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MULTILINEAR POLYNOMIALS AND POWER VALUES OF DERIVATIONS ON RIGHT IDEALS

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Abstract: Let K be a commutative ring with unity, R a prime K-algebra, with extended centroid C, d a non-zero derivation of R, $f(x_1,\dots,x_n)$ a multilinear polynomial over K, I a non-zero right ideal of R, $a\in R$ and $m\geq 1$ a fixed integer. If $a(d(f(r_1,\dots,r_n) - f(r_1,\dots,r_n))^m = 0$, for all $r_1,\dots,r_n\in I$, then one of the following holds: (i) $f(x_1,\dots,x_n)x_{n+1}$ is an identity for I; (ii) $aI = ad(I) = 0$.

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1. Introduction

Throughout this paper, R always denotes a prime ring with center $Z(R)$ and extended centroid C, Q its two-sided Martindale quotients ring. Here we will consider some related problems concerning annihilators of power values of derivations on multilinear polynomials in prime rings. Recently Bell and Daif [3] proved that if R is a semiprime ring with a non-zero ideal I such that $d([x,y]) - [x,y] = 0$, or $d([x,y]) + [x,y] = 0$, for all $x,y\in I$, then I is central. Later, in [1], Ashraf and Rehman showed that R is commutative if and only if it satisfies any one of the properties $d(xy) - xy\in Z(R)$, $d(xy) + xy\in Z(R)$. Subsequently in [4], we studied the subset $T = \{d(xy) - xy, x,y\in I\}$, where $I\neq 0$ is an ideal of R by examining its left annihilator $l(T) = \{x\in R : xt = 0, \forall t\in T\}$ and proved: (i) Let R be a prime ring, $a\in R$, $d : R\longrightarrow R$ a derivation of R

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and $I \neq 0$ an ideal of R. If, for any $x, y \in I$, $a(d(xy) - xy) = 0$, then $a = 0$; (ii) Let R be a prime ring, $0 \neq a \in R$, $d : R \longrightarrow R$ a derivation of R and $I \neq 0$ an ideal of R. If, for any $x, y \in I$, $a(d(xy) - xy) \in Z(R)$, then R is commutative.

Our purpose here is to continue this line of investigation and extend the previuos cited results. In particular here we study the subset $A_S = \{d(f(x_1, \ldots, x_n)) - f(x_1, \ldots, x_n), x_1, \ldots, x_n \in S\}$, where $f(x_1, \ldots, x_n)$ is a multilinear polynomial over C in n non-commuting variables and S is any subring of R. An approach that can be used in studying A_S is to examine its size and a reasonable criteria for studying the size of A_S is to examine its left annihilator

$$l(A_S) = \{a \in R : a(d(f(x_1, \ldots, x_n)) - f(x_1, \ldots, x_n)) = 0, x_1, \ldots, x_n \in S\}.$$

If A_S is large, we would expect that $l(A_S) = 0$. In [7] we proved that if I is a right ideal of R and $m \geq 1$ a fixed integer such that $a^m = 0$, for all $a \in A_I$, then $f(x_1, \ldots, x_n)x_{n+1}$ is an identity for I. The above quoted result says that the size of the set A_I could be rather large in R. Our goal is to confirm this fact by proving the following theorem.

Theorem 2. *Let K be a commutative ring with unity, R a prime K-algebra, with extended centroid C, d a non-zero derivation of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over K, I a non-zero right ideal of R, $a \in R$ and $m \geq 1$ a fixed integer. If $a(d(f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n))^m = 0$, for all $r_1, \ldots, r_n \in I$, then one of the following holds:*

(i) $f(x_1, \ldots, x_n)x_{n+1}$ is an identity for I;

(ii) $aI = ad(I) = 0$.

We first dispose of the case when R is not a domain. In fact, if R is a domain, by supposing $a \neq 0$, we get $(d(f(r_1, \ldots, r_n)) - f(r_1, \ldots, r_n)) = 0$, for any $r_1, \ldots, r_n \in I$. In this situation, by [7], we conclude that $f(x_1, \ldots, x_n)x_{n+1}$ must be an identity for I.

In order to simplify the arguments which will be used in the prof of our theorem, we premit some notations, definitions and results which are well known in the literature concerning generalized polynomial identities (GPI) and differential identities (DI).

Remark 1. In all that follows let $T = Q *_C C\{X\}$ be the free product over C of the C-algebra Q and the free C-algebra $C\{X\}$, with X the countable set consisting of non-commuting indeterminates $x_1, x_2, \ldots, x_n, \ldots$. We refer the reader to [2] for the definitions and the related properties of these objects.

