. Hence equation (2.3) gives the class of generated equations:

$$\ddot{y} + (q_1(t) + \frac{1}{\psi^4})y = 0, \tag{2.20}$$

which is by (2.1) has the solution of the form:

$$y = \psi e^{\pm i \int \frac{dt}{\psi^2}}, \tag{2.21}$$

Example 3.6. The equation $\ddot{\psi} - \alpha^2\psi = 0$ has $q_1(t) = -\alpha^2$ and $\psi(t) = e^{\alpha t}$. Hence equations (2.20) – (2.21) give $q_2(t) = e^{-4\alpha t}$ and the class of generated equations is of the form:

$$\ddot{y} - (\alpha^2 - e^{-4\alpha t})y = 0,$$

whose solution is of the form $y = e^{\alpha t}e^{\mp \int \frac{idt}{\psi^2}}$.

Remark. Observe from the examples above that the initial equation (1.2) generate different classes of differential equations.

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