International Journal of Pure and Applied Mathematics
Volume 20 No. 2 2005, 223-226

COVERING MAPS OF BIMEROMORPHICALLY EQUIVALENT COMPACT COMPLEX SURFACES

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: We classify the smooth and connected compact complex surfaces (or projective surfaces) X such that there is an étale morphism $f : X' \to X$ with X' bimeromorphic (or birational) to X.

AMS Subject Classification: 14J25, 14H20, 32J15
Key Words: self-map, étale map, compact complex surface, birational surfaces, bimeromorphic functions

1. Birational Surfaces and Ètale Maps

We propose the classification (in low dimension) of the triples (X, X', f) such that X and X' are smooth and connected projective varieties (or compact complex manifolds), X and X' are birational (or bimeromorphic), $f : X' \to X$ is étale (or a local isomorphism) and $\deg(f) \geq 2$. The case $\dim(X) = 1$ is trivial ($X \cong X'$ is an elliptic curve). If $X' \cong X$ and $\dim(X) = 2$, see [1]. We would like also to iterate the procedure in the following sense. Take $\{(X_n, f_n)\}_{n\geq 0}$, where each X_n is a smooth and connected projective variety (or a compact complex manifold), all X_n's are birational (or bimeromorphic), each $f_n : X_{n+1} \to X_n$, $n \geq 0$ is étale (or a local isomorphism) and $\deg(f_n) \geq 2$ for all n. To make iteration backward and forward it wolud be nice to study the families $\{(X_n, f_n)\}_{n\in\mathbb{Z}}$ such that each X_n is a smooth and connected projective variety (or a compact complex manifold), all X_n's are birational (or bimeromorphic), each $f_n : X_{n+1} \to X_n$, $n \geq 0$ is étale (or a local isomorphism) and $\deg(f_n) \geq 2$ for all n. We work either in the algebraic geometric category over

Received: March 15, 2005

an algebraically closed field $\mathbb{K}$ or in the complex analytic category. The aim of this note is to prove the following theorem.

Theorem 1. *Let X, X' be either birational smooth and connected projective surfaces or bimeromorphic smooth compact complex surfaces and $f : X' \to X$ an étale map. Then (X, X', f) belongs to of the following classes:*

(i) $\kappa(X) = \kappa(X') = -\infty$, $q(X) = q(X') = 1$; call $h : X \to C$ and $h' : X' \to C'$ the rulings of X and X' with C and C' elliptic curves; $C' \cong C$ and f is induced from an étale degree $\deg(f)$ morphism $C' \to C$.

(ii) (analytic case) $a(X) = a(X') = 0$, the model Y of X and X' is one of the three classes of surfaces constructed by Inoue in [2] and $f : X' \to X$ is induced by an étale degree $\deg(f)$ holomorphic map $f' : Y \to Y$. Every Y and X' occur in this way; (X', f) are uniquely determined by X, the blowing-down map $X \to Y$ and f'.

(iii) the birational (or bimeromorphic) model of X and X' is a torus Y and f is induced from an étale $\deg(f)$ morphism $f' : Y \to Y$; every Y and every X' occur in this way; (X', f) are uniquely determined by X, the blowing-down map $X \to Y$ and f'.

(iv) the birational (or bimeromorphic) model of X and X' is a hyperelliptic surface Y and f is induced from an étale $\deg(f)$ morphism $f' : Y \to Y$; every Y and every X' occur in this way; (X', f) are uniquely determined by X, the blowing-down map $X \to Y$ and f'.

(v) X is an (perhaps non-minimal) ellitic surface, the birational (or bimeromorphic) model Y of X and X' is isomorphic to $E \times B$ with E elliptic curve and B smooth curve of genus ≥ 2, and f is induced from an étale $\deg(f)$ morphism $f' : Y \to Y$; every Y and every X' occur in this way; (X', f) are uniquely determined by X, the blowing-down map $X \to Y$ and f'.

Remark 1. In all the cases listed in Theorem 1 we may fit the triple (X', X, f) in a family $\{(X_n, f_n)\}_{n \geq 0}$ just taking the pull-back by the blowing-dowm map $X \to Y$ of the iterated of the étale map $f' : Y \to Y$.

At the end of this note we will give a few remark in the case of singular normal surfaces.

Proof of Theorem 1. Set $x := \deg(f)$. Notice that if X is algebraic, then X' is an algebraic. Since X is not simply connected, X is not rational.

(a) Here we assume X algebraic and $\kappa(X) = -\infty$. Hence X is birationally ruled. Since X is not rational, $q(X) > 0$ and there is a unique morphism $h : X \to C$ such that C is a smooth curve of genus $q(X)$ and a general fiber of h is connected, smooth and rational. Since f is étale and $\mathbf{P}^1$ is algebraically simply connected, $f^{-1}(T)$ is the disjoint union of x smooth rational curves.

Hence $\kappa(X) = -\infty$. Since $q(X') \geq q(X) > 0$, there is a unique morphism $h' : X' \to C'$ such that a general fiber of h' is smooth and rational. Since X and X' are birational, $C \cong C'$. We just saw that f is induced from an étale degree $\deg(f)$ morphism $C' \to C$ at least on the general fiber of h. Since C is a smooth curve and X, X' are normal, this is true globally.