We recall that every derivation of R can be uniquely extended to a derivation of Q [10]. Moreover, since R is a prime ring, we may assume $K \subseteq C$ and so for any $\alpha \in K$ one has $d(\alpha) \in C$.

We will use the following notation:

$$f(x_1,\dots,x_n)=\alpha x_1x_2\dots x_n+\sum_{\sigma\neq 1}\alpha_\sigma x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(n)}$$

for some $\alpha,\alpha_\sigma\in C$ and moreover we denote by $f^d(x_1,\dots,x_n)$ the polynomial obtained from $f(x_1,\dots,x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma)$. Thus we write $d(f(r_1,\dots,r_n))=f^d(r_1,\dots,r_n)+\sum_i f(r_1,\dots,\ d(r_i),\dots,r_n)$, for all $r_1,r_2,\dots,r_n$ in R.

Remark 2. We will also write a multilinear polynomial $f(x_1,\dots,x_n)$ as follows:

$$f(x_1,\dots,x_n)=\sum_i t_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)x_i\,, \qquad [\mathbf{A}]$$

where t_i are multilinear polynomials in $n-1$ variables, and x_i never appears in any monomials of t_i.

Analogously you can use the following notation:

$$f(x_1,\dots,x_n)=t_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)x_i+h(x_1,\dots,x_n)\,, \qquad [\mathbf{B}]$$

where any t_i is defined as above and h is a multilinear polynomial in n variables such that x_i never appear as last variable in any monomial of h.

Remark 3. Recall that if B is a basis of Q over C, then any element of $T=Q*_C C\{x_1,\dots,x_n\}$ can be written in the form $g=\sum_i\alpha_im_i$, where $\alpha_i\in C$ and m_i are B-monomials, that is $m_i=q_0y_1\cdots\cdot y_nq_n$, with $q_i\in B$ and $y_i\in\{x_1,\dots,x_n\}$. In [6] it is showed that a generalized polynomial $g=\sum_i\alpha_im_i$ is the zero element of T if and only if any α_i is zero.

Before beginning the proof of our result we premit the following lemma.

Lemma 1. *If $d(I)I=0$ then Theorem 2 holds.*

Proof. Let d an inner derivation induced by the element $q\in Q$ and $x,y\in I$, $r\in R$. Then $[q,xr]y=0$, whence $qxry-xrqy=0$, that is $qx=\beta_xx$, with $\beta_x\in C$, and analogously $qy=\beta_yy$, $q(x+y)=\beta_{x+y}(x+y)$. From this, it is easy to see that β_x is independent from the choice of $x\in I$, therefore there exists $\beta\in C$ such that $(q-\beta)I=0$.

Let $p=q-\beta$, then q and p induce the same inner derivation and, for all $r_1,\dots,r_n\in I$,

$$\begin{aligned}0&=a\left([p,(f(r_1,\dots,r_n))]-f(r_1,\dots,r_n)\right)^m\\&=(-1)^ma\left(f(r_1,\dots,r_n)p+f(r_1,\dots,r_n)\right)^m.\end{aligned}$$

Right multiplying by $r_{n+1} \in I$, we get $(-1)^m a f(r_1, \ldots, r_n)^m r_{n+1} = 0$. Therefore, by main result in [6], it follows that either $aI = 0$ or $f(x_1, \ldots, x_n)x_{n+1}$ is an identity for I, and we are done.

Let now d an outer derivation. Then, for any $0 \neq c \in I$, R satisfies $d(cx)cy = d(c)xcy + cd(x)cy$. By using Kharchenko's result in [9], it follows that R satisfies $d(c)x_1cx_2 + cx_3cx_2$ and in particular R satisfies the blended component cx_3cx_2. This means, since R is prime, that $c = 0$, a contradiction. □

In the light of this last lemma, in what follows we may assume that $d(I)I \neq 0$.

Lemma 2. *Let d an inner derivation induced by $q \in Q$. If R does not satisfy any non-trivial GPI, then Theorem 2 holds.*

Proof. By our assumption, for any $r_1, \ldots, r_n \in I$, $a([q, f(r_1, \ldots, r_n)] - f(r_1, \ldots, r_n))^m = 0$. Since the condition $qI = 0$ is equivalent to $d(I)I = 0$, thanks to Lemma 1, we consider the only case when $qI \neq 0$.

