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ITERATED COVERING MAPS IN FAMILIES OF COMPACT COMPLEX SURFACES

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Abstract: Here we consider (in the algebraic geometry set-up, too!) the existence of families of data $(T, \{X_t\}_{t\in T}, s : T \in T, \{f_t\}_{t\in T})$ such that:

(i) T is a reduced and irreducible complex space;

(ii) $\{X_t\}_{t\in T}$ is a smooth family of connected and compact complex manifolds;

(iii) $s : T \to T$ is a set-theoretic map;

(iv) $f_t : X_{s(t)} \to X_t$ is a locally invertible, but not globally invertible analytic map.

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1. Introduction

We work either in the algebraic geometric category over an algebraically closed field $\mathbb{K}$ or in the complex analytic category. Here we consider the existence of families of data $(T, \{X_t\}_{t\in T}, s : T \in T, \{f_t\}_{t\in T})$ such that:

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(iv) $f_t : X_{s(t)} \to X_t$ is a locally invertible, but not globally invertible analytic map.

In the algebraic category instead of (iv) we require that each f_t is ètale. When T is a point, the corresponding problem (existence of locally invertible self-maps which are not isomorphisms) was considered (in several different categories) in [2] (stimultated by [6]). For singular spaces, the requirement "locally invertible" must be relaxed (see [1] for the case T a point and $\dim(X_t) = 1$, see here in Section 2 for a few remarks). The aim of conditions (iii) and (iv) is to allow the iterations of the maps f_t's: take f_t, then $f_{s(t)} \circ f_t$, then $f_{s(s(t))} \circ f_{s(t)} \circ f_t$, and so on. The following additional condition allows us to make "backward iterations in a non-unique way":

(v) the map s is surjective.

Indeed, take any $t_1 \in$ such that $s(t_1) = t$, any $t_2 \in T$ such that $s(t_2) = t_1$, and so on and take f_t, $f_t \circ f_{t_1}$, $f_t \circ f_{t_1} \circ f_{t_2}$, and so on. If we want to do "backward iterations in a unique way" it is sufficient to take the condition:

(vi) s is bijective.

As a by-product of [2], Theorem 0.2, and of a few remarks we will prove the following result.

Theorem 1. *Take $(T, \{X_t\}_{t\in T}, s : T \in T, \{f_t\}_{t\in T})$ (for some T) (in the complex analytic category or in the algebraic category) with $\dim(X_t) = 2$. Then each X_t is one of the following surfaces:*

(a) $X_t \cong E \times B$ with E an elliptic curve and B a smooth curve with genus at least two.

(b) X_t is a torus.

(c) X_t is a hyperelliptic surface.

(d) X_t is a $\mathbf{P}^1$-bundle over an elliptic curve.

(e) X_t is one of the non-käler surfaces without curves and with $b_1(X_t) = 1$ constructed by Inoue in [8].

Conversely, every surface X_t in the classes (a), (b), (c) and (e) arises in this way and we may even take as T just one point. Some of the X_t in case (d) arise in this way; the one for which we may take T a point are classified in [2]

Remark 1. Take $(T, \{X_t\}_{t\in T}, s : T \in T, \{f_t\}_{t\in T})$ satisfying (i), (ii), (iii) and (iv). Then all compact manifolds X_t are diffeomorphic and in particular they are homeomorphic. Hence no such X_t may be simply connected. Furthermore each group $\pi_1(X_t, *)$ has a proper subgroup with finite index isomorphic to it. Hence $\pi_1(X_t, *)$ is infinite.

Remark 2. Assume $\dim(X_t) = 1$. Then all curves X_t must have genus one. We may take as T either a point or a parameter space for all elliptic curves.

Lemma 1. *Take $(T, \{X_t\}_{t\in T}, s : T \in T, \{f_t\}_{t\in T})$ satisfying (i), (ii), (iii) and (iv) (in the complex analytic or in the algebraic category). Assume* $\dim(X_t) = 2$. *Then no surface X_t has an exceptional curve of the first kind, i.e. a smooth and rational curve $D \subset X_t$ such that $D^2 = D \cdot \omega_{X_t} = -1$.*

Proof. By Remark 1 X_t is not rational. Hence it has at most finitely many exceptional curves and the number $\alpha(t)$ of these exceptional curves of the first kind is the same for all $t \in T$ by the local rigidity of exceptional curves of the first kind. Assume the existence of an exceptional curve $D \subset X_t$. Let $z > 1$ be the degree of the ètale map $f_t : X_{s(t)} \to X_t$. Since $D \cong \mathbf{P}^1$, it is algebraically simply connected and hence ${f_t}^{-1}(D)$ is the disjoint union of z curves, say $D_1, \dots D_z$, each of them mapped isomorphically onto D by f_t. Since f_t is étale, we also have $D_i \cdot \omega_{s(t)} = -1$ for all i. By the adjunction formula we obtain ${D_i}^2 = -1$. Hence each D_i is an exceptional curve of the first kind. Hence $\alpha(s(t)) \geq z(\alpha(t)) > \alpha(t)$, contradiction. □

