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THE NONLINEAR SCHRÖDINGER'S EQUATION AND SOLITON-SOLITON INTERACTION

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Abstract: The intra-channel collision of optical solitons, with non-Kerr law nonlinearities, that is governed by the nonlinear Schrödinger's equation, is studied in this paper by the aid of quasi-particle theory. The perturbations terms that are considered in this paper are both of Hamiltonian as well as non-Hamiltonian type. The suppression of soliton-soliton interaction, in presence of these perturbation terms, is acheived. The nonlinearities that are studied in this paper are Kerr, power, parabolic and dual-power laws. The numerical simulations support the quasi-particle theory.

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1. Introduction

The theoretical possibility of existence of optical solitons in a dielectric dispersive

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fiber was first predicted by Hasegawa and Tappert [12]. A couple of years later Mollenauer et al [12] succesfully performed the famous experiment to verify this prediction. Important characterestic properties of these solitons are that they posses a localized waveform which remains intact upon interaction with another soliton. Because of their remarkable robustness, they attracted enormous interest in optical and telecommunication community. At present optical solitons are regarded as the natural data bits for transmission and processing of information in future, and an important alternative for the next generation of ultra high speed optical communication systems.

The fundamental mechanism of soliton formation namely the balanced interplay of linear group velocity dispersion (GVD) and nonlinearity induced self-phase modulation (SPM) is well understood. In the pico second regime, the nonlinear evolution equation that takes into account this interplay of GVD and SPM and which describes the dynamics of soliton is the well known nonlinear Schrödinger's equation (NLSE). The NLSE, which is the ideal equation in an ideal Kerr media, is in its original form found to be completely integrable by the method of Inverse Scattering Transform (IST) and tremendous success has been acheived in the development of soliton theory in the framework of the NLSE model.

However, communication grade optical fibers or as a matter of fact any optical transmitting medium does posses finite attenuation coefficient, thus optical loss in inevitable and the pulse is often detoriated by this loss. Therefore, optical amplifiers have to be employed to compensate for this loss. When the gain bandwidth of the amplifier is comparable to the spectral width of the ultrashort optical pulse, the frequency and intensity dependent gain must be considered. Another hindrance to the stable propagation in a practical system is the noise induced Gordon-Hauss timing jitter. An important aspect that has not been addressed with proper perspective is the fact that due to its nonsaturable nature, Kerr nonlinearity is inadequate to describe the soliton dynamics in the ultrahigh bit rate transmission. For example, when transmission bit rate is very high, for soliton formation the peak power of the incident field accordingly become very large. On the other hand higher order nonlinearities may become significant even at moderate intensities in certain materials such as semiconductor doped glass fibers. Under circumstances, as mentioned above, non-Kerr law nonlinearities come into play changing essentially the physical features of optical soliton propagation. Therefore when very high bit rate transmission or transmission through materials with higher nonlinear coefficients are considered, it is necessary to take into account higher order nonlinearities. This problem can be addressed by incorporating various non-Kerr law nonlinearities in the NLSE.

It has been realized that the Gordon-Haus timing jitter can be reduced by introducing bandpass filtering. Stabilazition of soliton propgation with the aid of nonlinear gain or under combined operation of gain and saturable absorption was recommended by Kodama et al [20-24]. Thus, in order to model these features in the soliton dynamics, in a practical situation, the NLSE should be modified by incorporating additional terms. Thus, the concept of control of soliton propagation described by the NLSE with non-Kerr law nonlinearities is new and important developments in the application of solitons for optical communication systems. Because the NLSE, with non-Kerr laws, is not integrable, perturbation methods or numerical techniques have to be applied. Therefore, the control of soliton and interction of two neighboring solitons incorporating perturbation terms like nonlinear gain, saturable amplification, filtering, higher order dispersion, self-steepening and nonlinear dispersions are going to be addressed in this paper.

The dimensionless form of the generalized NLSE is given by

$$iq_Z + \frac{1}{2}q_{TT} + F\left(|q|^2\right)q = 0\,, \tag{1}$$

where Z represents the nondimensional distance along the fiber while, T represents time in dimensionless form. Also, in (1), F is a real-valued algebraic function and it is necessary to have the smoothness of the complex function $F\left(|q|^2\right)q : C \mapsto C$. Considering the complex plane C as a two-dimensional linear space R^2, the function $F\left(|q|^2\right)q$ is k times continously differentiable, so that

$$F\left(|q|^2\right)q \in \bigcup_{m,n=1}^{\infty} C^k\left((-n,n)\times(-m,m);R^2\right)\,. \tag{2}$$

Equation (1) is a nonlinear partial differential equation (PDE) of parabolic type that is not integrable, in general. The special case, $F(s) = s$, also known as Kerr law of nonlinearity, is integrable by the method of Inverse Scattering Transform (IST) [11]. The IST is the nonlinear analog of Fourier transform that is used for solving the linear partial differential equations. Schematically, the IST and the technique of Fourier transform are similar [11]. The solutions are known as *solitons*.

The general case $F(s) \neq s$ takes (1) away from the IST picture as it is not of Painleve type [2]. In the anomalous dispersion regime [7], the particularly relevant solutions to (1) are called solitons, or nontopological solitons. In a rigorous sense, the pulses of the non-integrable systems are not solitons. However, the term solitons has been used broadly for the solutions of the nonintegrable system as well, and this has become common. So, in this paper, the pulses shall be refered to as 'solitons'. Although stationary pulses exist, and some solutions can be written in the analytic form, their behaviour is different from that of the solutions of the cubic NLSE.

In most cases, the interest is confined to a single pulse described by the 1-soliton solution of the NLSE. However, in this paper, the effects of the perturbation terms in NLSE on two soliton interaction will be studied. It is necessary to launch the solitons close to each other for enhancing the information carrying capacity of the fiber. If two solitons are placed close to each other then it can lead to their mutual interaction thus providing a very serious hinderance to the performance of the soliton transmission system. However, the presence of the perturbation terms of the NLSE can lead to the suppression of the two soliton interaction thus solving our problem.

2. Mathematical Formulation

The perturbed NLSE that is going to be studied in this paper for the SSI is

$$i\frac{\partial q}{\partial Z}+\frac{1}{2}\frac{\partial^2 q}{\partial T^2}+F\left(|q|^2\right)q=i\epsilon R[q,q^*]\,, \tag{3}$$

where

$$\begin{aligned}R=\delta\,|q|^{2m}\,q+\sigma q\int_{-\infty}^{T}|q|^2\,dt-\alpha\frac{\partial q}{\partial T}+\beta\frac{\partial^2 q}{\partial T^2}&\\ -\gamma\frac{\partial^3 q}{\partial T^3}-i\sigma\frac{\partial^4 q}{\partial T^4}+\lambda\frac{\partial}{\partial T}\left(|q|^2 q\right)+\mu q\frac{\partial}{\partial T}\left(|q|^2\right)\,.&\end{aligned} \tag{4}$$

In fiber optics ϵ is called the relative width of the spectrum, that arises due to quasi-monochromaticity, and is assumed that $0<\epsilon\ll 1$. For the perturbation terms given by R in (4), δ represents the coefficient of nonlinear gain, while σ is the coefficient of saturable amplifiers. Also, in (3), m could be 0, 1 or 2. For $m=0$, there is linear gain, while for $m=1$ it is called quadratic gain and for $m=2$, it is quintic gain or gain saturation. The coefficient of β is called the bandpass filtering term. Also, in (4), α is the frequency separation between the soliton carrier and the frequency at the peak of EDFA gain. Moreover, λ is the self-steepening coefficient for short pulses [3, 4, 11, 25] (typically ≤ 100 femto seconds), μ is the higher order dispersion coefficient [11, 25] and γ is the coefficient of the third order dispersion [11, 19, 25]. Finally, σ represents the coefficient of fourth order dispersion. It needs to be noted that the coefficients of δ, β and σ represents the non-Hamiltonian perturbation while the remaining terms in (4) represents the Hamiltonian type perturbation.

It is known that the NLSE, as given by (1), does not give correct prediction for pulse widths smaller than 1 picosecond. For example, in solid state solitary lasers, where pulses as short as 10 femtoseconds are generated, the approximation breaks down. Thus, quasi-monochromaticity is no longer valid and higher order dispersion terms come in. If the group velocity dispersion is close to zero, one needs to consider the third order dispersion for performance enhancement along trans-oceanic distances. Also, for short pulse widths, where group velocity dispersion changes within the spectral bandwidth of the signal can no longer be neglected, one needs to take into account the presence of the third order dispersion.

Equation (1), unlike the Kerr law case, does not have infinitely many conserved quantities. In fact, it has as few as three integrals of motion [3, 4, 5]. They are the energy (E) also known as the wave power, linear momentum (M) and the Hamiltonian (H) that are respectively given by

$$E = \int_{-\infty}^{\infty} |q|^2 dT\,, \tag{5}$$

$$M = \frac{i}{2}\int_{-\infty}^{\infty} (q^* q_T - q q_T^*) dT\,, \tag{6}$$

$$H = \int_{-\infty}^{\infty} \left[\frac{1}{2}|q_T|^2 - f(I)\right] dT\,, \tag{7}$$

where

$$f(I) = \int_0^I F(\xi) d\xi \tag{8}$$

and the intensity I is given by $I = |q|^2$. One can see that (1) can be written in a canonical form

$$iq_Z = \frac{\delta H}{\delta q^*}\,, \tag{9}$$

$$iq_Z^* = -\frac{\delta H}{\delta q}\,. \tag{10}$$

This defines a Hamiltonian dynamical system on an infinite dimensional phase space. It can be analyzed using the theory of Hamiltonian systems. This means that a behaviour of the solution is defined, to a large extent, by the singular points of the system, namely the stationary solutions of (1) and depends on the nature of these points as determined by the stability of its stationary solutions [3].

The quasi-particle theory (QPT) of soliton-soliton interaction (SSI) has been investigated [2, 3, 7, 8] and it is proved by virtue of it that the interaction can be supressed due to nonlinear gain and filters. Also it has been proved that the sliding frequency guiding filters [8, 9] leads to the suppression of the SSI. Here, by virtue of the QPT, it will be proved that SSI can be suppressed due to the NLSE given by (1) and also in presence of the perturbation terms in (2), for Kerr, power, parabolic and dual-power laws.

The soliton solution of (3), for $\epsilon = 0$, although not integrable, for any law of nonlinearity, is assumed to be given in the form

$$q(Z,T) = \eta(Z) g\left[\zeta(Z)\left(T - vZ - T_0\right)\right] e^{(-i\kappa T + i\omega Z + i\sigma_0)}\,, \tag{11}$$

where

$$\kappa = -v\,, \tag{12}$$

$$\zeta(Z) = \chi\left(\eta(Z)\right)\,, \tag{13}$$

$$\omega(Z) = \psi\left(\eta(Z), \kappa(Z)\right)\,. \tag{14}$$

In (11), g represents the shape of the soliton described by the GNLSE and it depends on the type of nonlinearity in (1). The parameters $\eta(Z)$ and $\zeta(Z)$, in (11), respectively represent the soliton amplitude and width, while $\kappa(Z)$ and $\omega(Z)$ are the frequency and wave number of the soliton respectively while v is the velocity. Also T_0 and σ_0 respectively represent the center of the soliton and center of soliton phase. In (13) and (14), the functional forms, χ and ψ, depend on the type of nonlinearity in (1).

The 2-soliton solution of the NLSE, given by (1), takes the asymptotic form

$$q(Z,T) = \sum_{l=1}^{2} \eta_l(Z) g\left[\zeta_l(Z)\left(T - v_l Z - T_l\right)\right] e^{(-i\kappa_l T + i\omega_l Z + i\sigma_l)}\,, \tag{15}$$

with

$$\kappa_l = -v_l\,, \tag{16}$$

$$\zeta_l(Z) = \chi\left(\eta_l(Z)\right) , \tag{17}$$

$$\omega_l(Z) = \psi\left(\eta_l(Z), \kappa_l(Z)\right) , \tag{18}$$

where $l = 1, 2$. In the study of SSI, the initial pulse waveform is taken to be of the form

$$q(0,T) = \eta_1 g\left[\zeta_1\left(T - \frac{T_0}{2}\right)\right] e^{i\phi_1} + \eta_2 g\left[\zeta_2\left(T + \frac{T_0}{2}\right)\right] e^{i\phi_2} , \tag{19}$$

which represents the injection of 2-soliton like pulses into a fiber. Here T_0 represents the initial separation of the solitons namely the center-to-center soliton separation. It is to be noted that for $T_0 \to \infty$ (15) represents exact soliton solutions, but for $T_0 \sim O(1)$ it does not represent an exact 2-soliton solution. The initial pulse form will modify depending on the type of perturbation considered as seen below.

Non-Hamiltonian Perturbations. For studying the SSI with non-Hamiltonian type perturbations, the case of in-phase injection of solitons with equal amplitudes will be considered. So, without any loss of generality, $\eta_1 = \eta_2 = 1$ and $\phi_1 = \phi_2 = 0$ is chosen so that (15) modifies to

$$q(0,T) = g\left[\zeta_1\left(T - \frac{T_0}{2}\right)\right] + g\left[\zeta_2\left(T + \frac{T_0}{2}\right)\right] . \tag{20}$$

Hamiltonian Perturbations. For studying the SSI with Hamiltonian type perturbations, the case of in-phase injection of solitons with unequal amplitudes will be considered. So without any loss of generality $\eta_1 = \eta_0$, $\eta_2 = 1$ and $\phi_1 = \phi_2 = 0$ is chosen so that (15) modifies to

$$q(0,T) = \eta_0 g\left[\zeta_0\left(T - \frac{T_0}{2}\right)\right] + g\left[\zeta\left(T + \frac{T_0}{2}\right)\right] , \tag{21}$$

where

$$\zeta_0 = \chi(\eta_0) , \tag{22}$$

and ζ is given by (7).