Suppose there exists $c \in I$ such that ac and aqc are linearly C-independent. Since R is not a GPI-ring

$$\begin{aligned} a([q, f(cx_1, cx_2, \ldots, cx_n)] &- f(cx_1, \ldots, cx_n))^m \\ &= (aqf(cx_1, \ldots, cx_n) - af(cx_1, \ldots, cx_n)q - af(cx_1, \ldots, cx_n)) \\ &\quad \cdot ([q, f(cx_1, cx_2, \ldots, cx_n)] - f(cx_1, \ldots, cx_n))^{m-1} = 0 \in T \end{aligned}$$

is a trivial generalized polynomial identity for R. Moreover, since ac and aqc are independent,

$$aqf(cx_1, \ldots, cx_n) \cdot ([q, f(cx_1, cx_2, \ldots, cx_n)] - f(cx_1, \ldots, cx_n))^{m-1} = 0 \in T,$$

that is

$$\begin{aligned} aq(\alpha cx_1 \cdot cx_2 \cdots cx_n &+ \sum_{\sigma \neq 1} cx_{\sigma(1)} \cdots cx_{\sigma(n)}) \\ &\cdot ([q, f(cx_1, cx_2, \ldots, cx_n)] - f(cx_1, \ldots, cx_n))^{m-1} = 0 \in T. \end{aligned}$$

Therefore

$$aq(\alpha cx_1 \cdot cx_2 \cdots cx_n) \cdot ([q, f(cx_1, cx_2, \ldots, cx_n)] - f(cx_1, \ldots, cx_n))^{m-1} = 0 \in T,$$

that is

$$\begin{aligned} aq(\alpha cx_1 \cdot cx_2 \cdots cx_n) &\cdot (qf(cx_1, \ldots, cx_n) - f(cx_1, \ldots, cx_n)q - f(cx_1, \ldots, cx_n)) \\ &\cdot ([q, f(cx_1, cx_2, \ldots, cx_n)] - f(cx_1, \ldots, cx_n))^{m-2} = 0 \in T. \end{aligned}$$

Since c and qc are linearly C-independent, we get

$$aq(\alpha cx_1 \cdot cx_2 \cdots cx_n) \cdot (qf(cx_1, \ldots, cx_n)) \\ \cdot ([q, f(cx_1, cx_2, \ldots, cx_n)] - f(cx_1, \ldots, cx_n))^{m-2} = 0 \in T.$$

As above

$$(\alpha aqcx_1 \cdot cx_2 \cdots cx_n) \cdot \alpha qcx_1 \cdot cx_2 \cdots cx_n \\ \cdot ([q, f(cx_1, cx_2, \ldots, cx_n)] - f(cx_1, \ldots, cx_n))^{m-2} = 0 \in T.$$

Repeating this process yields

$$a(\alpha qcx_1 \cdot cx_2 \cdots cx_n)^m = 0\,,$$

a contradiction.

Thus, for all $c \in I$, ac and aqc are linearly C-dependent. This means that, for all $c \in I$, there exists $\alpha = \alpha_c \in C$ such that $\alpha_c ac = aqc$. In this case, standard arguments show that there exists $\alpha \in C$, such that $\alpha ac = aqc$, for all $c \in I$, that is $a(q - \alpha)c = 0$. Replace q by $(q - \alpha)$, so the condition is now $aqc = 0$.

For this reason, the main assumption says that

$$(-af(cx_1, \ldots, cx_n)q - af(cx_1, \ldots, cx_n)) \\ \cdot ([q, f(cx_1, cx_2, \ldots, cx_n)] - f(cx_1, \ldots, cx_n))^{m-1} = 0 \in T.$$

Since $c \notin C$, then

$$(-af(cx_1, \ldots, cx_n)q - af(cx_1, \ldots, cx_n)) \cdot ([q, f(cx_1, cx_2, \ldots, cx_n)] \\ - f(cx_1, \ldots, cx_n))^{m-2} \cdot (-f(cx_1, \ldots, cx_n)q - f(cx_1, \ldots, cx_n)) = 0 \in T.$$

Moreover, since qc and c are linearly C-independent

$$(-af(cx_1, \ldots, cx_n)q - af(cx_1, \ldots, cx_n)) \cdot ([q, f(cx_1, cx_2, \ldots, cx_n)] \\ - f(cx_1, \ldots, cx_n))^{m-3} \cdot (-f(cx_1, \ldots, cx_n)q - f(cx_1, \ldots, cx_n))^2 = 0 \in T.$$

Repeating this process, we get

$$a(-f(cx_1, \ldots, cx_n)q - f(cx_1, \ldots, cx_n))^m = 0 \in T\,,$$

which is a contradiction, because R does not satisfy any non-trivial generalized polynomial identity, unless when $ac = 0$.

Finally suppose that for any $c \in I$ there exists $\alpha_c \in C$ such that $qc = \alpha_c c$. So it follows that $0 = aqc = \alpha_c ac$. In any case $ac = 0$, for all $c \in I$, i.e. $aI = 0$.

