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CURVILINEAR JOINS AND CURVILINEAR SECANT VARIETIES TO SUBVARIETIES OF A PROJECTIVE SPACE

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Abstract: Fix integral subvarieties $X_1, \dots X_s$ of $\mathbf{P}^n$ and integers $m_i > 0$, $1 \leq i \leq s$. Here we introduce and study the variety $\mathcal{C}(X_1, m_1, \dots, X_s, m_s) \subseteq \mathbf{P}^n$ defined as the closure in $\mathbf{P}^n$ of the union of all linear subspaces $\langle Z_1 \cup \cdots \cup Z_s \rangle$, where Z_i is a length m_i connected curvilinear subscheme of X_i.

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1. Curvilinear Secant Varieties

Here we discuss a geometrically significant modification of the secant varieties of projective varieties and of the join of two (or more) subvarieties of $\mathbf{P}^n$. Fix integral subvarieties $X_1, \dots X_s, Y_1, \dots, Y_k$ of $\mathbf{P}^n$ and integers $m_i > 0$, $1 \leq i \leq s$. Let $\mathcal{C}(X_1, m_1, \dots, X_s, m_s) \subseteq \mathbf{P}^n$ denote the closure in $\mathbf{P}^n$ of the union of all linear subspaces $\langle Z_1 \cup \cdots \cup Z_s \rangle$, where Z_i is a length m_i connected curvilinear subscheme of X_i. Let $[Y_1; Y_2]$ denote the join of the varieties Y_1, Y_2, i.e. set $[Y_1; Y_2] := \{P\}$ if $Y_1 = Y_2 = \{P\}$ for some $P \in \mathbf{P}^n$, while in all other cases let $[Y_1; Y_2]$ be the closure in $\mathbf{P}^n$ of the union of all lines $\langle \{P, Q\} \rangle$ spanned by some $P \in Y_1$ and some $Q \in Y_2$ such that $P \neq Q$. If $k \geq 3$ define inductively $[Y_1; \dots; Y_k]$ by the formula $[Y_1; \dots; Y_k] := [[Y_1; \dots; Y_{k-1}]; Y_k]$. Hence $\mathcal{C}(X_1, m_1, \dots, X_s, m_s)$ and $[Y_1; \dots; Y_k]$ are integral varieties. The aim of this

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short note is the proof of the following result.

Theorem 1. *Let $X \subset \mathbf{P}^n$ be an integral variety and $C \subset \mathbf{P}^n$ an integral non-degenerate curve. Then for all integers $m > 0$ we have* $\dim([X; \mathcal{C}(C,m)]) = \min\{n, \dim(X) + m + 2\}$.

Theorem 1 has two parts:

(i) The variety $\mathcal{C}(C,m)$ has the expected dimension;

(ii) The join of the variety $\mathcal{C}(C,m)$ with any other subvariety of $\mathbf{P}^n$ has the expected dimension.

If we take C instead of $\mathcal{C}(C,m)$, then part (ii) is known in arbitrary characteristic ([1]; Part (ii) of Proposition 1.3). The proof of part (i) will be easy. It is the combination of both parts which is our main point. We strongly believe that the curvilinear set-up is important even if instead of C we take a variety of larger dimension, but of course, Theorem 1 cannot be true in that generality, even if we take just a point instead of X.

Remark 1. Assume $s \geq 2$. It is easy to check that $\mathcal{C}(X_1, m_1, \dots, X_s, m_s) = [\mathcal{C}(X_1, m_1); \dots; \mathcal{C}(X_s, m_s)]$. Hence the study of all varieties $\mathcal{C}(X_1, m_1, \dots, X_s, m_s)$ (or at least of their dimension) may be divided in two parts:

(i) the study of the varieties $\mathcal{C}(X,m)$;

(ii) the study of the joins of the varieties arising in (i) (for certain classes of varieties X and certain integers m).

By Remark 1 Theorem 1 has the following corollary.

Corollary 1. *Fix integers $m > 0$, $s \geq 0$ and $a_i > 0$, $1 \leq i \leq s$. Let $X_i \subset \mathbf{P}^n$, $1 \leq i \leq s$, be integral varieties and $C \subset \mathbf{P}^n$ an integral non-degenerate curve. Then*

$$\dim(\mathcal{C}(X_1, n_1, \dots, X_s, n_s, C, m)) \\ = \min\{n, \dim(\mathcal{C}(X_1, n_1, \dots, X_s, n_s)) + m + 2\}\,.$$

After finitely many steps Corollary 1 gives the following result.

Corollary 2. *Fix integers $s > 0$ and $m_i > 0$, $1 \leq i \leq s$. Let $C_i \subset \mathbf{P}^n$, $1 \leq i \leq s$, be integral non-degenerate curves. Then*

$$\dim(\mathcal{C}(C_1, m_1, \dots, C_s, m_s)) = \min\{n, m_1 + \cdots + m_s + 2s - 1\}\,.$$

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. For the reason of this restriction on $\mathrm{char}(\mathbb{K})$, see Example 1 and Example 2 below. To prove Theorem 1 we need the following very easy result.

Lemma 1. *Let $C \subset \mathbf{P}^n$ be an integral non-degenerate curve. For all integers $m > t \geq 0$ we have* $\dim(\mathcal{C}(X,m)) = \min\{n,m\}$ *and* $\dim(\mathcal{C}(X,m;t)) = \min\{(n-t)((t+1), 1+(m-t-1)(t+1)\}$.