The special cases with regards to the four laws of nonlinearity will now be individually discussed in the following four subsections.

2.1. Kerr Law

For the case of Kerr law of nonlinearity $F(s) = s$ so that equation (3) becomes

$$i\frac{\partial q}{\partial Z} + \frac{1}{2}\frac{\partial^2 q}{\partial T^2} + |q|^2 q = i\epsilon R[q, q^*]\,. \tag{23}$$

The 1-soliton solution of (15), for $\epsilon = 0$, that can be obtained by IST has the form [1, 2]

$$q(Z,T) = \frac{\eta}{\cosh\left[\zeta(T - vZ - T_0)\right]} e^{i(-\kappa T + \omega Z + \sigma_0)}\,, \tag{24}$$

where

$$\zeta \equiv \chi(\eta) = \eta\,, \tag{25}$$

$$\omega \equiv \psi(\eta, \kappa) = \frac{\eta^2 - \kappa^2}{2}\,. \tag{26}$$

Also, the 2-soliton solution of the NLSE (1) takes the asymptotic form [2]

$$q(Z,T) = \sum_{l=1}^{2} \frac{\eta_l}{\cosh\left[\eta_l(T - v_l Z - T_{0_l})\right]} e^{i\left(-\kappa_l T + \omega_l Z + \sigma_{0_l}\right)}\,, \tag{27}$$

where

$$\zeta_l \equiv \chi(\eta_l) = \eta_l\,, \tag{28}$$

$$\omega_l \equiv \psi(\eta_l, \kappa_l) = \frac{\eta_l^2 - \kappa_l^2}{2} \tag{29}$$

and $l = 1, 2$. In the study of SSI, with non-Hamiltonian perturbations, the initial pulse form is taken to be

$$q(0,T) = \frac{\eta_1}{\cosh\left[\eta_1\left(T - \frac{T_0}{2}\right)\right]} e^{i\phi_1} + \frac{\eta_2}{\cosh\left[\eta_2\left(T + \frac{T_0}{2}\right)\right]} e^{i\phi_2}\,. \tag{30}$$

For *non-Hamiltonian* type perturbation, the choice $\eta_1 = \eta_2 = 1$ and $\phi_1 = \phi_2 = 0$, gives

$$q(0,T) = \frac{1}{\cosh\left[\left(T - \frac{T_0}{2}\right)\right]} + \frac{1}{\cosh\left(T + \frac{T_0}{2}\right)}\,, \tag{31}$$

which represents an in-phase injection of pulses with equal amplitudes.

For *Hamiltonian* type perturbations, the choice $\eta_1 = \eta_0$, $\eta_2 = 1$ and $\phi_1 = \phi_2 = 0$, gives

$$q(0,T) = \frac{\eta_0}{\cosh\left[\eta_0\left(T - \frac{T_0}{2}\right)\right]} + \frac{1}{\cosh\left(T + \frac{T_0}{2}\right)}\,, \tag{32}$$

which represents an in-phase injection of pulses with unequal amplitudes.

2.2. Power Law

For the case of power law nonlinearity, $F(s) = s^p$, where $0 < p < 2$ to prevent wave collapse. So equation (3) modifies to

$$i\frac{\partial q}{\partial Z} + \frac{1}{2}\frac{\partial^2 q}{\partial T^2} + |q|^{2p} q = i\epsilon R[q, q^*]\,. \tag{33}$$

In this case, equation (33) with $\epsilon = 0$ is not integrable by IST. However, (33), for $\epsilon = 0$, supports solitons of the form [3]

$$q(Z,T) = \frac{\eta}{\cosh^{\frac{1}{p}}\left[\zeta(T - vZ - T_0)\right]} e^{i(-\kappa T + \omega Z + \sigma_0)}\,, \tag{34}$$

where

$$\zeta \equiv \chi(\eta) = \eta^p \left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}}\,, \tag{35}$$

$$\omega \equiv \psi(\eta, \kappa) = \frac{\zeta^2}{2p^2} - \frac{\kappa^2}{2}\,. \tag{36}$$

The 2-soliton solution of the NLSE (1) takes the asymptotic form

$$q(Z,T) = \sum_{l=1}^{2} \frac{\eta_l}{\cosh^{\frac{1}{p}}\left[\zeta_l(T - v_l Z - T_{0_l})\right]} e^{i\left(-\kappa_l T + \omega_l Z + \sigma_{0_l}\right)}\,, \tag{37}$$

where

$$\zeta_l \equiv \chi(\eta_l) = \eta_l^p \left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}}\,, \tag{38}$$

$$\omega_l \equiv \psi(\eta_l, \kappa_l) = \frac{\zeta_l^2}{2p^2} - \frac{\kappa_l^2}{2}\,. \tag{39}$$

In the study of SSI for power law, the initial pulse waveform is assumed to be

$$q(0,T) = \frac{\eta_1}{\cosh^{\frac{1}{p}}\left[\zeta_1\left(T - \frac{T_0}{2}\right)\right]} e^{i\phi_1} + \frac{\eta_2}{\cosh^{\frac{1}{p}}\left[\zeta_2\left(T + \frac{T_0}{2}\right)\right]} e^{i\phi_2}\,. \tag{40}$$

For *non-Hamiltonian* type perturbation, the choice $\eta_1 = \eta_2 = 1$ and $\phi_1 = \phi_2 = 0$, gives

$$q(0,T) = \frac{1}{\cosh^{\frac{1}{p}}\left[\zeta\left(T - \frac{T_0}{2}\right)\right]} + \frac{1}{\cosh^{\frac{1}{p}}\left[\zeta\left(T + \frac{T_0}{2}\right)\right]}\,, \tag{41}$$

where

$$\zeta = \sqrt{\frac{2p^2}{1+p}}\,, \tag{42}$$

which represents an in-phase injection of pulses with equal amplitudes.

For *Hamiltonian* type perturbations, the choice $\eta_1 = \eta_0$, $\eta_2 = 1$ and $\phi_1 = \phi_2 = 0$, gives

$$q(0,T) = \frac{\eta_0}{\cosh^{\frac{1}{p}}\left[\zeta_0\left(T - \frac{T_0}{2}\right)\right]} + \frac{1}{\cosh^{\frac{1}{p}}\left[\zeta\left(T + \frac{T_0}{2}\right)\right]}\,, \tag{43}$$

where

$$\zeta_0 = \eta_0^p \left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}}\,, \tag{44}$$

which represents in-phase injection of solitons with unequal amplitudes.

2.3. Parabolic Law

For the parabolic law of nonlinearity, $F(s) = s + \nu s^2$, where the parameter $\nu > 0$, and so the NLSE is

$$i\frac{\partial q}{\partial Z} + \frac{1}{2}\frac{\partial^2 q}{\partial T^2} + \left(|q|^2 + \nu|q|^4\right) q = i\epsilon R[q, q^*]\,. \tag{45}$$

Equation (45) is not integrable by the IST. However, (45), for $\epsilon = 0$ supports solitons of the form [3]

$$q(Z,T) = \frac{\eta}{\left[1 + a\cosh\left\{\zeta\left(T - vZ - T_0\right)\right\}\right]^{\frac{1}{2}}} e^{i(-\kappa T + \omega Z + \sigma_0)}\,, \tag{46}$$

where

$$\zeta \equiv \chi(\eta) = \eta\sqrt{2}\,, \tag{47}$$

$$\omega \equiv \psi(\eta,\kappa) = \frac{\eta^2}{4} - \frac{\kappa^2}{2}\,, \tag{48}$$

$$a = \sqrt{1 + \frac{4}{3}\nu\eta^2}\,. \tag{49}$$

In this paper it will be assumed that $\nu > 0$ although ν could be negative as well. In fact, it is to be noted that for (45), for $\epsilon = 0$, solitons exist for $\nu \in (-3/4A^2, \infty)$. Also, the 2-soliton solution of the parabolic law takes the asymptotic form [2]

$$q(Z,T) = \sum_{l=1}^{2} \frac{\eta_l}{\left[1 + a_l \cosh\left\{\zeta_l\left(T - vZ - T_l\right)\right\}\right]^{\frac{1}{2}}} e^{i\left(-\kappa_l T + \omega_l Z + \sigma_{0_l}\right)}\,, \tag{50}$$

with

$$\zeta_l \equiv \chi(\eta_l) = \eta_l\sqrt{2}\,, \tag{51}$$

$$\omega_l \equiv \psi(\eta_l,\kappa_l) = \frac{\eta_l^2 - 2\kappa_l^2}{4}\,, \tag{52}$$

$$a_l = \sqrt{1 + \frac{4}{3}\nu\eta_l^2}\,. \tag{53}$$

In the study of SSI, with parabolic law nonlinearity, the initial pulse waveform is taken to be of the form

$$q(0,T) = \frac{\eta_1}{\left[1 + a_1 \cosh\left\{\zeta_1\left(T - \frac{T_0}{2}\right)\right\}\right]^{\frac{1}{2}}} e^{i\phi_1} + \frac{\eta_2}{\left[1 + a_2 \cosh\left\{\zeta_2\left(T + \frac{T_0}{2}\right)\right\}\right]^{\frac{1}{2}}} e^{i\phi_2}\,. \tag{54}$$

For *non-Hamiltonian* type perturbation, the choice $\eta_1 = \eta_2 = 1$ and $\phi_1 = \phi_2 = 0$, gives

$$q(0,T)=\frac{1}{\left[1+\sqrt{1+\frac{4}{3}\nu}\cosh\left\{\sqrt{2}\left(T-\frac{T_0}{2}\right)\right\}\right]^{\frac{1}{2}}} + \frac{1}{\left[1+\sqrt{1+\frac{4}{3}\nu}\cosh\left\{\sqrt{2}\left(T+\frac{T_0}{2}\right)\right\}\right]^{\frac{1}{2}}}\,. \tag{55}$$

For *Hamiltonian* type perturbations, the choice $\eta_1=\eta_0$, $\eta_2=1$ and $\phi_1=\phi_2=0$, gives

$$q(0,T)=\frac{\eta_0}{\left[1+\sqrt{1+\frac{4}{3}\nu\eta_0}\cosh\left\{\eta_0\sqrt{2}\left(T-\frac{T_0}{2}\right)\right\}\right]^{\frac{1}{2}}} + \frac{1}{\left[1+\sqrt{1+\frac{4}{3}\nu}\cosh\left\{\sqrt{2}\left(T+\frac{T_0}{2}\right)\right\}\right]^{\frac{1}{2}}}\,, \tag{56}$$

which represents an in-phase injection of pulses with unequal amplitudes.

2.4. Dual-Power Law

For the dual-power law nonlinearity, $F(s)=s^p+\nu s^{2p}$, where in this case, $\nu<0$ so that the NLSE is

$$i\frac{\partial q}{\partial Z}+\frac{1}{2}\frac{\partial^2 q}{\partial T^2}+\left(|q|^{2p}+\nu|q|^{4p}\right)q=i\epsilon R[q,q^*]\,. \tag{57}$$

Equation (57), for $\epsilon=0$, is not integrable by IST. However, it supports solitary waves of the form

$$q(Z,T)=\frac{\eta}{\left[1+a\cosh\left\{\zeta\left(T-vZ-T_0\right)\right\}\right]^{\frac{1}{2p}}}e^{i(-\kappa T+\omega Z+\sigma_0)}\,, \tag{58}$$

where

$$\zeta\equiv\chi(\eta)=\eta^p\left(\frac{2p^2}{1+p}\right)^{\frac{1}{2p}}\,, \tag{59}$$

$$\omega\equiv\psi(\eta,\kappa)=\frac{\eta^{2p}}{2p+2}-\frac{\kappa^2}{2}\,, \tag{60}$$

$$a=\sqrt{1+\frac{\nu\zeta^2}{2p^2}\frac{(1+p)^2}{1+2p}}\,. \tag{61}$$

For dual-power law nonlinearity, solitons exist for

$$-\frac{2p^2}{\zeta^2}\frac{1+2p}{(1+p)^2} < \nu < 0\,. \tag{62}$$

In this case, the 2-soliton solution of the NLSE (1) takes the asymptotic form

$$q(Z,T) = \sum_{l=1}^{2} \frac{\eta_l}{\left[1 + a_l \cosh\left\{\zeta_l\left(T - vZ - T_l\right)\right\}\right]^{\frac{1}{2p}}} e^{i\left(-\kappa_l T + \omega_l Z + \sigma_{0_l}\right)}\,, \tag{63}$$

with

$$\zeta_l \equiv \chi(\eta_l) = \eta_l^p \left(\frac{2p^2}{1+p}\right)^{\frac{1}{2p}}\,, \tag{64}$$

$$\omega_l \equiv \psi(\eta_l, \kappa_l) = \frac{\eta_l^{2p}}{2p+2} - \frac{\kappa_l^2}{2}\,, \tag{65}$$

$$a_l = \sqrt{1 + \frac{\nu\zeta_l^2}{2p^2}\frac{(1+p)^2}{1+2p}}\,. \tag{66}$$

In the study of SSI, with dual-power law nonlinearity, the initial pulse waveform is taken to be of the form

$$q(0,T) = \frac{\eta_1}{\left[1 + a_1 \cosh\left\{\zeta_1\left(T - \frac{T_0}{2}\right)\right\}\right]^{\frac{1}{2p}}} e^{i\phi_1} + \frac{\eta_2}{\left[1 + a_2 \cosh\left\{\zeta_2\left(T + \frac{T_0}{2}\right)\right\}\right]^{\frac{1}{2p}}} e^{i\phi_2}\,. \tag{67}$$

For *non-Hamiltonian* type perturbation, the choice $\eta_1 = \eta_2 = 1$ and $\phi_1 = \phi_2 = 0$, gives

$$q(0,T) = \frac{1}{\left[1 + a_1 \cosh\left\{\zeta\left(T - \frac{T_0}{2}\right)\right\}\right]^{\frac{1}{2p}}} + \frac{1}{\left[1 + a_2 \cosh\left\{\zeta\left(T + \frac{T_0}{2}\right)\right\}\right]^{\frac{1}{2p}}}\,, \tag{68}$$

where ζ is given by (59). For *Hamiltonian* type perturbations, the choice $\eta_1 = \eta_0$, $\eta_2 = 1$ and $\phi_1 = \phi_2 = 0$, gives

$$q(0,T) = \frac{\eta_0}{\left[1 + a_1 \cosh\left\{\zeta_0\left(T - \frac{T_0}{2}\right)\right\}\right]^{\frac{1}{2p}}} + \frac{1}{\left[1 + a_2 \cosh\left\{\zeta\left(T + \frac{T_0}{2}\right)\right\}\right]^{\frac{1}{2p}}}, \quad (69)$$

where

$$\zeta_0 = \eta_0^p \left(\frac{2p^2}{1+p}\right)^{\frac{1}{2p}}, \quad (70)$$

which represents in-phase injection of pulses with unequal amplitudes.