Remark 3. Take $(T, \{X_t\}_{t\in T}, s : T \in T, \{f_t\}_{t\in T})$ satisfying (i), (ii), (iii) and (iv) in the algebraic category. Assume $\dim(X_t) = 2$ and $\kappa(X_t) = -\infty$ for at least one t. Hence $\dim(X_x) = 2$ and $\kappa(X_x) = -\infty$ for every $x \in T$. By Remark 1 no X_t is rational. By Lemma 1 each X_t is minimal. By the classification of surfaces and the knownledge of algebraic fundamental group of all smooth projective curves, we obtain that for every $t \in T$ there is a $\mathbf{P}^1$-bundle $\beta_t : X_t \to C_t$ with C_t smooth elliptic curve. Since $\mathbf{P}^1$ is algebraically simply connected, we also see that each f_t is induced by an étale covering $C_{s(t)} \to C_t$. Conversely, it is very easy to check that in this way we get examples. For the classification of all cases in which we may take T a point, see [2], Theorem 0.2.

Remark 4. Take $(T, \{X_t\}_{t\in T}, s : T \in T, \{f_t\}_{t\in T})$ satisfying (i), (ii), (iii) and (iv) either in the complex analytic or in the algebraic category. Assume

$\dim(X_t) = 0$. Since T is connected, we have $\omega^2_{X_a} = \omega^2_{X_b}$ for all $a, b \in T$. Since f_t is étale, ${f_t}^*(\omega_{X_t}) \cong \omega_{X_{s(t)}}$. Hence $\omega^2_{X_{s(t)}} = \deg(f_t) \cdot \omega^2_{X_t}$. Hence $\omega^2_{X_t} = 0$ (i.e. $c_1(X_t)^2 = 0$) for all $t \in T$.

Remark 5. Take $(T, \{X_t\}_{t \in T}, s : T \in T, \{f_t\}_{t \in T})$ satisfying (i), (ii), (iii) and (iv) either in the complex analytic or in the algebraic category. Assume $\dim(X_t) = 2$. By Remark 1 each X_t is minimal. Thus Remark 4 implies that no X_t has general type.

Remark 6. Take $(T, \{X_t\}_{t \in T}, s : T \in T, \{f_t\}_{t \in T})$ satisfying (i), (ii), (iii) and (iv) (in the complex analytic or in the algebraic category). Assume $\dim(X_t) = 2$. Since f_t is étale, ${f_t}^*(TX_t) \cong TX_{s(t)}$. Hence $c_2(X_{s(t)} = \deg(f_t) \cdot c_2(X_t)$. Thus $c_2(X_t) = 0$. Since $c_1(X_t)^2 = 0$ (Remark 4), Noether's formula ([4], p. 339, or [3], p. 20) implies $\chi(\mathcal{O}_{X_t}) = 0$.

Remark 7. Obviuosly, complex tori and Abelian varieties gives examples satisfying (i), (ii), (iii) and (iv).

Proof of Theorem 1. For the existence part for classes (b), (c) and (e), see [2], Theorem 0.2. Let $a(X_t)$ denote the algebraic dimension of X_t. We divide the remaining parts of the proof into six steps.

(1) Here we assume that each X_t is projective and $\kappa(X_t) = 0$. Since $\chi(\mathcal{O}_X) = 0$ (Remark 6) Enriques' classification ([4], p. 373) implies that either X_t is Abelian or a classical hyperelliptic surface or a nonclassical hyperelliptic surface (only in $\mathrm{char}(\mathbb{K} = 2, 3)$.

(2) Here we assume that some X_t is non-algebraic and $\kappa(X_t) = 0$. Since $c_1(X_t)^2 = c_2(X_t) = 0$ (Remark 6), we obtain that either X_t is a torus for each t (and some of them may even be algebraic) or X_t is a Kodaira surface (a primary Kodaira surface or a secondary Kodaira surface) ([3], p. 188).

(3) Assume $a(X_t) = 0$. Hence X_t has only finitely many compact curves ([5], Theorem 2.16). Furthermore, the irreducibility of T implies the existence of integer y, y' such that the number of such curves in X_t is at most y and the sum of thir arithmetic genera at most y' for all $t \in T$; indeed, we may take $y \leq b_2(X_t)$ for any $t \in T$. Fix $t \in T$ and any irreducible compact curve $A \subset X_t$ (if any). Notice that $p_a(B) \geq p_a(A)$ for every irreducible component B of ${f_t}^{-1}(A)$. Furthermore, if $p_a(A) \geq 2$, then $p_a(B) > p_a(A)$, unless $f_t|B : B \to A$ is an isomorphism; in this case ${f_t}^{-1}(A)$ must have another ireducible component. We may take the iterates f_t, $f_{s(t)} \circ f_t$, $f_{s(s(t))} \circ f_{s(t)} \circ f_t$, and so on. By the finiteness of y and y' we obtain $p_a(A) \leq 1$ for all curves $A \subset X_t$. Since f_t is étale of degree x, we have ${f_t}^{-1}(A) \cdot {f_t}^{-1}(A) = x(A \cdot A)$. Since the numbers $A \cdot A$ are bounded for all $t \in T$, iterating we obtain $A \cdot A = 0$ for all $A \subset X_t$ and all t. It is very easy to exclude the case $p_a(A) = 0$, showing that in this case X_t