(b) Assume $a(X) = a(X') = 0$. Since $b_1(X) > 0$, part (ii) follows from a classification theorem due to Bogomolov and Teleman (see [4]) and the discussion in [1], 2.1, 2.4, 2.5 and 2.6.

(c) By (a) and (b) we may assume $\kappa(X) = \kappa(X') \geq 0$. Hence X and X' have a unique minimal model, Y. Take the pull-back by the blowing-down morphism $X \to Y$ of any étale morphism $f' : Y \to Y$. In this way using respectively cases (i), (ii) and (iii) of [1], Theorem 0.2, we obtain the existence part of cases (iii), (iv) and (v) of Theorem 1. For the uniqueness part (i.e. that any pair (X', f) is induced by some pair (X, f') taking a pull-back, just use that $\mathbf{P}^1$ is algebraically simply connected. □

Now we consider pairs (X, f) with X reduced and connected compact complex space (or integral projective variety) and the collowing condition is satisfied:

($\langle\alpha\rangle$) $f : X \to X$ is a finite morphism such that $\deg(f) \geq 2$, $f(\mathrm{Sing}(X)) = \mathrm{Sing}(X)$, $f(X_{reg}) = X_{reg}$ and $f|X_{reg}$ is a local biholomorphism (or it is étale).

Remark 2. Take $(X, \tilde{f})$ satisfying ($\langle\alpha\rangle$) and assume $\mathrm{Sing}(X)$ finite. Set $x := \sharp(\mathrm{Sing}(X)$. Condition ($\langle\alpha\rangle$) implies that $\tilde{f}$ induces a permutation of the finite set $\mathrm{Sing}(X)$. Hence the composition $f := \tilde{f} \circ \cdots \circ \tilde{f}$ (x times) fixes each singular point of X. Hence the pair (X, f) satifies the following condition:

($\langle\beta\rangle$) $f : X \to X$ is a finite morphism such that $\deg(f) \geq 2$, $f|\mathrm{Sing}(X))$ is the identity, $f(X_{reg}) = X_{reg}$ and $f|X_{reg}$ is a local biholomorphism (or it is étale).

Remark 3. It is easy to check the non-existence of any pair (X, f) satisfying ($\langle\alpha\rangle$) with X normal projective surface whose smooth birational model has general type (see e.g. [6], last two lines of p. 626).

Motivated by Remark 2 we consider the local case (in the complex analytic category) for normal surface singularities, i.e. triples (A, o, f) such that:

($\langle\gamma\rangle$) (A, o) is the germ at o of a normal surface singularity, $f : (A, o) \to (A, o)$ is a finite map with $\deg(f) \geq 2$, $f^{-1}(o) = o$ and $f|A\backslash\{o\} \to A\backslash\{o\}$ an unramified covering.

Everything is known and due to J. Wahl ([6], last Corollary of p. 626 and parts (b) and (c) of the theorem at p. 626); for a classification of the corresponding singularities, see [3], §9, or (from a different point of view) [5].

Take (A, o, f) satisfying $(\langle\gamma\rangle)$. Obviously, the fundamental group of $A\backslash\{o\}$ is not finite and hence (A, o) is not a quotient singularity.

Remark 4. Let X be a connected and normal compact complex surface (or projective variety) such that there is $\tilde{f} : X \to X$ satisfying $(\langle\alpha\rangle)$. Hence there is $f : X \to X$ satisfying $(\langle\beta\rangle)$. Let $u : \tilde{X} \to X$ a good desingularization of X. We assume that $\tilde{X}$ is not rational. Hence there is a unique minimal model $v : \tilde{X} \to Y$ of $\tilde{X}$. The étale map f induces a finite deg(f) map $h : Y \to Y$. First assume $\tilde{X}$ algebraic, $\kappa(\tilde{X}) = -\infty$ and $q(\tilde{X}) > 0$. Hence there is a uniquely determined $\mathbf{P}^1$-bundle $w : Y \to C$ with C a smooth curve of genus $q(X)$. Since C is the Albanese variety of Y, the morphism h induces a finite degree deg(f) morphism $h' : C \to C$. If $q(X) \geq 2$, this is impossible. If $q(X) = 1$, this implies h' étale. The same proof works if $\kappa(Y) = 1$ and the base of the elliptic fibration is the Albanese variety of Y.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, Globally invertible differentiable or homorphic maps, *Rend. Sem. Mat. Univ. Padova*, **105** (2001), 25-35.

[2] M. Inoue, On surfaces of class VII_0, *Invent. Math.*, **24** (1974), 269-310.

[3] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degeneration of surfaces, *Ann. of Math.*, **127**, No. (1988), 93-163.

[4] A.-D. Teleman, Projectively flat surfaces and Bogomolov's Theorem on class VII_0 surfaces, *Int. J. Math.*, **5** (1995), 253-264.

[5] P. Wagreich, Singularities of complex surfaces with solvable local findamental group, *Topology*, **11**, No. 1 (1972), 51-72.

[6] J. Wahl, A characteristic number for links of surfaces singularities, *J. Amer. Math. Soc.*, **3**, No. 3 (1990), 341-348.