3. Quasi-Particle Theory

The QPT dates back to 1981 since the appearance of the paper by Karpman and Solov'ev [13]. The mathematical approach to SSI will be studied using the QPT. Here, the solitons are treated as particles. If two pulses are separated and each of them is close to a soliton they can be written as the linear superposition of two soliton like pulses as [2]

$$q(Z,T) = q_1(Z,T) + q_2(Z,T), \quad (71)$$

with

$$q_l(Z,T) = A_l g\left[D_l\left(T - T_l\right)\right] e^{-i\{B_l(T-T_l)-\delta_l\}}, \quad (72)$$

where

$$D_l = \chi(A_l) \quad (73)$$

and $l = 1, 2$ while A_l, B_l, D_l, T_l and δ_l are functions of Z. Here, A_l, D_l and B_l do not represent the amplitude, width and the frequency of the full wave form. However, they approach the amplitude, width and frequency respectively for large separation namely if $\Delta T = T_1 - T_2 \to \infty$, then $A_l \to \eta_l$, $D_l \to \zeta_l$ and $B_l \to \kappa_l$. Since the waveform is assumed to remain in the form of two pulses, the method is called the quasi-particle approach. First, the equations for A_l, B_l,

T_l and δ_l using the soliton perturbation theory (SPT) [11, 22] will be derived. Substituting (71) into (3) yields

$$i\frac{\partial q_l}{\partial Z} + \frac{1}{2}\frac{\partial^2 q_l}{\partial T^2} = i\epsilon R[q_l, q_l^*] - F\left(|q_l + q_{\bar{l}}|^2\right)|q_l + q_{\bar{l}}| \ , \tag{74}$$

where $l = 1, 2$ and $\bar{l} = 3 - l$. By SPT, the evolution equations are

$$\frac{dA_l}{dZ} = F_1^{(l)}(A, \Delta T, \Delta\phi) + \epsilon M_l \,, \tag{75}$$

$$\frac{dB_l}{dZ} = F_2^{(l)}(A, \Delta T, \Delta\phi) + \epsilon N_l \,, \tag{76}$$

$$\frac{dT_l}{dZ} = -B_l - F_3(A, \Delta T, \Delta\phi) + \epsilon Q_l \,, \tag{77}$$

$$\frac{d\delta_l}{dZ} = \psi\left(A_l, B_l\right) + F_4(A, \Delta T, \Delta\phi) + \epsilon P_l \,, \tag{78}$$

where the functions $F_1^{(l)}$, $F_2^{(l)}$, F_3 and F_4 formulate on using the SPT in (74), with the right side being treated as perturbation terms. The exact form of these functions can be obtained when a specific law of nonlinearity is considered. Also,

$$M_l = h_1(A_l)\int_{-\infty}^{\infty} \Re\left\{\hat{R}[q_l, q_l^*]e^{-i\phi_l}\right\} g\left(\tau_l\right) d\tau_l \,, \tag{79}$$

$$N_l = h_2(A_l)\int_{-\infty}^{\infty} \Im\left\{\hat{R}[q_l, q_l^*]e^{-i\phi_l}\right\} g'\left(\tau_l\right) d\tau_l \,, \tag{80}$$

$$Q_l = h_3(A_l)\int_{-\infty}^{\infty} \Re\left\{\hat{R}[q_l, q_l^*]e^{-i\phi_l}\right\} \tau_l g\left(\tau_l\right) d\tau_l \,, \tag{81}$$

$$P_l = h_4(A_l)\int_{-\infty}^{\infty} \Im\left\{\hat{R}[q_l, q_l^*]e^{-i\phi_l}\right\} \left[g\left(\tau_l\right) - \tau_l g'\left(\tau_l\right)\right] d\tau_l \,, \tag{82}$$

where the functions $h_j(A_l)$ for $1 \leq j \leq 4$ are by virtue of (75)-(78) and the type of nonlinearity that is considered. Also, in (79)-(82), $\Re$ and $\Im$ stands for the real and imaginary parts respectively. Moreover, the following notations are used

$$\hat{R}[q_l, q_l^*] = R[q_l, q_l^*] - F\left(|q_l + q_{\bar{l}}|^2\right)|q_l + q_{\bar{l}}| \ , \tag{83}$$

$$\tau_l = A_l\left(T - T_l\right) \ , \tag{84}$$

$$\phi_l = B_l\left(T - T_l\right) - \delta_l \, , \tag{85}$$

$$\Delta\phi = B\Delta T + \Delta\delta \, , \tag{86}$$

$$\Delta T = T_1 - T_2 \, , \tag{87}$$

$$\Delta\delta = \delta_1 - \delta_2 \, , \tag{88}$$

$$A = \frac{1}{2}\left(A_1 + A_2\right) \ , \tag{89}$$

$$B = \frac{1}{2}\left(B_1 + B_2\right) \ , \tag{90}$$

$$\Delta A = A_1 - A_2 \, , \tag{91}$$

$$\Delta B = B_1 - B_2 \, . \tag{92}$$

Finally, it is assumed that

$$|\Delta A| \ll A \, , \tag{93}$$

$$|\Delta B| \ll 1 \, , \tag{94}$$

$$|\Delta D| \ll D \, , \tag{95}$$

$$A\Delta T \gg 1 \, , \tag{96}$$

$$D\Delta T \gg 1\,, \tag{97}$$

$$|\Delta A|\Delta T \ll 1\,, \tag{98}$$

$$|\Delta D|\Delta T \ll 1\,. \tag{99}$$

From (84)-(92), one can now obtain

$$\frac{dA}{dZ} = \epsilon M\,, \tag{100}$$

$$\frac{dB}{dZ} = \epsilon N\,, \tag{101}$$

$$\frac{d\,(\Delta A)}{dZ} = F_1^{(1)}(A, \Delta T, \Delta\phi) - F_1^{(2)}(A, \Delta T, \Delta\phi) + \epsilon\Delta M\,, \tag{102}$$

$$\frac{d\,(\Delta B)}{dZ} = F_2^{(1)}(A, \Delta T, \Delta\phi) - F_2^{(2)}(A, \Delta T, \Delta\phi) + \epsilon\Delta N\,, \tag{103}$$

$$\frac{d\,(\Delta T)}{dZ} = -\Delta B + \epsilon\Delta Q\,, \tag{104}$$

$$\frac{d\,(\Delta\phi)}{dZ} = \psi\,(A_1, B_1) - \psi\,(A_2, B_2) - B\Delta B + \frac{\Delta T}{2}\left(F_2^{(1)} + F_2^{(2)}\right) + \epsilon\Delta P + \epsilon B\Delta Q\,, \tag{105}$$

where

$$M = \frac{1}{2}\,(M_1 + M_2)\ , \tag{106}$$

$$N = \frac{1}{2}\,(N_1 + N_2) \tag{107}$$

and ΔM, ΔN, ΔQ and ΔP are the variations of M, N, Q and P which are written as, for example

$$\Delta M = \frac{\partial M}{\partial A}\Delta A + \frac{\partial M}{\partial B}\Delta B\,. \tag{108}$$

Assuming that they are functions of A and B only, which is, in fact, true for most of the cases of interest, otherwise, the equations for

$$T = \frac{1}{2}(T_1 + T_2) \tag{109}$$

and

$$\phi = \frac{1}{2}(\phi_1 + \phi_2) \tag{110}$$

would have been necessary. The results in this section, that are derived so far, will now be utilized to show that the SSI can indeed be suppressed in presence of the perturbation terms given by (4) for the four cases of nonlinearity. In all the four types of nonlinearities, the initial conditions for the Kerr law nonlinearity, corresponding to the initial waveform (19), are

$$A = 1,\ B = 0,\ \Delta A_0 = 0,\ \Delta B_0 = 0,\ \Delta T_0 = T_0 \text{ and } \Delta\phi_0 = 0\,.$$

The four laws of nonlinearity are studied in the following subsections.

3.1. Kerr Law

In this case, (11) reduces to

$$q_l(Z,T) = \frac{A_l}{\cosh\left[D_l\left(T - T_l\right)\right]} e^{-iB_l(T-T_l)+i\delta_l}\,, \tag{111}$$

where

$$D_l \equiv \chi(A_l) = A_l\,, \tag{112}$$

so that (74) transforms to

$$i\frac{\partial q_l}{\partial Z} + \frac{1}{2}\frac{\partial^2 q_l}{\partial T^2} + |q_l|^2 q_l = i\epsilon R[q_l, q_l^*] - \left(q_l^2 q_{\bar{l}}^* + 2|q_l|^2 q_{\bar{l}}\right)\,, \tag{113}$$

where $l = 1, 2$ and $\bar{l} = 3 - l$ and the separation

$$|q|^2 q = \left(|q_1|^2 q_1 + q_1^2 q_2^* + 2\,|q_1|^2 q_2\right) + \left(|q_2|^2 q_2 + q_2^2 q_1^* + 2\,|q_2|^2 q_1\right) \tag{114}$$

was used based on the degree of overlapping. By SPT, the evolution equations are

$$\frac{dA_l}{dZ} = (-1)^{l+1} 4A^3 e^{-A\Delta T}\sin(\Delta\phi) + \epsilon M_l\,, \tag{115}$$

$$\frac{dB_l}{dZ} = (-1)^{l+1} 4A^3 e^{-A\Delta T} \cos(\Delta\phi) + \epsilon N_l \,, \tag{116}$$

$$\frac{dT_l}{dZ} = -B_l - 2Ae^{-A\Delta T} \sin(\Delta\phi) + \epsilon Q_l \,, \tag{117}$$

$$\frac{d\delta_l}{dZ} = \frac{1}{2}\left(A_l^2 + B_l^2\right) - 2ABe^{-A\Delta T} \sin(\Delta\phi) + 6A^2 e^{-A\Delta T} \cos(\Delta\phi) + \epsilon P_l \,, \tag{118}$$

where

$$M_l = \int_{-\infty}^{\infty} \Re\left\{\hat{R}[q_l, q_l^*] e^{-i\phi_l}\right\} \frac{1}{\cosh\tau_l} d\tau_l \,, \tag{119}$$

$$N_l = -\int_{-\infty}^{\infty} \Im\left\{\hat{R}[q_l, q_l^*] e^{-i\phi_l}\right\} \frac{\tanh\tau_l}{\cosh\tau_l} d\tau_l \,, \tag{120}$$

$$Q_l = \frac{1}{A_l^2}\int_{-\infty}^{\infty} \Re\left\{\hat{R}[q_l, q_l^*] e^{-i\phi_l}\right\} \frac{\tau_l}{\cosh\tau_l} d\tau_l \,, \tag{121}$$

$$P_l = \frac{1}{A_l}\int_{-\infty}^{\infty} \Im\left\{\hat{R}[q_l, q_l^*] e^{-i\phi_l}\right\} \frac{(1-\tau_l\tanh\tau_l)}{\cosh\tau_l} d\tau_l \,, \tag{122}$$

and

$$\hat{R}[q_l, q_l^*] = R[q_l, q_l^*] - i\left(q_l^2 q_l^* + 2|q_l|^2 q_l^*\right) \,. \tag{123}$$

The study for the Kerr law case will now be split into the following two subsections