Then we conclude that, if R does not satisfy any non-trivial generalized polynomial identity, must be $aI = ad(I) = 0$. □

2. The Case $I = R$

In this first section we will assume that $a(d(f(r_1,\dots,r_n) - f(r_1,\dots,r_n))^m = 0$, for all $r_1,\dots,r_n \in R$. Now our aim is to prove the following

Theorem 1. *Let R be a prime K-algebra, d a nonzero derivation of R, $f(x_1,\dots,x_n)$ a multilinear polynomial over K in n non-commuting variables, $a \in R$ and $m \geq 1$. Suppose that $f(x_1,\dots,x_n)$ is not an identity on R. If $a\,(d(f(r_1,\dots,r_n) - f(r_1,\dots,r_n))^m = 0$, for any $r_1,\dots,r_n \in R$, then $a = 0$.*

Lemma 3. *If d is an outer derivation of R then Theorem 1 holds.*

Proof. Suppose by contradiction that $a \neq 0$. We denote by $f^d(x_1,\dots,x_n)$ the polynomial obtained from $f(x_1,\dots,x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma\cdot 1)$. Thus we write $d(f(x_1,\dots,x_n)) = f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,d(x_i),\dots,x_n)$. Since R satisfies the generalized differential identity

$$a\,(d(f(x_1,\dots,x_n)) - f(x_1,\dots,x_n))^m = a\left(f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,d(x_i),\dots,x_n) - f(x_1,\dots,x_n)\right)^m$$

and d is an outer derivation, by [9] R satisfies the generalized polynomial identity

$$a\left(f^d(x_1,\dots,x_n) + \sum_i f(x_1,\dots,y_i,\dots,x_n) - f(x_1,\dots,x_n)\right)^m$$

and in particular R satisfies $a\,(f(y_1,x_2,\dots,x_n))^m$. As a consequence of [6], since R is prime and $f(x_1,\dots,x_n)$ is not an identity for R, we get $a = 0$. □

Lemma 4. Let d be an inner derivation induced by the element $q \in Q$. If R is a dense ring of linear transformations over an infinite dimensional right vector space V over a division ring D, then $a = 0$.

Proof. Suppose $a \neq 0$.

Since $f(x_1,\dots,x_n)$ is a multilinear polynomial and $a([q,f(x_1,\dots,x_n)] - f(x_1,\dots,x_n))^m$ is a generalized identity for R, by [14, Lemma 2] we have $a([q,r]-r)^m = 0$, for all $r \in R$. Suppose there exists $v \in V$ such that v and qv are linearly D-independent. By the density of R, there exists $x \in R$ such that $xv = 0$ and $x(qv) = v$. Thus $0 = a([q,x]-x)^m v = (-1)^m av$. Since $av \neq 0$, we get a contradiction. Therefore v and qv are linearly independent, for all $v \in V$, that is $q \in C$, a contradiction again. This means that $a = 0$. □

Proof of Theorem 1. Since Lemma 3 gives a complete proof in case d is an outer derivation, now we assume that d is inner, induced by $q \in Q$.

Since by [10] R and Q satisfy the same differential identities, we have that $a\left(d(f(x_1,\ldots,x_n)) - f(x_1,\ldots,x_n)\right)^m = 0$ also in Q. Moreover Q is prime, by the primeness of R, and replacing R by Q we may assume, without loss of generality, $C = Z(R)$ and R is a C-algebra centrally closed, that is $R = RC$. If R does not satisfy any non-trivial generalized polynomial identity then, by Lemma 1, $a = 0$. Thus we may suppose that R satisfies a non-trivial generalized polynomial identity. By Martindale's Theorem in [12], R is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over a division ring D. If $\dim_D V = \infty$, then, by Lemma 4, we get the conclusion required.

Therefore consider the case $\dim_D(V) = k$, with k finite positive integer ≥ 2, because R is not a domain. In this condition R is a simple ring which satisfies a non-trivial generalized polynomial identity. By [13 Theorem 2.3.29] $R \subseteq M_t(F)$, for a suitable field F and $t \geq 2$, moreover $M_t(F)$ satisfies the same generalized identity of R. Suppose $f(x_1,\ldots,x_n)$ is not central on R.

By [11] there exist $u_1,\ldots,u_n \in M_t(F)$, such that $f(u_1,\ldots,u_n) = \beta e_{ij}$, for some distinct i,j, with $\beta \in F - \{0\}$ and e_{ij} the usual matrix unit with 1 in (i,j)-entry and zero elsewhere. Moreover, since the set $\{f(x_1,\ldots,x_n) : x_1,\ldots,x_n \in M_t(F)\}$ is invariant under the action of all F-automorphisms of $M_t(F)$, then for any $i \neq j$ there exist $r_1,\ldots,r_n \in M_t(F)$ such that $f(r_1,\ldots,r_n) = \beta e_{ij}$.