Proof. We assume $m < n$, the case $m \geq n$ being even easier. By [2], Theorem 17, for a general $P \in C$ the osculating $(m-1)$-dimensional space of C at P intersects C in a zero-dimensional scheme whose connected component supported by P has length m. Since C is non-degenerate, there is a one-dimensional family of distinct osculating spaces to C. □

Proof of Theorem 1. For the main definitions and properties concerning principal bundles and osculating bundles of a bundle on a smooth curve (see [4] and [3]). Set $a := \dim(X)$. By induction on m it is sufficient to do the case $m \leq n-a-2$. Let $f : X \to C$ be the normalization map. Set $L := f^*(\mathcal{O}_C(1))$ and $V := f^*(H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))) \subseteq H^0(X,L)$. Hence there is a surjection $u : V \otimes \mathcal{O}_X \to L$. For every integer $t \geq 0$ let $P^t(L)$ the principal bundle of L. $P^t(L)$ is a rank $t+1$ vector bundle on X. The surjection u induces a map $u_t : V \otimes \mathcal{O}_X \to P^t(L)$ and the sheaf $\mathrm{Im}(u_t)$ is called the osculating bundle of order t of the pair (L,V). Since X is a smooth curve, $G^t(L)$ is locally free. Since $\mathrm{char}(\mathbf{K}) = 0$, $\mathrm{rank}(G^t(L)) = \min\{n+1, t+1\}$ (or use Lemma 1). There is a surjective morphism $a_t : \mathbf{P}(G^t(L)) \to \mathcal{C}(C,t)$. Since $\mathrm{char}(\mathbb{K}) = 0$, for every $t \leq n-2$ the morphism a_t is birational and hence $a_t$is just the normalization map of the variety $\mathcal{C}(C,t)$. Since $\mathrm{char}(\mathbb{K}) = 0$, the tangent space to $[X;\mathcal{C}(C,m)]$ at a general point of it is computed taking a general $A \in X$ and a general $B \in \mathcal{C}(C,m)$ and then making the linear span of the corresponding tangent spaces T_AX and $T_B\mathcal{C}(C,m)$ (Terracini's Lemma proved for instance in [1], Corollary 1.11). Hence Theorem 1 is equivalent to $T_AX \cap T_B\mathcal{C}(C,m) = \emptyset$. Since u_m is invertible near $u_t^{-1}(B)$, the point B corresponds to a point B' on X and a point B'' on the fiber of $G^m(L)$ over B': the unique point such that $u_m(B'') = B'$. By [3], Proposition 8.3, $T_B\mathcal{C}(C,m)$ is the $(m+1)$-dimensional osculating space to C at $f(B')$. We fix A and move B. Let $v : \mathbf{P}^n \backslash T_AX \to \mathbf{P}^{n-a-1}$ be the linear projection. Assume $T_AX \cap T_B\mathcal{C}(C,m) \neq \emptyset$ for general $B \in \mathcal{C}(C,m)$. Then the m-osculating space of the curve $v(C)$ at its general point $v(f(B))$ has contact order at least $m+1$ with $v(C)$ at $v(f(B))$, contradicting the characteristic zero assumption ([2], Theorem 17). □

Example 1. Let $C \subset \mathbf{P}^n$ be an integral non-degenerate curve. C is said to be strange if there is $P \in \mathbf{P}^n$ (called the strange point of C) such that all the tangent lines to smooth points of C passes through P. Hence for all integers $x \geq 2$ and all $P_i \in C_{reg}$ we have $\dim(\langle T_{P_1}C \cup \cdots \cup T_{P_x}C\rangle) \leq x+1$. Thus $\mathcal{C}(C,2,\ldots,C,2)$ (x times) has at most dimension $2x$. Strange curves exist only in positive characteristic. In any positive characteristic and for any integer

$n \geq 2$ there are huge families of singular strange subcurves of $\mathbf{P}^n$.

Example 2. Let $C \subset \mathbf{P}^n$ be an integral non-degenerate curve. There is a sequence of $n+1$ integers $\{b_i\}_{0 \leq i \leq n}$ (often called the gap sequence or the Hasse sequence or the generic osculating sequence of C) such that $b_0 = 0$, $b_i < b_{i+1}$ for $0 \leq i \leq n-1$, and for a general $P \in C$ the osculating i-dimensional linear space to C at P as order of contact $b_i + 1$ with C at P ($1 \leq i \leq n$) ([2], p. 49). If either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > \deg(C)$, then $b_i = i$ for every i ([2], Theorem 17). If $0 < \mathrm{char}(\mathbb{K}) < \deg(C)$ this may be false (even for $i = 1$), but we always have $b_{n-1} \leq \deg(C)$ and the first integer m such that $b_m \geq m+2$ (if any) is always divisible by p. Assume the existence of such an integer m and let $(m+1)P$ be the effective degree $m+1$ Cartier divisor of C with a general $P \in C$ as support. Since $b_m \geq m+1$, the linear span $\langle (m+1)P \rangle$ has dimension at most $m+1$. Hence our definition of $\dim(\mathcal{C}(C, m+1))$ does not make sense.

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