3.1.1. Non-Hamiltonian Perturbations

In presence of non-Hamiltonian perturbation terms, as given by (4), the dynamical system of the soliton parameters, by SPT are

$$\frac{dA}{dZ} = \epsilon\delta \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(m+1\right)}{\Gamma\left(\frac{2m+3}{2}\right)} A^{2m+1} + 2\epsilon\sigma A^2 - \frac{2}{3}\epsilon\beta A\left(A^2 + 3B^2\right) \,, \tag{124}$$

$$\frac{dB}{dZ} = -\frac{4}{3}\epsilon\beta A^2 B\,, \tag{125}$$

so that by virtue of (102)-(105)

$$\begin{aligned}\frac{d(\Delta A)}{dZ} &= 8A^3 e^{-A\Delta T}\sin(\Delta\phi)\\ &+\frac{\epsilon\delta}{2^{2m}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(m+1)}{\Gamma\left(\frac{2m+3}{2}\right)}\sum_{r=0}^{2m+1}\binom{2m+1}{2r+1}(2A)^{2r+1}(\Delta A)^{2m-2r}\\ &+4\epsilon\sigma A\Delta A - 2\epsilon\beta\left(A^2+B^2\right)\Delta A - 4\epsilon\beta AB\Delta B\,,\end{aligned} \tag{126}$$

$$\frac{d(\Delta B)}{dZ} = 8A^3 e^{-A\Delta T}\cos(\Delta\phi) - \frac{8}{3}\epsilon\beta AB\Delta A - \frac{4}{3}\epsilon\beta A^2\Delta B\,, \tag{127}$$

$$\frac{d(\Delta T)}{dZ} = -\Delta B + \epsilon\sigma\,, \tag{128}$$

$$\frac{d(\Delta\phi)}{dZ} = A\Delta A - \frac{4}{3}\epsilon\beta A^2 B\Delta T\,, \tag{129}$$

where in (126)

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 3.2.1}\,. \tag{130}$$

For the fixed point of the dynamical system, given by (126) and (127), with $A = 1$ and $B = 0$, one recovers

$$\beta = 3\sigma + \frac{3\delta}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(m+1)}{\Gamma\left(\frac{2m+3}{2}\right)}\,. \tag{131}$$

From (128) and (129), one has the coupled system of equations for $\Delta\phi$, the phase difference, and ΔT, the soliton separation, with the fixed point $A = 1$ and $B = 0$ as follows:

$$\frac{d^2(\Delta T)}{dZ^2} + \frac{4}{3}\epsilon\beta\frac{d(\Delta T)}{dZ} + 8e^{-\Delta T}\cos(\Delta\phi) = 0\,, \tag{132}$$

$$\frac{d^2(\Delta\phi)}{dZ^2} + 2\epsilon(\beta - 2\sigma)\frac{d(\Delta\phi)}{dZ}$$

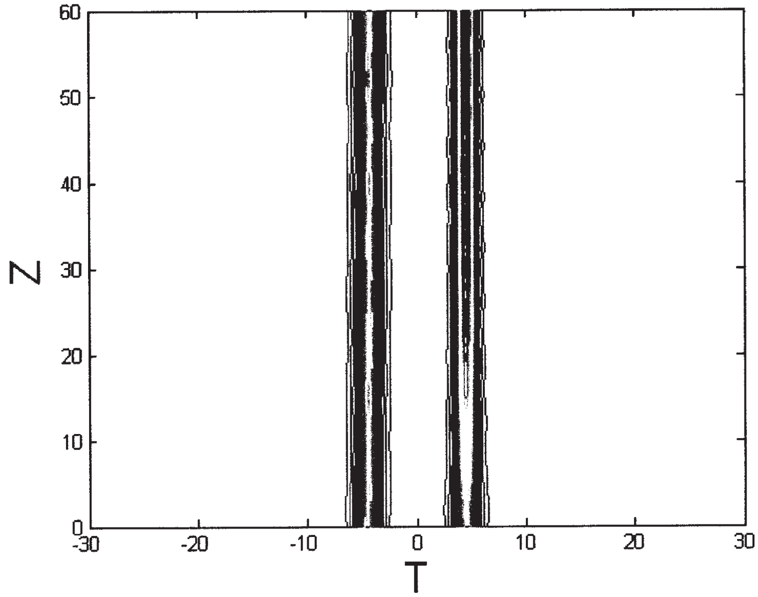

Figure 1(a): $m = 0$, $\sigma = \delta = 0.005$

$$-\frac{\epsilon\delta}{2^{2m}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(m+1\right)}{\Gamma\left(\frac{2m+3}{2}\right)}\sum_{r=0}^{2m+1}\binom{2m+1}{2r+1}\left(\frac{d(\Delta\phi)}{dZ}\right)^{2m-2r} - 8e^{-\Delta T}\sin(\Delta\phi) = 0\,, \quad (133)$$

where in (132) and (133) β is given by (131). Equations (132) and (133) shows that inserting filters produces a damping in both pulse separation and phase difference as seen in Figures 1(a), 1(b) and 1(c).

3.1.2. Hamiltonian Perturbations

In case of Hamiltonian perturbations, the dynamical system of the soliton parameters is

$$\frac{dA}{dZ} = 0\,, \quad (134)$$

$$\frac{dB}{dZ} = 0\,, \quad (135)$$

$$\frac{dT_0}{dZ} = -B - 3\epsilon\gamma B^2 - \frac{\epsilon}{3}\left\{A^2\left(3\lambda + 2\mu + 3\gamma\right) + 3\alpha\right\}\,, \quad (136)$$

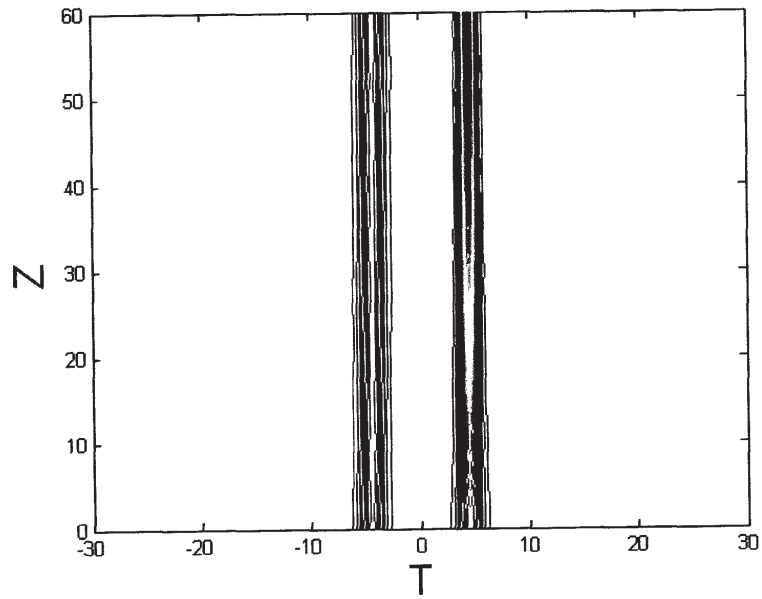

Figure 1(b): $m = 1$, $\sigma = \delta = 0.005$

so that by virtue of (102)-(105)

$$\frac{d\,(\Delta A)}{dZ} = 8A^3 e^{-A\Delta T} \sin(\Delta\phi)\,, \tag{137}$$

$$\frac{d\,(\Delta B)}{dZ} = 8A^3 e^{-A\Delta T} \cos(\Delta\phi)\,, \tag{138}$$

$$\frac{d\,(\Delta T)}{dZ} = -\Delta B - \frac{3}{2}\epsilon\gamma B\Delta B - \frac{\epsilon}{6}(3\lambda + 2\mu + 3\gamma)A\Delta A\,, \tag{139}$$

$$\frac{d\,(\Delta\phi)}{dZ} = A\Delta A\,. \tag{140}$$

Now,

$$A = \frac{1}{2}\left(A_0 + 1\right)\,, \tag{141}$$

$$B = 0\,, \tag{142}$$

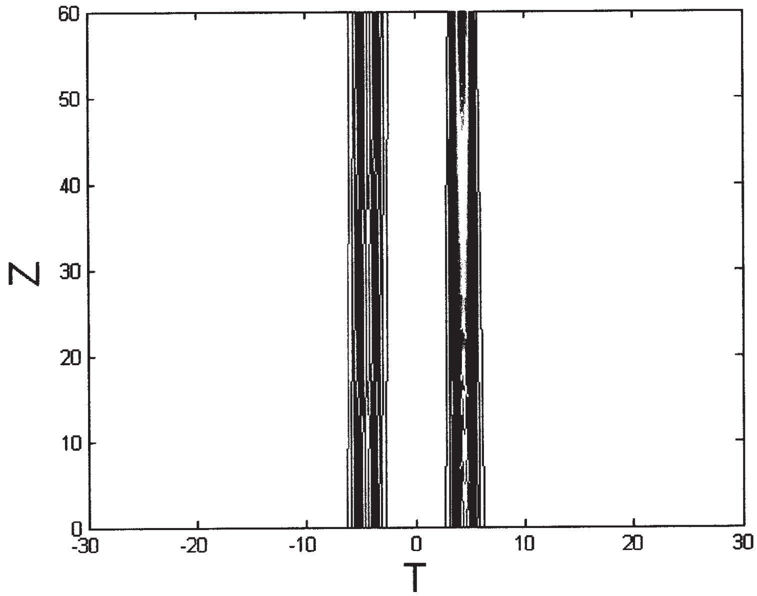

Figure 1(c): $m = 2$, $\sigma = \delta = 0.005$

$$\Delta A_0 = A_0 - 1\,, \tag{143}$$

$$\Delta B_0 = 0\,, \tag{144}$$

$$\Delta T_0 = T_0\,, \tag{145}$$

$$\Delta \phi_0 = 0\,, \tag{146}$$

$$\Delta \phi = \Delta \delta\,, \tag{147}$$

so that

$$\frac{d\,(\Delta T)}{dZ} + \frac{\epsilon}{6}(3\lambda + 2\mu + 3\gamma)\frac{d\,(\Delta\phi)}{dZ} = -\Delta B\,. \tag{148}$$

For $\Delta B = 0$

$$\Delta T = T_0 - \frac{\epsilon}{6}(3\lambda + 2\mu + 3\gamma)\Delta\delta\,. \tag{149}$$

Now, $T_0 \sim O(1)$ so that $\Delta T \not\to 0$ and thus the pulses do not collide during the transmission. This is observed in the following numerical simulation, see Figure 2.

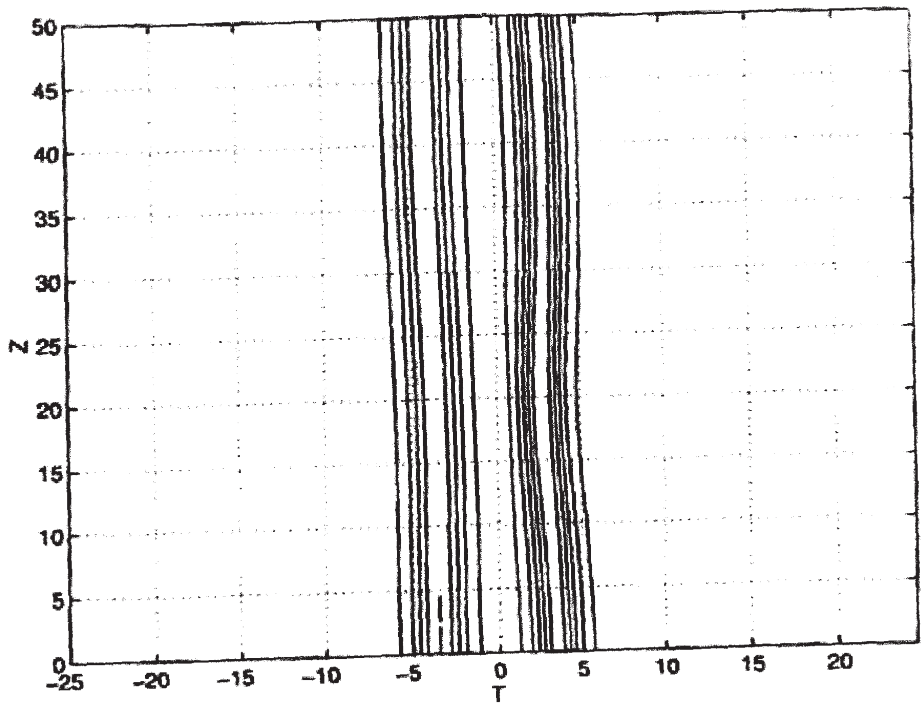

Figure 2: $\varepsilon = 0.1$, $\gamma = 0.14$

3.2. Power Law

In this case, (11) is

$$q_l(Z,T) = \frac{A_l}{\cosh^{\frac{1}{p}}\left[D_l\left(T - T_l\right)\right]} e^{-iB_l(T-T_l)+i\delta_l} , \tag{150}$$

where

$$D_l \equiv \chi(A_l) = A_l^p \left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}} , \tag{151}$$

so that (74) gives

$$i\frac{\partial q_l}{\partial Z} + \frac{1}{2}\frac{\partial^2 q_l}{\partial T^2} = i\epsilon R[q_l, q_l^*] - \left[\sum_{r=0}^{p}\binom{p}{r} q_1^{p-r} q_2^r\right]\left[\sum_{r=0}^{p}\binom{p}{r}\left(q_1^*\right)^{p-r}\left(q_2^*\right)^r\right](q_1 + q_2) . \tag{152}$$