Suppose by contradiction that the matrix $a = \sum a_{hl}e_{hl}$ is not zero. Let $q = \sum q_{hl}e_{hl}$, with $q_{hl} \in F$ and fix i and $j \neq i$. Then

$$\begin{aligned} 0 &= a([q, f(r_1,\ldots,r_n)] - f(r_1,\ldots,r_n))^m \\ &= a\left(qf(r_1,\ldots,r_n) - f(r_1,\ldots,r_n)q - f(r_1,\ldots,r_n)\right)^m \\ &= a\left(q\beta e_{ij} - \beta e_{ij}q - \beta e_{ij}\right)^m . \end{aligned}$$

In particular, right multiplying by $e_{ij}q$ we have

$$0 = a\left(q\beta e_{ij} - \beta e_{ij}q - \beta e_{ij}\right)^m e_{ij}q = a(-\beta)^m(e_{ij}q)^{m+1}.$$

Then, for all $j \neq i$, either $q_{ji} = 0$ or the i-th column of the matrix a is zero.

Case 1: $t = 2$. Since $f(x_1, \ldots, x_n)$ is not central on R, by [11, Lemma 2 and Lemma 9], there exists a sequence of matrices $r = (r_1, \ldots, r_n)$ such that $f(r) = \beta e_{21}$ is not zero.

Suppose that q is not diagonal, say $q_{12} \neq 0$, then the 2-nd column of a is zero. In other words we are in the following situation:

$$q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \; q_{12} \neq 0, \quad a = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}, \; f(r) = \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix}.$$

By calculation it follows that

$$([q, f(r)] - f(r))^{2m} = \begin{bmatrix} (q_{12}\beta)^{2m} & 0 \\ 0 & (q_{12}\beta)^{2m} \end{bmatrix}.$$

Then $0 = a[q, f(r) - f(r)]^{2m} = a(q_{12}\beta)^{2m} = 0$ and so $a = 0$.

Moreover we get the same conclusion if suppose $q_{21} \neq 0$. Thus we conclude that if $k = 2$, either q is a diagonal matrix or $a = 0$.

Case 2: $t \geq 3$. Also in this case we want to prove that if a is not zero then q is a diagonal matrix. Suppose there exists $q_{ji} \neq 0$, $i \neq j$, then the i-th column of a is zero. For all $l \neq i, j$ let $\varphi_{li} \in \mathrm{Aut}_F(M_t(F))$ such that $\varphi_{li}(x) = (1 + e_{li})x(1 - e_{li})$. Consider the following valutations of $f(x_1, \ldots, x_n)$:

$$f(r) = \gamma e_{ij}, \quad f(s) = \varphi_{li}(f(r)) = \gamma e_{ij} + \gamma e_{lj}, \quad \gamma \neq 0.$$

Since the i-th column of a is zero, by $a([q, f(s)] - f(s))^m = 0$ and right multiplying by $e_{ij} + e_{lj}$, we have:

$$0 = a([q, f(s)] - f(s))^m (e_{ij} + e_{lj}) = a(-\gamma)^m (q_{ji} + q_{jl})^m (e_{ij} + e_{lj}). \quad (1)$$

Notice that if $q_{ji} + q_{jl} = 0$, then $q_{jl} = -q_{ji} \neq 0$ and, as in the first part of the proof, the l-th column of a is zero. On the other hand, if $q_{ji} + q_{jl} \neq 0$, by (1), for all k, $a_{kl} = -a_{ki}$ and, since the i-th column of a is zero, it follows again that the l-th one is also zero. Hence we can say that the matrix a has at most one nonzero column, the j-th one.

Thus $a = ae_{jj}$ and so

$$\begin{aligned} 0 = a([q, f(r)] - f(r))^m &= ae_{jj}([q, \gamma e_{ij}] - \gamma e_{ij})^m \\ &= ae_{jj}(q\gamma e_{ij} - \gamma e_{ij} q - \gamma e_{ij})([q, \gamma e_{ij}] - \gamma e_{ij})^{m-1} \\ &= ae_{jj} q\gamma e_{ij}([q, \gamma e_{ij}] - \gamma e_{ij})^{m-1} = \cdots = ae_{jj}(q\gamma e_{ij})^m = a(q_{ji}\gamma)^m. \end{aligned}$$

Hence $a = 0$ follows from $q_{ji} \neq 0$.