Here, the separation

$$|q|^{2p}q = \left[\sum_{r=0}^{p}\binom{p}{r}q_1^{p-r}q_2^r\right]\left[\sum_{r=0}^{p}\binom{p}{r}\left(q_1^*\right)^{p-r}\left(q_2^*\right)^r\right](q_1+q_2) \tag{153}$$

was used based on the degree of overlapping. By SPT, the evolution equations are

$$\frac{dA_l}{dZ} = F_1^{(l)}(A,\Delta T,\Delta\phi;p) + \epsilon M_l\,, \tag{154}$$

$$\frac{dB_l}{dZ} = F_2^{(l)}(A,\Delta T,\Delta\phi;p) + \epsilon N_l\,, \tag{155}$$

$$\frac{dT_l}{dZ} = -B_l - F_3(A,\Delta T,\Delta\phi;p) + \epsilon Q_l\,, \tag{156}$$

$$\frac{d\delta_l}{dZ} = \frac{B_l^2}{2} + \frac{A_l^{2p}}{p+1} + F_4(A,\Delta T,\Delta\phi;p) + \epsilon P_l\,, \tag{157}$$

where, for power law

$$M_l = \frac{1}{2-p}\left(\frac{2p^2}{1+p}\right)^{\frac{p-3}{2p}} \times \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-\infty}^{\infty}\Re\left\{\hat{R}[q_l,q_l^*]e^{-i\phi_l}\right\}\frac{1}{\cosh^{\frac{1}{p}}\tau_l}d\tau_l\,, \tag{158}$$

$$N_l = \frac{2}{p}A_l^{p-1}\left(\frac{2p^2}{1+p}\right)^{\frac{p-1}{2p}} \times \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-\infty}^{\infty}\Im\left\{\hat{R}[q_l,q_l^*]e^{-i\phi_l}\right\}\frac{\tanh\tau_l}{\cosh^{\frac{1}{p}}\tau_l}d\tau_l\,, \tag{159}$$

$$Q_l = \frac{1}{A_l^{p+1}}\left(\frac{p+1}{2p^2}\right)^{\frac{p+2}{2p}}$$

$$\times \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-\infty}^{\infty} \Re\left\{\hat{R}[q_l, q_l^*]e^{-i\phi_l}\right\} \frac{\tau_l}{\cosh^{\frac{1}{p}} \tau_l} d\tau_l \,, \quad (160)$$

$$P_l = \frac{1}{A_l^{p+1}} \left(\frac{2p^2}{p+1}\right)^{\frac{p+1}{2p}} \times \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-\infty}^{\infty} \Im\left\{\hat{R}[q_l, q_l^*]e^{-i\phi_l}\right\} \frac{(1-\tau_l \tanh \tau_l)}{\cosh^{\frac{1}{p}} \tau_l} d\tau_l \,. \quad (161)$$

In addition, the following notations are used

$$\hat{R}[q_l, q_l^*] = R[q_l, q_l^*] - i\left[\sum_{r=0}^{p} \binom{p}{r} q_1^{p-r} q_2^r\right] \times \left[\sum_{r=0}^{p} \binom{p}{r} (q_1^*)^{p-r} (q_2^*)^r\right] (q_1+q_2) + i|q_l|^{2p} q_l \,. \quad (162)$$

For the power law, the study will now be split into the following two subsections.

3.2.1. Non-Hamiltonian Perturbations

In presence of non-Hamiltonian perturbation terms, as given by (4), the dynamical system of the soliton parameters, by virtue of SPT, are

$$\begin{aligned}\frac{dA}{dZ} &= \frac{2\epsilon\delta}{2-p}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{2p}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)} \frac{\Gamma\left(\frac{m+1}{p}\right)}{\Gamma\left(\frac{m+1}{p}+\frac{1}{2}\right)} A^{2m+1} \\ &+ \frac{\epsilon\sigma}{2-p}\left(\frac{1+p}{2p^2}\right)^{\frac{2p}{p+1}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-\infty}^{\infty} \frac{1}{\cosh^{\frac{2}{p}} \tau} \left(\int_{-\infty}^{\tau} \frac{ds}{\cosh^{\frac{2}{p}} s}\right) d\tau A^{3-p} \\ &+ \frac{2\epsilon\beta}{p^2(2-p)}\left(\frac{2p^2}{p+1}\right)^{\frac{2p-1}{2p}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)} \frac{\Gamma\left(\frac{p+1}{p}\right)}{\Gamma\left(\frac{p+1}{p}+\frac{1}{2}\right)} A^{2p+1} \\ &- \frac{2\epsilon\beta}{2-p}\left(\frac{2p^2}{p+1}\right)^{\frac{1}{2p}} AB^2 - \frac{2\epsilon\beta}{p^2(2-p)}\left(\frac{2p^2}{p+1}\right)^{\frac{2p-1}{2p}} A^{2p+1} \,, \quad (163)\end{aligned}$$

$$\frac{dB}{dZ} = \frac{4\epsilon\beta}{p^2}\left(\frac{2p^2}{p+1}\right)^{\frac{3p-2}{2p}}\left(\frac{p-2}{p+2}\right)BA^{2p}\,, \tag{164}$$

so that by virtue of (102)-(105)

$$\begin{aligned}
\frac{d\left(\Delta A\right)}{dZ} &= F_1^{(1)}(A,\Delta T,\Delta\phi) - F_1^{(2)}(A,\Delta T,\Delta\phi) \\
&\quad + \frac{\epsilon\delta}{2-p}\left(\frac{1+p}{2p^2}\right)^{\frac{1}{2p}}\frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)}\frac{\Gamma\left(\frac{m+1}{p}\right)}{\Gamma\left(\frac{m+1}{p}+\frac{1}{2}\right)} \\
&\quad \times\sum_{r=0}^{2m+1}\binom{2m+1}{2r+1}(2A)^{2r+1}(\Delta A)^{2m-2r} + \frac{\epsilon\sigma}{2-p}\left(\frac{1+p}{2p^2}\right)^{\frac{2p}{p+1}}\frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{2}\right)} \\
&\quad \times\int_{-\infty}^{\infty}\frac{1}{\cosh^{\frac{2}{p}}\tau}\left(\int_{-\infty}^{\tau}\frac{ds}{\cosh^{\frac{2}{p}}s}\right)d\tau\sum_{r=0}^{2m+1}\binom{3-p}{2r+1}(2A)^{2-p+2r}(\Delta A)^{2r+1} \\
&\quad + \frac{\epsilon\beta}{p^2(2-p)}\left(\frac{2p^2}{p+1}\right)^{\frac{2p-1}{2p}}\frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)}\frac{\Gamma\left(\frac{p+1}{p}\right)}{\Gamma\left(\frac{p+1}{p}+\frac{1}{2}\right)} \\
&\quad \times\sum_{r=0}^{2m+1}\binom{2m+1}{2r+1}(2A)^{2r+1}(\Delta A)^{2m-2r} \\
&\quad - \frac{2\epsilon\beta}{2-p}\left(\frac{2p^2}{p+1}\right)^{\frac{1}{2p}}AB\Delta B - \frac{\epsilon\beta}{p^2(2-p)}\left(\frac{2p^2}{p+1}\right)^{\frac{2p-1}{2p}} \\
&\quad \times\sum_{r=0}^{2m+1}\binom{2m+1}{2r+1}(2A)^{2r+1}(\Delta A)^{2m-2r}\ ,
\end{aligned} \tag{165}$$

$$\begin{aligned}
\frac{d\left(\Delta B\right)}{dZ} &= F_2^{(1)}(A,\Delta T,\Delta\phi) - F_2^{(2)}(A,\Delta T,\Delta\phi) \\
&\quad + \frac{4\epsilon\beta}{p^2}\left(\frac{2p^2}{p+1}\right)^{\frac{3p-2}{2p}}\left(\frac{p-2}{p+2}\right)\left[\Delta BA^{2p} + 2pBA^{2p+1}\Delta A\right]\ ,
\end{aligned} \tag{166}$$

$$\frac{d\left(\Delta T\right)}{dZ} = -\Delta B + 2\epsilon\sigma\left(\frac{A_1^{p+2}}{D_1^3} - \frac{A_2^{p+2}}{D_2^3}\right)\left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{2}\right)}$$

$$\times \int_{-\infty}^{\infty} \frac{\tau}{\cosh^{\frac{2}{p}} \tau} \left(\int_{-\infty}^{\tau} \frac{ds}{\cosh^{\frac{2}{p}} s} \right) d\tau \,, \tag{167}$$

$$\frac{d\,(\Delta\phi)}{dZ} = \frac{A_1^{2p} - A_2^{2p}}{p+1} - \frac{4\epsilon\beta}{p^2} \left(\frac{2p^2}{p+1} \right)^{\frac{3p-2}{2p}} \left(\frac{p-2}{p+2} \right) A^{2p} B \Delta T \,. \tag{168}$$

For the fixed point of the dynamical system given by (164) and (165) with $A = 1$ and $B = 0$, one gets

$$\beta = \frac{1}{2\Gamma\left(\frac{2}{p}\right) - \Gamma\left(\frac{1}{p} + \frac{1}{2}\right)} \left[\delta(p+1) \frac{\Gamma\left(\frac{2}{p}\right)\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)} \frac{\Gamma\left(\frac{m+1}{p}\right)}{\Gamma\left(\frac{m+1}{p}+\frac{1}{2}\right)} \right.$$
$$+\sigma p^2 \left(\frac{p+1}{2p^2} \right)^{\frac{(3p-1)(2p+1)}{2p(p+1)}}$$
$$\left. \times \frac{\Gamma\left(\frac{2}{p}\right)\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-\infty}^{\infty} \frac{1}{\cosh^{\frac{2}{p}} \tau} \left(\int_{-\infty}^{\tau} \frac{ds}{\cosh^{\frac{2}{p}} s} \right) d\tau \right] . \tag{169}$$

Thus, by (167) and (168) one gets the coupled system of equations in ΔT and $\Delta\phi$ for the fixed point with $A = 1$ and $B = 0$ as

$$\frac{d^2\,(\Delta T)}{dZ^2} - \frac{4\epsilon\beta}{p^2} \left(\frac{2p^2}{p+1} \right)^{\frac{3p-2}{2p}} \left(\frac{p-2}{p+2} \right) \frac{d\,(\Delta T)}{dZ} + F_2^{(1)} - F_2^{(1)} = 0 \,, \tag{170}$$

$$\frac{d^2\,(\Delta\phi)}{dZ^2} = \frac{2p}{p+1} \left[A_1^{2p-1} \frac{dA_1}{dZ} - A_2^{2p-1} \frac{dA_2}{dZ} \right] , \tag{171}$$

where in (170) β is given by (169). From (137), one can observe that there is a damping introduced in the soliton separation and the coefficient of the damping term is positive as $0 < p < 2$. In the following figures, numerical simulations show that the suppression of the SSI is acheived, for power law, as proved in the QPT, see Figure 3(a) and Figure 3(b).

3.2.2. Hamiltonian Perturbations

In presence of Hamiltonian perturbation terms, as given by (4), the dynamical system of the soliton parameters, by virtue of SPT, are

$$\frac{dA}{dZ} = 0 \,, \tag{172}$$

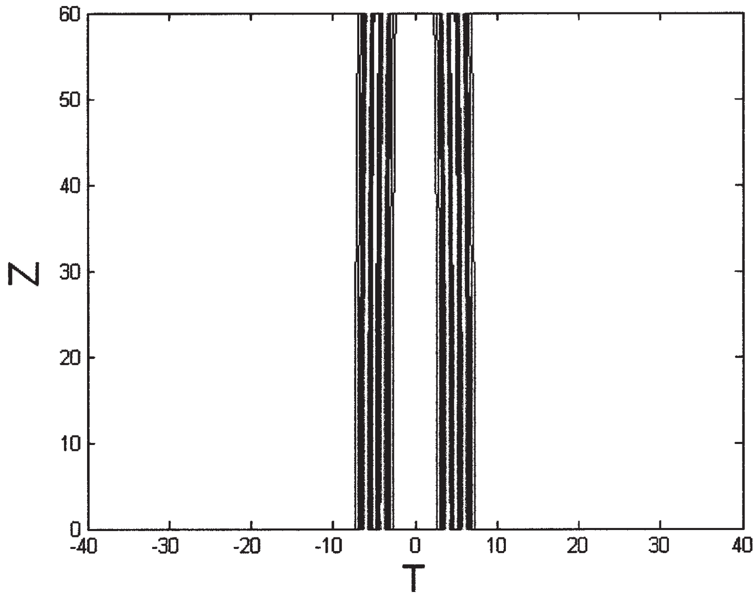

Figure 3(a): $m = 1$, $p = 1/2$, $\beta = (18/7)\delta$, $\delta = 0.001$, $\beta = 0.00257$

$$\frac{dB}{dZ} = 0\,, \tag{173}$$

$$\frac{dT_0}{dZ} = -B - \frac{\epsilon}{2}A^2\left(3\lambda + 2\mu\right)\frac{\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{2}{p}\right)}\frac{\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}$$

$$-\epsilon\left(\mu + 3\gamma B^2\right) + \frac{3\epsilon\gamma D^3}{p^2}\left[\frac{\Gamma\left(\frac{p-1}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}{\Gamma\left(\frac{3}{2}+\frac{2}{p}\right)} + 1\right]\,, \tag{174}$$

so that by virtue of (102)-(105),

$$\frac{d\left(\Delta A\right)}{dZ} = F_1^{(1)}(A, \Delta T, \Delta\phi) - F_1^{(2)}(A, \Delta T, \Delta\phi)\,, \tag{175}$$

$$\frac{d\left(\Delta B\right)}{dZ} = F_2^{(1)}(A, \Delta T, \Delta\phi) - F_2^{(2)}(A, \Delta T, \Delta\phi)\,, \tag{176}$$

$$\frac{d\left(\Delta T\right)}{dZ} = -\Delta B - \frac{\epsilon}{4}A\Delta A\left(3\lambda + 2\mu\right)\frac{\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{2}{p}\right)}\frac{\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}$$

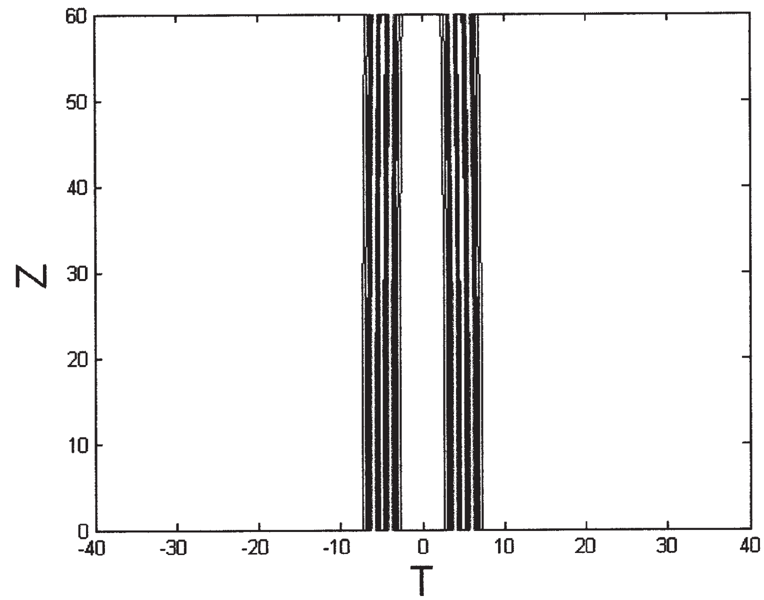

Figure 3(b): $m = 2$, $p = 1/2$, $\beta = (480/231)\delta$, $\delta = 0.001$, $\beta = 0.002$

$$- \frac{3\epsilon\gamma}{4p^2}\Delta D\left[12D^2 + (\Delta D)^2\right] + \frac{3\epsilon\gamma}{p^2}\Delta D\left[12D^2 + (\Delta D)^2\right] \times \frac{\Gamma\left(\frac{p-1}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{p}\right)}, \quad (177)$$

which can be rewritten, by virtue of (105), as

$$\frac{d(\Delta T)}{dZ} = -\Delta B - \frac{\epsilon}{4}(3\lambda + 2\mu)\, g_1\left\{\frac{d(\Delta\phi)}{dZ}\right\}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{2}{p}\right)}\frac{\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)} - \frac{3\epsilon\gamma}{4p^2}g_2\left\{\frac{d(\Delta\phi)}{dZ}\right\} + \frac{3\epsilon\gamma}{p^2}g_2\left\{\frac{d(\Delta\phi)}{dZ}\right\}\frac{\Gamma\left(\frac{p-1}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{p}\right)}, \quad (178)$$

$$\frac{d(\Delta\phi)}{dZ} = \frac{A_1^{2p} - A_2^{2p}}{p+1}. \quad (179)$$

For in-phase injection of solitons with unequal amplitudes,

$$A = \frac{1}{2}(A_0 + 1), \quad (180)$$

$$B = 0\,, \tag{181}$$

$$\Delta A_0 = A_0 - 1\,, \tag{182}$$

$$\Delta B_0 = 0\,, \tag{183}$$

$$\Delta T_0 = T_0\,, \tag{184}$$

$$\Delta \phi_0 = 0\,, \tag{185}$$

$$\Delta \phi = \Delta \delta\,, \tag{186}$$

so that for $\Delta B = 0$

$$\begin{aligned}\Delta T = T_0 &- \frac{\epsilon}{6}(3\lambda + 2\mu) \\ &- \frac{\epsilon}{4}\,(3\lambda + 2\mu)\,h_1\left\{\frac{d(\Delta\phi)}{dZ}\right\}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{2}{p}\right)}\frac{\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)} \\ &- \frac{3\epsilon\gamma}{4p^2}h_2\left\{\frac{d(\Delta\phi)}{dZ}\right\} + \frac{3\epsilon\gamma}{p^2}h_2\left\{\frac{d(\Delta\phi)}{dZ}\right\}\frac{\Gamma\left(\frac{p-1}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}\frac{\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{p}\right)}\,,\end{aligned} \tag{187}$$

where

$$h_j(s) = \int g_j(s)ds\,, \tag{188}$$

for $j = 1, 2$. Thus

$$\Delta T = T_0 + O(\epsilon)\,. \tag{189}$$

Now, $T_0 \sim O(1)$ so that $\Delta T \not\to 0$ and thus the pulses do not collide during the transmission. This will be observed in the following numerical simulation, see Figure 4.

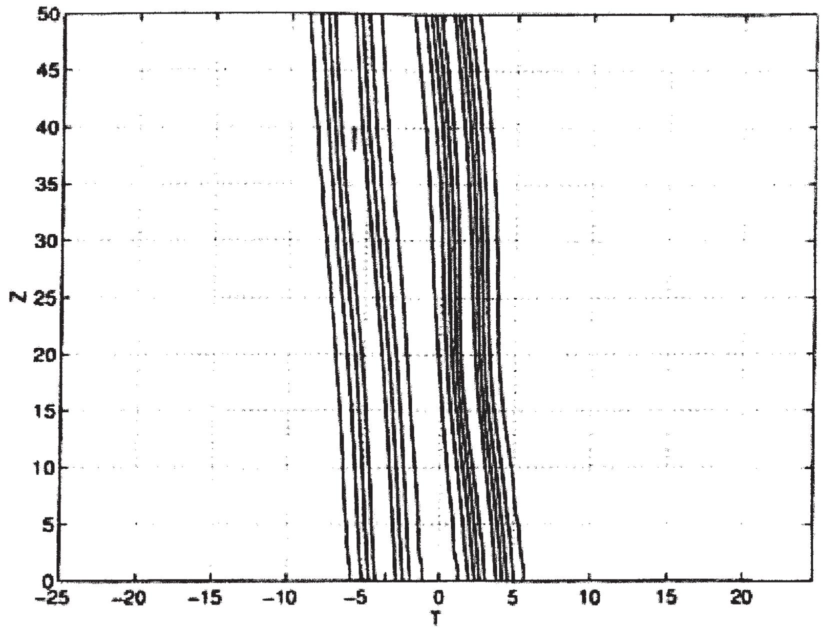

Figure 4: $\varepsilon = 0.1$, $p = 1/2$, $\gamma = 0.14$, $\lambda = \mu = 0.5$

3.3. Parabolic Law

In this case, (29) yields

$$q_l(Z,T) = \frac{A_l}{\left[1 + a_l \cosh\left\{D_l\left(T - T_l\right)\right\}\right]^{\frac{1}{2}}} e^{-iB_l(T-T_l)+i\delta_l}\,, \tag{190}$$

where

$$D_l \equiv \chi(A_l) = A_l\sqrt{2}\,. \tag{191}$$

By (74), one gets

$$\begin{aligned} i\frac{\partial q_l}{\partial Z} + \frac{1}{2}\frac{\partial^2 q_l}{\partial T^2} + \left(|q_l|^2 + \nu\,|q_l|^4\right) q_l = i\epsilon R[q_l, q_{\bar{l}}^*] - \left(q_l^2 q_{\bar{l}}^* + 2\,|q_l|^2\, q_{\bar{l}}\right) \\ - \nu\left[q_l^3\left(q_{\bar{l}}^*\right)^2 + 2\,|q_l|^2\, q_l^2 q_{\bar{l}}^* + 3\,|q_l|^4\, q_{\bar{l}} + 3\,|q_l|^2\, q_l^* q_{\bar{l}}^2 + 6\,|q_l|^2\,|q_{\bar{l}}|^2\, q_l\right]\,, \end{aligned} \tag{192}$$

where $l = 1, 2$ and $\bar{l} = 3 - l$. Here, the separation

$$|q|^2q + \nu|q|^4q = \left(|q_1|^2 q_1 + q_1^2 q_2^* + 2|q_1|^2 q_2\right) + \left(|q_2|^2 q_2 + q_2^2 q_1^* + 2|q_2|^2 q_1\right) + \nu\left[|q_1|^4 q_1 + q_1^3 (q_2^*)^2 + 2|q_1|^2 q_1^2 q_2^* + 3|q_1|^4 q_2 + 3|q_1|^2 q_1^* q_2^2 + 6|q_1|^2 |q_2|^2 q_1\right] + \nu\left[|q_2|^4 q_2 + q_2^3 (q_1^*)^2 + 2|q_2|^2 q_2^2 q_1^* + 3|q_2|^4 q_1 + 3|q_2|^2 q_2^* q_1^2 + 6|q_1|^2 |q_2|^2 q_2\right] \quad (193)$$

was used based on the degree of overlapping. By SPT, the evolution equations are

$$\frac{dA_l}{dZ} = F_1^{(l)}(A, \Delta T, \Delta\phi; \nu) + \epsilon M_l\,, \quad (194)$$

$$\frac{dB_l}{dZ} = F_2^{(l)}(A, \Delta T, \Delta\phi; \nu) + \epsilon N_l\,, \quad (195)$$

$$\frac{dT_l}{dZ} = -B_l - F_3(A, \Delta T, \Delta\phi; \nu) + \epsilon Q_l\,, \quad (196)$$

$$\frac{d\delta_l}{dZ} = \frac{A_l^2}{4} + \frac{B_l^2}{2} + F_4(A, \Delta T, \Delta\phi; \nu) + \epsilon P_l\,, \quad (197)$$

where, in this case

$$M_l = h_1^{(1)}(A_l)\int_{-\infty}^{\infty} \Re\left\{\hat{R}[q_l, q_l^*]e^{-i\phi_l}\right\} \frac{d\tau_l}{(1 + a_l \cosh\tau_l)^{\frac{1}{2}}} d\tau_l\,, \quad (198)$$

$$N_l = h_2^{(1)}(A_l)\int_{-\infty}^{\infty} \Im\left\{\hat{R}[q_l, q_l^*]e^{-i\phi_l}\right\} \frac{\sinh\tau_l}{(1 + a_l \cosh\tau_l)^{\frac{1}{2}}} d\tau_l\,, \quad (199)$$

$$Q_l = h_3^{(1)}(A_l)\int_{-\infty}^{\infty} \Re\left\{\hat{R}[q_l, q_l^*]e^{-i\phi_l}\right\} \frac{\tau_l}{(1 + a_l \cosh\tau_l)^{\frac{1}{2}}} d\tau_l\,, \quad (200)$$

$$P_l = h_4^{(1)}(A_l)\int_{-\infty}^{\infty} \Im\left\{\hat{R}[q_l, q_l^*]e^{-i\phi_l}\right\} \frac{(1 - a_l\tau_l \sinh\tau_l)}{(1 + a_l \cosh\tau_l)^{\frac{1}{2}}} d\tau_l\,. \quad (201)$$

Also, the following notations are used

$$\hat{R}[q_l, q_l^*] = R[q_l, q_l^*] - \left(q_l^2 q_{\bar{l}}^* + 2|q_l|^2 q_{\bar{l}}\right) - \nu\left[q_l^3 \left(q_{\bar{l}}^*\right)^2 + 2|q_l|^2 q_l^2 q_{\bar{l}}^* + 3|q_l|^4 q_{\bar{l}} + 3|q_l|^2 q_l^* q_{\bar{l}}^2 + 6|q_l|^2 |q_{\bar{l}}|^2 q_l\right]. \quad (202)$$

For the parabolic law case, the study will be split into the following two subsections.