The previous two cases show that if a is a nonzero matrix then q is a diagonal one, $q = \sum q_{kk}e_{kk}$. Now let $\varphi_{ij} \in \mathrm{Aut}_F(M_t(F))$ such that $\varphi_{ij}(x) = (1+e_{ij})x(1-e_{ij})$, with $i \neq j$. Since $0 = \varphi_{ij}(a)([\varphi_{ij}(q), \varphi(f(x_1, \ldots, x_n))] - \varphi(f(x_1, \ldots, x_n)))^m = \varphi_{ij}(a)([\varphi_{ij}(q), f(y_1, \ldots, y_n)] - f(y_1, \ldots, y_n))^m$ and $\varphi_{ij}(a) \neq 0$, we have that $\varphi(q)$ is also diagonal. On the other hand $\varphi_{ij}(q) = q + (q_{jj} - q_{ii})e_{ij}$, i.e. $q_{jj} = q_{ii}$ and q is central in $M_t(F)$, which is a contradiction. Therefore if $f(x_1, \ldots, x_n)$ is not central then $a = 0$.

Consider now the case when $f(x_1, \ldots, x_n)$ is central on R. In this situation we have that

$$0 = a([q, f(x_1, \ldots, x_n)] - f(x_1, \ldots, x_n))^m = a(f(x_1, \ldots, x_n))^m.$$

Since $f(x_1, \ldots, x_n)^m$ is also central in R but $f(x_1, \ldots, x_n)$ is not an identity in R, it follows again that $a = 0$. □

3. The One-Sided Case

Now we consider the case when $a(d(f(x_1, \ldots, x_n)) - f(x_1, \ldots, x_n))^m = 0$ holds for all $x_1, \ldots, x_n \in I$, a right ideal of R.

Remark 4. Without loss of generality, when d is an inner derivation of R, R is simple and equal to its own socle, $IR = I$ and $a \in I$. In fact, by Lemma 1, R is GPI (otherwise we are done). So Q has non-zero socle H with non-zero right ideal $J = IH$ [12]. Note that H is simple, $J = JH$ and J satisfies the same basic conditions as I, in view of [10]. Since $Ja \neq 0$, we may replace a by $0 \neq c_1 a$, for some $c_1 \in J$. Now just replace R by H, I by J, a by $c_1 a$ and we are done.

Remark 5. Thanks to Lemma 2, in all that follows we assume that there exist $c_1, c_2 \in I$ such that $d(c_1)c_2 \neq 0$. Moreover we assume that $f(x_1, \ldots, x_n)x_{n+1}$ is not an identity for I, i.e. there exist $b_1, \ldots, b_{n+1} \in I$ such that $f(b_1, \ldots, b_n)b_{n+1} \neq 0$.

Lemma 5. *If d is an inner derivation, then Theorem 2 holds.*

Proof. Let $d(x) = [q, x]$, for all $x \in R$. By Remark 4, we will assume that R is a GPI-ring. As the definition [A] in Remark 2, write :

$$f(x_1, \ldots, x_n) = \sum_i t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)x_i.$$

Obviously $f(x_1, \ldots, x_n)$ is not an identity for I, thus there exists some i such that $t_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)x_i$ is not an identity for I. For simplicity fix

$i = n$ and $t_n(x_1, \ldots, x_{n-1}) = t(x_1, \ldots, x_{n-1})x_n$. So there exist $a_1, \ldots, a_n \in I$ such that $t(a_1, \ldots, a_{n-1})a_n \neq 0$.

Assume first that $aI = 0$ and suppose $ad(I) \neq 0$. Our aim is to get a contradiction. There exists $c \in I$ such that $ad(c) \neq 0$.

By Remark 4, R is regular, so there exists $e^2 = e \in R$ such that $eR = cR + a_1R + \cdots + a_nR$ and $ec = c$, $ea_i = a_i$, for all $i = 1, \ldots, n$, hence $e \in IR$. For any $r_1, \ldots, r_n \in R$

$$f(er_1, er_2, \ldots, er_{n-1}, er_n(1-e)) = t(er_1, \ldots, er_{n-1})er_n(1-e)$$

and

$$\begin{aligned} 0 = a([q, t(er_1, \ldots, er_{n-1})er_n(1-e)] - t(er_1, \ldots, er_{n-1})er_n(1-e))^m \\ = aqt(er_1, \ldots, er_{n-1})er_n(1-e) \\ \cdot ([q, t(er_1, \ldots, er_{n-1})er_n(1-e)] - t(er_1, \ldots, er_{n-1})er_n(1-e))^{m-1} \\ = aqt(er_1, \ldots, er_{n-1})er_n(1-e)\left(qt(er_1, \ldots, er_{n-1})er_n(1-e)\right)^{m-1}, \end{aligned}$$

which implies

$$aqt(er_1, \ldots, er_{n-1})e\left(r_n(1-e)qt(er_1, \ldots, er_{n-1})e\right)^m = 0.$$