3.3.1. Non-Hamiltonian Perturbations

In presence of non-Hamiltonian perturbation terms, as given by, (4), the dynamical system of the soliton parameters, by SPT are

$$\begin{aligned}\frac{dA}{dZ} &= \frac{\epsilon\delta A^{2m+1}}{2^m a^{m-1}} F\left(m+1, m+1, m+\frac{3}{2}; \frac{a-1}{2a}\right) B\left(\frac{m+1}{p}, \frac{1}{2}\right) \\ &+ \epsilon\sigma a^2 \frac{\sqrt{2}}{2} \int_{-\infty}^{\infty} \frac{1}{1+a\cosh\tau} \left(\int_{-\infty}^{\tau} \frac{ds}{1+a\cosh s}\right) d\tau \\ &- \epsilon\beta a^2 A^3 \left[\frac{a^2}{(a^2-1)^{\frac{3}{2}}} \tan^{-1}\sqrt{\frac{a-1}{a+1}} - \frac{1}{2(a^2-1)}\right] \\ &- 4\sqrt{2}\epsilon\beta \frac{a^2 B^2}{\sqrt{a^2-1}} \tan^{-1}\sqrt{\frac{a-1}{a+1}}\,, \end{aligned} \tag{203}$$

$$\frac{dB}{dZ} = -\epsilon\beta \frac{\sqrt{2}A^3 B}{E} \left[\frac{a^2}{(a^2-1)^{\frac{3}{2}}} \tan^{-1}\sqrt{\frac{a-1}{a+1}} - \frac{1}{2(a^2-1)}\right], \tag{204}$$

where E is the energy of the soliton given by

$$E = \int_{-\infty}^{\infty} |q|^2 dT = \frac{2\sqrt{2}A}{\sqrt{a^2-1}} \tan^{-1}\sqrt{\frac{a-1}{a+1}} \tag{205}$$

and $F(\alpha, \beta; \gamma; z)$ is the Gauss' hypergeometric function defined as

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{z^n}{n!} \tag{206}$$

and $B(l, m)$ is the beta function defined as

$$B(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1} dx\,. \tag{207}$$

For the fixed point of the dynamical system, given by (203) and (204), with $A = 1$ and $B = 0$, one recovers

$$\beta = \frac{\sigma\sqrt{2}}{2} \frac{\int_{-\infty}^{\infty} \frac{1}{1+a\cosh\tau} \left(\int_{-\infty}^{\tau} \frac{ds}{1+a\cosh s}\right) d\tau}{\frac{a^2}{(a^2-1)^{\frac{3}{2}}} \tan^{-1}\sqrt{\frac{a-1}{a+1}} - \frac{1}{2(a^2-1)}}$$

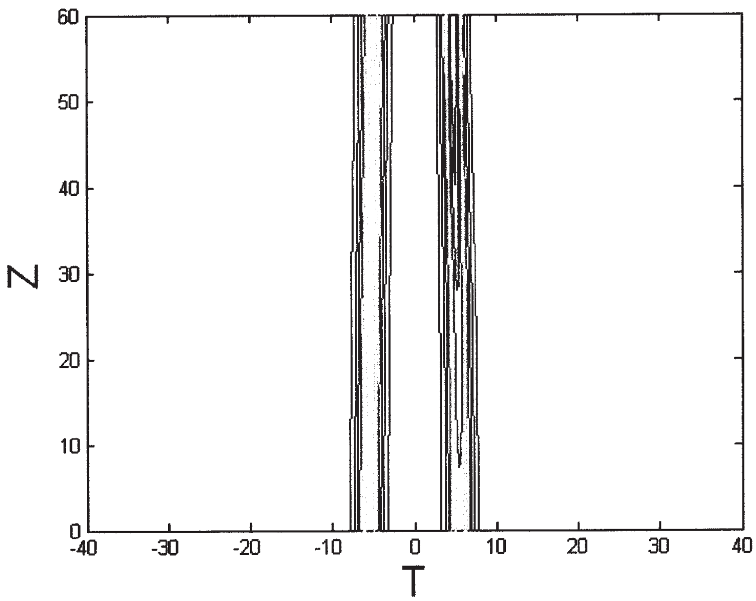

Figure 5: $m = 0$, $\sigma \neq 0$, $\delta \neq 0$, $\beta \neq 0$, $\delta = 0.001$, $\sigma = 0.001$, $\beta = 0.00895$

$$+\frac{\delta}{2^m a^{m+1}} \frac{F\left(m+1, m+1, m+\frac{3}{2}; \frac{a-1}{2a}\right) B\left(\frac{m+1}{p}, \frac{1}{2}\right)}{\frac{a^2}{(a^2-1)^{\frac{3}{2}}} \tan^{-1}\sqrt{\frac{a-1}{a+1}} - \frac{1}{2(a^2-1)}}. \tag{208}$$

Thus, one can obtain using, (104)-(105)

$$\frac{d^2(\Delta T)}{dZ^2} + \epsilon\beta G \frac{d(\Delta T)}{dZ} + F_2^{(1)} - F_2^{(2)} = 0\,, \tag{209}$$

where β is given by (208) and $G > 0$ represents the coefficient of $-\epsilon\beta\Delta B$ in $d(\Delta B)/dZ = dB_1/dZ - dB_2/dZ$. Now, equation (209) shows that there is a damping in the separation of solitons thus proving that there will be a suppression of the SSI in presence of the perturbation terms given by (4). The following numerical simulations show that the suppression of the SSI is achieved, for the parabolic law, as proved in the QPT, see Figure 5.

3.3.2. Hamiltonian Perturbations

In presence of Hamiltonian perturbation terms, as given by (4), the dynamical system of the soliton parameters, by SPT are

$$\frac{dA}{dZ} = 0\,, \tag{210}$$

$$\frac{dB}{dZ} = 0\,, \tag{211}$$

$$\begin{aligned}\frac{dT_0}{dZ} = &-B - \frac{\epsilon\alpha}{E}\frac{2\sqrt{2}A}{\sqrt{a^2-1}}\tan^{-1}\sqrt{\frac{a-1}{a+1}}\\ &- \frac{\epsilon\sqrt{2}}{2E}\frac{A^3}{(a^2-1)^{\frac{3}{2}}}(3\lambda+2\mu)\left[\sqrt{a^2-1} - 2\tan^{-1}\sqrt{\frac{a-1}{a+1}}\right] - 3\sqrt{2}\frac{\epsilon\gamma A}{E}\\ &\times\left[\left(\frac{a^4A^2}{2} + \frac{2\kappa^2}{\sqrt{a^2-1}}\right)\tan^{-1}\sqrt{\frac{a-1}{a+1}} - \frac{a^2A^2}{4}\sqrt{a^2-1}\right]\,. \end{aligned} \tag{212}$$

From (102)-(105) one can now conclude that

$$\frac{d\,(\Delta A)}{dZ} = F_1^{(1)}(A,\Delta T,\Delta\phi) - F_1^{(2)}(A,\Delta T,\Delta\phi)\,, \tag{213}$$

$$\frac{d\,(\Delta B)}{dZ} = F_2^{(1)}(A,\Delta T,\Delta\phi) - F_2^{(2)}(A,\Delta T,\Delta\phi)\,, \tag{214}$$

$$\frac{d\,(\Delta T)}{dZ} = \frac{dT_1}{dZ} - \frac{dT_2}{dZ}\,, \tag{215}$$

$$\frac{d\,(\Delta\phi)}{dZ} = \frac{1}{2}A\Delta A\,. \tag{216}$$

Equation (215) can be rewritten as

$$\frac{d\,(\Delta T)}{dZ} = -\Delta B + \epsilon G\,(\alpha,\lambda,\mu,\gamma,\sigma)\,g\left\{\frac{d(\Delta\phi)}{dZ}\right\}\,, \tag{217}$$

where G is the functional form that depends on the said parameters. For in-phase injection of solitons with unequal amplitudes,

$$A = \frac{1}{2}\left(A_0 + 1\right) , \tag{218}$$

$$B = 0\,, \tag{219}$$

$$\Delta A_0 = A_0 - 1\,, \tag{220}$$

$$\Delta B_0 = 0\,, \tag{221}$$

$$\Delta T_0 = T_0\,, \tag{222}$$

$$\Delta \phi_0 = 0\,, \tag{223}$$

$$\Delta \phi = \Delta \delta\,, \tag{224}$$

so that, one can obtain from (217) and (222)-(224), for $\Delta B = 0$,

$$\Delta T = T_0 + \epsilon G\left(\alpha, \lambda, \mu, \gamma, \sigma\right) h \left\{\frac{d(\Delta\phi)}{dZ}\right\} , \tag{225}$$

where

$$h_j(s) = \int g_j(s) ds\,, \tag{226}$$

for $j = 1, 2$. Thus,

$$\Delta T = T_0 + O(\epsilon)\,. \tag{227}$$

Now, $T_0 \sim O(1)$ so that $\Delta T \nrightarrow 0$ and thus the pulses do not collide during the transmission. This is observed in the following numerical simulations, see Figure 6.

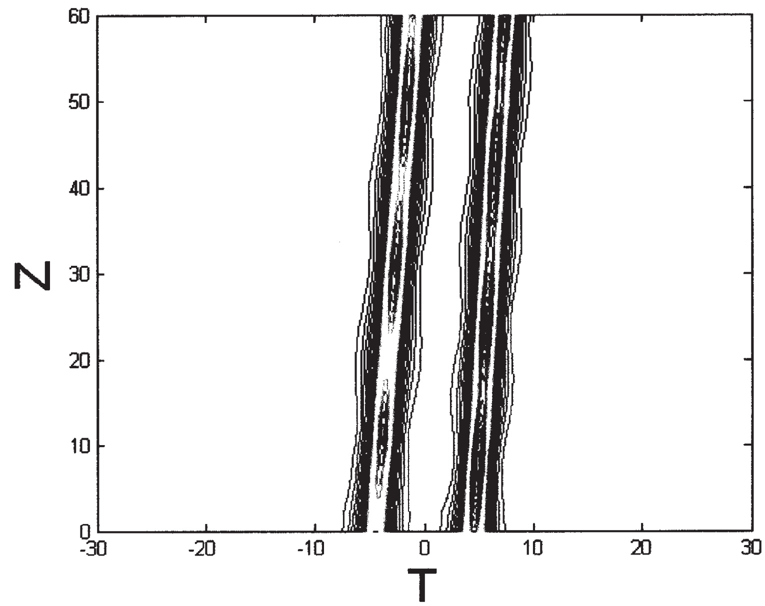

Figure 6: Parameters: $\nu = 0.3$, $\varepsilon = 0.1$, $\alpha = 0.80$, $\gamma = -0.35$, $\sigma = 0.03$

3.4. Dual-Power Law

In this case,

$$q_l(Z,T) = \frac{A_l}{\left[1 + a_l \cosh\left\{D_l\left(T - T_l\right)\right\}\right]^{\frac{1}{2}}} e^{-iB_l(T-T_l)+i\delta_l} , \tag{228}$$

where

$$D_l \equiv \chi(A_l) = A_l^p \left(\frac{2p^2}{1+p}\right)^{\frac{1}{2}} . \tag{229}$$

Here (74) modifies to

$$\begin{aligned} i\frac{\partial q_l}{\partial Z} + \frac{1}{2}\frac{\partial^2 q_l}{\partial T^2} &= i\epsilon R[q_l, q_l^*] \\ &\quad - \left[\sum_{r=0}^{p} \binom{p}{r} q_1^{p-r} q_2^r\right] \left[\sum_{r=0}^{p} \binom{p}{r} \left(q_1^*\right)^{p-r} \left(q_2^*\right)^r\right] (q_1 + q_2) \\ &\quad - \nu \left[\sum_{r=0}^{2p} \binom{2p}{r} q_1^{2p-r} q_2^r\right] \left[\sum_{r=0}^{2p} \binom{2p}{r} \left(q_1^*\right)^{2p-r} \left(q_2^*\right)^r\right] (q_1 + q_2) , \end{aligned} \tag{230}$$

where, the separation

$$
\begin{aligned}
&|q|^{2p}q+\nu|q|^{4p}q \\
&\qquad =\left[\sum_{r=0}^{p}\binom{p}{r}q_1^{p-r}q_2^{r}\right]\left[\sum_{r=0}^{p}\binom{p}{r}\left(q_1^*\right)^{p-r}\left(q_2^*\right)^{r}\right](q_1+q_2) \\
&\qquad +\nu\left[\sum_{r=0}^{2p}\binom{2p}{r}q_1^{2p-r}q_2^{r}\right]\left[\sum_{r=0}^{2p}\binom{2p}{r}\left(q_1^*\right)^{2p-r}\left(q_2^*\right)^{r}\right](q_1+q_2) \qquad (231)
\end{aligned}
$$

was used based on the degree of overlapping. By SPT, the evolution equations are

$$\frac{dA_l}{dZ}=F_1^{(l)}(A,\Delta T,\Delta\phi;\nu,p)+\epsilon M_l\,, \tag{232}$$

$$\frac{dB_l}{dZ}=F_2^{(l)}(A,\Delta T,\Delta\phi;\nu,p)+\epsilon N_l\,, \tag{233}$$

$$\frac{dT_l}{dZ}=-B_l-F_3(A,\Delta T,\Delta\phi;\nu,p)+\epsilon Q_l\,, \tag{234}$$

$$\frac{d\delta_l}{dZ}=\frac{A_l^{2p}}{2p+2}+\frac{B_l^2}{2}+F_4(A,\Delta T,\Delta\phi;\nu,p)+\epsilon P_l\,, \tag{235}$$

where, for the case of dual-power law,

$$M_l=h_1^{(2)}(A_l)\int_{-\infty}^{\infty}\Re\left\{\hat{R}[q_l,q_l^*]e^{-i\phi_l}\right\}\frac{d\tau_l}{\left(1+a_l\cosh\tau_l\right)^{\frac{1}{2p}}}d\tau_l\,, \tag{236}$$

$$N_l=h_2^{(2)}(A_l)\int_{-\infty}^{\infty}\Im\left\{\hat{R}[q_l,q_l^*]e^{-i\phi_l}\right\}\frac{\sinh\tau_l}{\left(1+a_l\cosh\tau_l\right)^{\frac{1}{2p}}}d\tau_l\,, \tag{237}$$

$$Q_l=h_3^{(2)}(A_l)\int_{-\infty}^{\infty}\Re\left\{\hat{R}[q_l,q_l^*]e^{-i\phi_l}\right\}\frac{\tau_l}{\left(1+a_l\cosh\tau_l\right)^{\frac{1}{2p}}}d\tau_l\,, \tag{238}$$

$$P_l=h_4^{(2)}(A_l)\int_{-\infty}^{\infty}\Im\left\{\hat{R}[q_l,q_l^*]e^{-i\phi_l}\right\}\frac{\left(1-a_l\tau_l\sinh\tau_l\right)}{\left(1+a_l\cosh\tau_l\right)^{\frac{1}{2p}}}d\tau_l\,. \tag{239}$$

In addition, the following notations are used

$$\begin{aligned}\hat{R}[q_l, q_l^*] &= R[q_l, q_l^*] + i|q_l|^{2p} q_l + i\nu |q_l|^{4p} q_l \\ &-i \left[\sum_{r=0}^{p} \binom{p}{r} q_1^{p-r} q_2^r \right] \left[\sum_{r=0}^{p} \binom{p}{r} (q_1^*)^{p-r} (q_2^*)^r \right] (q_1 + q_2) \\ &-i \left[\sum_{r=0}^{2p} \binom{2p}{r} q_1^{2p-r} q_2^r \right] \left[\sum_{r=0}^{2p} \binom{2p}{r} (q_1^*)^{2p-r} (q_2^*)^r \right] (q_1 + q_2) . \end{aligned} \tag{240}$$

For the case of dual-power law nonlinearity, the study is split into two subsections.