By a result in [8],

$$aqet(r_1e, \ldots, r_{n-1}e)eR(1-e)qet(r_1e, \ldots, r_{n-1}e)e = 0$$

and by the primeness of R, either $aqet(er_1, \ldots, er_{n-1})e = 0$ or $(1-e)qet(er_1, \ldots, er_{n-1})e = 0$. Since $ae = 0$, in both cases we have

$$0 = a(1-e)qet(er_1, \ldots, er_{n-1})e = aqet(er_1, \ldots, er_{n-1})e.$$

By the main result in [6], either $t(x_1, \ldots, x_{n-1})x_n$ is an identity for eR, or $aqe = 0$, that is $ae = ad(e) = 0$. On the other hand $0 = ad(e)c = ad(ec) = ad(c) \neq 0$, and also $0 \neq t(a_1, \ldots, a_{n-1})a_n = t(ea_1, \ldots, ea_{n-1})\ ea_n = 0$, a contradiction. Hence we conclude that $aI = 0$ forces also $ad(I) = 0$.

The second part of this lemma is dedicated to prove that the case $aI \neq 0$ cannot occur. To show this, we assume $aI \neq 0$ and get a contradiction. Recall that $f(I)I \neq 0$ and $d(I)I \neq 0$, as stated in Remark 5.

Let $a_{n+1} \in I$ such that $aa_{n+1} \neq 0$. Since $R = H$ is a regular ring, there exists $e^2 = e \in R$ such that $eR = c_1R + c_2R + \sum_i a_iR + \sum_j b_jR$, thus $e \in IR = I$ and $c_1 = ec_1$, $c_2 = ec_2$, $a_i = ea_i$ for all $i = 1, \ldots, n+1$, $b_j = eb_j$ for all $j = 1, \ldots, n+1$.

For any $r_1, \ldots, r_n \in R$ we have

$$f(er_1, er_2, \ldots, er_{n-1}, er_n(1-e)) = t(er_1, \ldots, er_{n-1})er_n(1-e)\,,$$

so that

$$a([q, t(er_1, \ldots, er_{n-1})er_n(1-e)] - t(er_1, \ldots, er_{n-1})er_n(1-e))^m = 0$$

and right multiplying by e

$$(-1)^m a\,(t(er_1, \ldots, er_{n-1})er_n(1-e)q)^m\, e = 0\,,$$

that is

$$at(er_1, \ldots, er_{n-1})e\,(r_n(1-e)qet(er_1, \ldots, er_{n-1})e)^{m+1} = 0.$$

Again by [8] we get

$$at(er_1, \ldots, er_{n-1})eR(1-e)qet(er_1, \ldots, er_{n-1})e = 0$$

and by the primeness of R

$$\text{either} \quad aet(er_1, \ldots, er_{n-1})e = 0, \quad \text{or} \quad (1-e)qet(er_1, \ldots, er_{n-1})e = 0.$$

Note that $ae \neq 0$, because $aea_{n+1} = aa_{n+1} \neq 0$. Thus, thanks to [6], either $(1-e)qe = 0$ or $t(x_1, \ldots, x_{n-1})x_n$ is an identity for eR. This last case cannot occur because $0 \neq t(a_1, \ldots, a_{n-1})a_n = t(ea_1, \ldots, ea_{n-1})ea_n$. Therefore we have $(1-e)qe = 0$.

By $qe = eqe$, it follows $d(eR) \subseteq eR$ and so the derivation d induces another one in the ring $\overline{eR} = \frac{eR}{eR \cap l_R(eR)}$, where $l_R(eR)$ is the left annihilator of eR in R. We have $\overline{d(x)} = \overline{d}(\overline{x})$, for all $x \in eR$. Moreover $a\overline{d}(\overline{f(r_1, \ldots, r_n)} - \overline{f(r_1, \ldots, r_n)})^m = \overline{0}$, for all $r_1, \ldots, r_n \in eR$. By Theorem 1, we have that either $a = \overline{0}$ or $\overline{d} = \overline{0}$, unless $\overline{f(x_1, \ldots, x_n)}$ is an identity in $\overline{eR}$.

In this last case we have that $f(eR)eR = 0$, which contradicts with $0 \neq f(b_1, \ldots, b_n)b_{n+1} = f(eb_1, \ldots, eb_n)eb_{n+1}$.

Moreover if $\overline{d} = \overline{0}$, then $d(eR)eR = 0$ which contradicts with $0 \neq d(c_1)c_2 = d(ec_1)ec_2$.

In the either cases we get $a = \overline{0}$, then $aeR = 0$ and again we get a contradiction by $0 \neq aa_{n+1} = aea_{n+1}$. □

Proof of Theorem 2. By Lemma 5 we may consider the only case when d in an outer derivation. By the assumption, for $c \in I$,

$$a\Big(f^d(cx_1,\dots,cx_n) + \sum_i f(cx_1,\dots,d(c)x_i + cd(x_i),\dots,cx_n) - f(cx_1,\dots,cx_n)\Big)^m$$

is a differential identity for R. By Kharchenko's Theorem [9], since d is not inner, R satisfies

$$a\Big(f^d(cx_1,\dots,cx_n) + \sum_i f(cx_1,\dots,d(c)x_i + cy_i,\dots,cx_n) - f(cx_1,\dots,cx_n)\Big)^m$$

and so R satisfies any blended component

$$a\left(f(cx_1,\dots,cy_i,\dots,cx_n)\right)^m ,$$

which means $a\left(f(cr_1,\dots,cr_n)\right)^m = 0$, for all $c \in I$ and $r_1,\dots,r_n \in R$.

Suppose that $aI \neq 0$, there exists $a_1 \in I$ such that $aa_1 \neq 0$. Then R is a non-trivial GPI-ring, so it is a regular ring. Thus $a_1R + \sum_i b_iR = eR$ for some idempotent element $e \in R$. Moreover $e \in IR$ and $ea_1 = a_1$, $eb_i = b_i$, for $i = 1,\dots,n+1$. Since $a\left(f(er_1,\dots,er_n)\right)^m = 0$, for all $r_1,\dots,r_n \in R$, then by [6], either $ae = 0$ or $f(eRe) = 0$. Both cases cannot occur, otherwise we should have either the contradiction $0 = ae = aea_1 = aa_1 \neq 0$, or the one $0 \neq f(b_1,\dots,b_n)b_{n+1} = f(eb_1,\dots,eb_n)eb_{n+1} = f(eb_1e,\dots,eb_ne)eb_{n+1} = 0$.

Let now $aI = 0$. By using the notation (B) in Remark 2, write

$$f(x_1,\dots,x_n) = t_j(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)x_j + h(x_1,\dots,x_n)\,,$$

where x_j never appears as last variable in any monomials of h. Without loss of generality, assume that $n = j$ and t_n is not an identity for I. Thus I satisfies the differential identity

$$a(d(f(x_1,\dots,x_na) - f(x_1,\dots,x_na))^m = a\left(d(t_n(x_1,\dots,x_{n-1})x_na) - t_n(x_1,\dots,x_{n-1})x_na\right)^m$$

and so I satisfies $(ad(t_n(x_1,\dots,x_{n-1}))x_n)^{m+1}$. Then, by [8], $ad(t_n(x_1,\dots,x_{n-1}))x_n$ is a differential identity for I.

Since $aI = 0$ and by the previous identity, we get $ad(t_n(x_1, \dots, x_{n-1})x_n) = 0$ in I. In particular it follows that $ad(t_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)x_i)$ is a differential identity for I, for all $i = 1, \dots, n$, so that $ad(f(x_1, \dots, x_n))$ is a differential identity for I.

Let now $c \in I$, such that $ad(c) \neq 0$. As above, since $ad(f(cx_1, \dots, cx_n))$ is a non-trivial GPI for R, then R is regular, $cR + \sum_i b_i R = eR$ for some idempotent element $e \in IR$. Moreover $e \in IR$ and $ec = c$, $eb_i = b_i$, for $i = 1, \dots, n+1$. Since $ae = 0$, we have that, for any $r_1, \dots, r_n \in R$

$$0 = ad(f(er_1e, \dots, er_ne)) = ad(ef(r_1e, \dots, r_ne)) = ad(e)f(r_1e, \dots, r_ne)$$

and again from [6] one has either $ad(e) = 0$, or $f(x_1, \dots, x_n)$ is an identity for $eRCe$. Also in this case, both conclusions cannot occur, if not we get either the contradiction $0 = ad(e)c = ad(ec) = ad(c) \neq 0$ or the one $0 \neq f(b_1, \dots, b_n)b_{n+1} = f(eb_1, \dots, eb_n)eb_{n+1} = f(eb_1e, \dots, eb_ne)eb_{n+1} = 0$. So that, for all $c \in I$, $ad(c) = 0$, that is $aI = ad(I) = 0$ and we are done. □

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