3.4.1. Non-Hamiltonian Perturbations

In presence of the perturbation terms, as given by, (4), the dynamical system of the soliton parameters, by SPT, are

$$\begin{aligned}\frac{dA}{dZ} &= \frac{4\epsilon\delta A^{2m+2}}{2^{\frac{m+1}{p}} a^{\frac{m+1}{p}} D} \\ &\times F\left(\frac{m+1}{p}, \frac{m+1}{p}, \frac{m+1}{p} + \frac{1}{2}; \frac{a-1}{2a}\right) B\left(\frac{m+1}{p}, \frac{1}{2}\right) \\ &+ \frac{2\epsilon\sigma A^4}{D^2} \int_{-\infty}^{\infty} \frac{1}{(1 + a\cosh\tau)^{\frac{1}{p}}} \left(\int_{-\infty}^{\tau} \frac{ds}{(1 + a\cosh s)^{\frac{1}{p}}} \right) d\tau \\ &- \frac{4\epsilon\beta A^2}{D a^{\frac{1}{p}} 2^{\frac{1}{p}}} \left[D^2 F\left(2 + \frac{1}{p}, \frac{1}{p}, \frac{3}{2} + \frac{1}{p}; \frac{a-1}{2a}\right) B\left(\frac{1}{p}, \frac{3}{2}\right) \right. \\ &\left. + B^2 F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{2} + \frac{1}{p}; \frac{a-1}{2a}\right) B\left(\frac{1}{p}, \frac{1}{2}\right) \right] , \end{aligned} \tag{241}$$

$$\frac{dB}{dZ} = -\frac{\epsilon\beta B D^2}{4p^2 A^2} \frac{F\left(2 + \frac{1}{p}, 1 + \frac{1}{p}, 2 + \frac{1}{p}; \frac{a-1}{2a}\right)}{F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{2} + \frac{1}{p}; \frac{a-1}{2a}\right)} \frac{B\left(1 + \frac{1}{p}, 1\right)}{B\left(\frac{1}{p}, \frac{1}{2}\right)} . \tag{242}$$

For the fixed point of the dynamical system, given by (241) and (242), with $A = 1$ and $B = 0$, one recovers

$$\beta = \frac{\delta}{2^{\frac{m}{p}} a^{\frac{m}{p}}} \left(\frac{1+p}{2p^2}\right)^{\frac{1}{p}} \frac{F\left(\frac{m+1}{p}, \frac{m+1}{p}, \frac{m+1}{p} + \frac{1}{2}; \frac{a-1}{2a}\right)}{F\left(2 + \frac{1}{p}, \frac{1}{p}, \frac{3}{2} + \frac{1}{p}; \frac{a-1}{2a}\right)} \frac{B\left(\frac{m+1}{p}, \frac{3}{2}\right)}{B\left(\frac{1}{p}, \frac{3}{2}\right)}$$

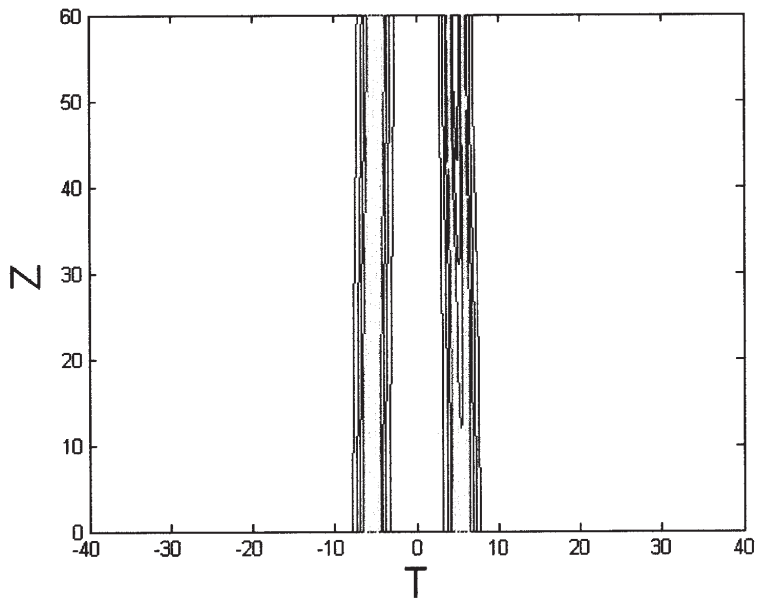

Figure 7: $m = 1$, $\sigma \neq 0$, $\delta \neq 0$, $\beta \neq 0$, $\delta = 0.001$, $\sigma = 0.001$, $\beta = 0.00496$

$$+\frac{\sigma a^{\frac{1}{p}}}{2^{\frac{p-1}{p}}}\left(\frac{1+p}{2p^2}\right)^{\frac{3}{2p}}\frac{1}{B\left(\frac{1}{p},\frac{3}{2}\right)F\left(2+\frac{1}{p},\frac{1}{p},\frac{3}{2}+\frac{1}{p};\frac{a-1}{2a}\right)}$$
$$\int_{-\infty}^{\infty}\frac{1}{(1+a\cosh\tau)^{\frac{1}{p}}}\left(\int_{-\infty}^{\tau}\frac{ds}{(1+a\cosh s)^{\frac{1}{p}}}\right)d\tau\,. \tag{243}$$

Thus, one can obtain using, (104)-105)

$$\frac{d^2(\Delta T)}{dZ^2}+\epsilon\beta G\frac{d(\Delta T)}{dZ}+F_2^{(1)}-F_2^{(2)}=0\,, \tag{244}$$

where β is given by (243) and $G > 0$ represents the coefficient of $-\epsilon\beta\Delta B$ in $d(\Delta B)/dZ = dB_1/dZ - dB_2/dZ$. Now, equation (244) shows that there is a damping in the separation of solitons thus proving that there will be a suppression of the SSI in presence of the perturbation terms given by (4). Thus, in the following figures, the numerical simulations show that the suppression of the SSI is acheived, for the dual-power law, as proved in the QPT.

3.4.2. Hamiltonian Perturbations

In presence of the perturbation terms, as given by (4), the dynamical system of the soliton parameters, by SPT, are

$$\frac{dA}{dZ} = 0\,, \tag{245}$$

$$\frac{dB}{dZ} = 0\,, \tag{246}$$

$$\begin{aligned}\frac{dT_0}{dZ} = -B - \epsilon\left(\mu + 3\gamma B^2\right) - 3\frac{\epsilon\gamma D^2}{2p^2}\frac{F\left(2+\frac{1}{p}, \frac{1}{p}, \frac{1}{p}+\frac{3}{2}; \frac{a-1}{2a}\right)}{F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{2}+\frac{1}{p}; \frac{a-1}{2a}\right)}\frac{B\left(\frac{1}{p}, \frac{3}{2}\right)}{B\left(\frac{1}{p}, \frac{1}{2}\right)} \\ - \epsilon\left(3\lambda + 2\mu\right)\frac{A^2}{2^{\frac{1}{p}}a^{\frac{1}{p}}}\frac{F\left(\frac{2}{p}, \frac{2}{p}, \frac{2}{p}+\frac{1}{2}; \frac{a-1}{2a}\right)}{F\left(\frac{1}{p}, \frac{1}{p}, \frac{1}{2}+\frac{1}{p}; \frac{a-1}{2a}\right)}\frac{B\left(\frac{2}{p}, \frac{1}{2}\right)}{B\left(\frac{1}{p}, \frac{1}{2}\right)}.\end{aligned} \tag{247}$$

From (102)-(105), one can now conclude that

$$\frac{d\left(\Delta A\right)}{dZ} = F_1^{(1)}(A, \Delta T, \Delta\phi) - F_1^{(2)}(A, \Delta T, \Delta\phi)\,, \tag{248}$$

$$\frac{d\left(\Delta B\right)}{dZ} = F_2^{(1)}(A, \Delta T, \Delta\phi) - F_2^{(2)}(A, \Delta T, \Delta\phi)\,, \tag{249}$$

$$\frac{d\left(\Delta T\right)}{dZ} = \frac{dT_1}{dZ} - \frac{dT_2}{dZ}\,, \tag{250}$$

$$\frac{d\left(\Delta\phi\right)}{dZ} = \frac{1}{2}A\Delta A\,. \tag{251}$$

For in-phase injection of solitons with unequal amplitudes,

$$A = \frac{1}{2}\left(A_0 + 1\right)\,, \tag{252}$$

$$B = 0\,, \tag{253}$$

$$\Delta A_0 = A_0 - 1\,, \tag{254}$$

$$\Delta B_0 = 0\,, \tag{255}$$

$$\Delta T_0 = T_0\,, \tag{256}$$

$$\Delta \phi_0 = 0\,, \tag{257}$$

$$\Delta \phi = \Delta \delta\,. \tag{258}$$

Equation (250) can be rewritten as

$$\frac{d\,(\Delta T)}{dZ} = -\Delta B + \epsilon G\,(\alpha, \lambda, \mu, \gamma, \sigma)\, g \left\{ \frac{d(\Delta\phi)}{dZ} \right\}\,, \tag{259}$$

where G is the functional form that depends on the said parameters. For in-phase injection of solitons with unequal amplitudes, so that for $\Delta B = 0$

$$\Delta T = T_0 + \epsilon G\,(\alpha, \lambda, \mu, \gamma, \sigma)\, h \left\{ \frac{d(\Delta\phi)}{dZ} \right\}\,, \tag{260}$$

where

$$h_j(s) = \int g_j(s) ds\,, \tag{261}$$

for $j = 1, 2$. Thus,

$$\Delta T = T_0 + O(\epsilon)\,. \tag{262}$$

Now, $T_0 \sim O(1)$ so that $\Delta T \not\to 0$ and thus the pulses do not collide during the transmission. This is observed in the following numerical simulations.

4. Conclusions

In this paper, the SSI of the NLSE in presence of Hamiltonian as well as non-Hamiltonian type perturbations was studied. The Hamiltinian perturbations included the third and fourth order dispersions, nonlinear dispersion term, self-steepening term and the frequency separation term. On the other hand, the

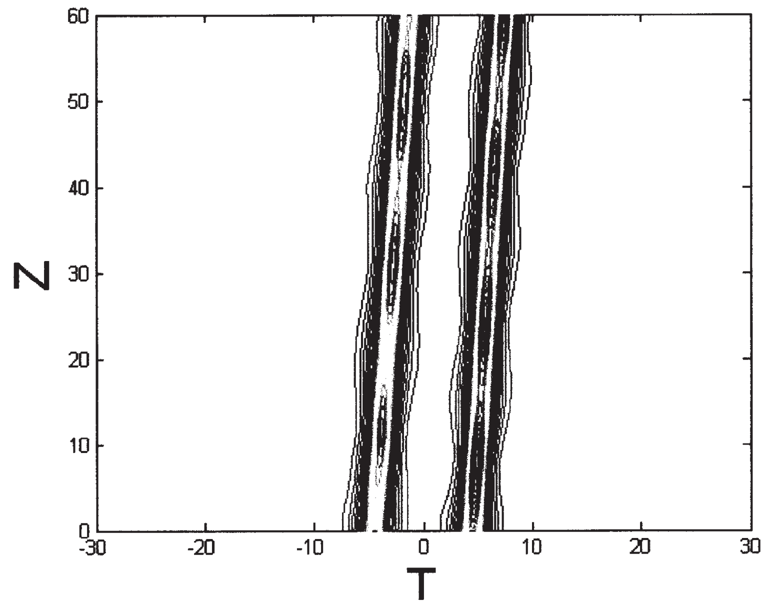

Figure 8: Parameters: $\nu = 0.5$, $\varepsilon = 0.1$, $p = 1/2$, $\alpha = 0.8$, $\gamma = -0.25$, $\sigma = 0.05$.

non-Hamiltonian type perturbation terms included the nonlinear gain, saturable amplifiers and the bandpass filters. The four laws that are considered in this paper are the Kerr law, power law, parabolic law and the dual-power law. It was observed that the SSI can be suppressed in presence of these perturbation terms for all the four laws. The QPT, due to these perturbation terms, for all four laws, were developed and especially for the laws beyond the Kerr law is seen here for the first time. The analytical reasoning of the suppression of the SSI was established.

Thus, in the applied soliton community, two solitons can be injected into a single channel, close to one another and also suppress their mutual interaction, so that performance enhancement can be acheived. This conclusion is based on analytical results due to the quasi-particle theory of SSI that is supported by numerical simulation.

In future, the SSI will be studied in the context of dispersion-managed solitons. Also, the study will be extended to the case of other laws of nonlinearities that are not considered in this paper namely saturable law, log law, exponential law and many others. Moreover, the case of SSI due to intra-pulse Raman scattering will be studied. Finally, this study of QPT can be extended to the case of 3-soliton, 4-soliton interaction and other higher numbers. All such studies are under way.

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