is ruled and hence $a(X_t) = 2$ (see [2], 2.1). It is very easy to exclude the case $p_a(A) = 1$, shoing that in this case $X - t$ must be an elliptic surface and thus $a(X_t) \geq 1$. In summary: if $a(X_t) = 0$ for some t, then no X_t contain a compact complex curve.

(4) (in the analytic category) By [10] and [8], Theorem in Section 5, every smooth complex compact surface without curves is an Inoue surface, i.e. belong to one of the three classes of surfaces constructed in [8]. As shown in [2], p. 34, all these surfaces may occur (even taking as T just one point). For the Inoue surfaces introduced in [8], Section 3, we may take $T = \mathbb{C}$ ([2], 2.5).

(5) (in the analytic category) By [4], table on bottom of p. 402, the case $a(X_t) \neq 2$ and $\kappa(X_t) = 0$ give no new case.

(6) Assume X_t is an elliptic surface. All cases with $\chi(\mathcal{O}_{X_t}) = 0$ are studied in [4], pp. 366–368. It is sufficient to consider the case $\kappa(X_t) = 1$. In the algebraic case when $\mathrm{char}(\mathbb{K}) = 0$, use the clasification of all surfaces with $c_1^2 = c_2 = 0$ given in [9], Theorem E.5.1; for the general case use also [7], Theorem 1.1. □

2. Singular Curves

Now we discuss the case of compact singular analytic space or singular projective varieties. As in [1] there is a natural condition (called here (vii)), which substitute the locally invertibility (or étale) condition:

(vii) $f_t : X_{s(t)} \to X_t$, $f_t((X_{s(t)})_{reg}) = (X_t)_{reg}$, $f_t(\mathrm{Sing}(X_{s(t)}) = \mathrm{Sing}(X_t)$ and $f_t|(X_{s(t)})_{reg}$ is étale.

We will always assume the following condition on the family $\{X_t\}_{t\in T}$:

(viii) the family $\{X_t\}_{t\in T}$ is flat and proper.

Instead of (viii) it is often more natural to require a stroger condition:

(ix) the family $\{X_t\}_{t\in T}$ is flat, proper and equisingular.

There are several non-equivalent definitions of equisingularity and each of them could be uses in (ix). In the one-dimensional case we may assume the following condition stronger than (viii), but weaker than (ix):

(x) Assume $\dim(X_t) = 1$. In the flat and proper family $\{X_t\}_{t\in T}$ the number of singular points of each fiber is constant.

We may even require the following stronger condition:

(xi) Assume $\dim(X_t) = 1$. In the flat and proper family $\{X_t\}_{t\in T}$ the number and total arithmetic genus of singular points of each fiber is constant.

In this section we prove the following result.

Proposition 1. *Fix an integer* $x \geq 2$ *and assume either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) > x$. *Take* $(T, \{X_t\}_{t\in T}, s : T \in T, \{f_t\}_{t\in T})$ *such that* $\dim(X_t) = 1$, $\deg(f_t) = x$ *and* $\mathrm{Sing}(X_t) \neq \emptyset$. *Assume condition (xi). Set* $z := \sharp({u_t}^{-1}(\mathrm{Sing}(X_t)))$. *Then* $\mathbf{P}^1$ *is the normalization of each* X_t *and either* $z = 1$ *and* X_t *has two branches at its singular point or* $z = 2$ *and each singular point of* X_t *is unibranch. Conversely, any such curve* X_{t_0} *fits in a family* $(T, \{X_t\}_{t\in T}, s : T \in T, \{f_t\}_{t\in T})$ *with* $\deg(f_t) = x$; *we may always take as* T *just one point, but we may even take positive dimensional parameter spaces.*

Remark 8. Let $u_t : Y_t \to X_t$ be the normalization map of X_t. Assume $\dim(X_t) = 1$. Condition (xi) is equivalent to the flatness of the family $\{Y_t\}_{t\in T}$. Furthermore, f_t induces a finite morphism $\tilde{f}_t : Y_{s(t)} \to Y_t$. Assume $\mathrm{Sing}(X_t) \neq 0$. Since $\sharp(\mathrm{Sing}(X_t)) = \sharp(\mathrm{Sing}(X_{s(t)})) \neq 0$ and $\deg(\tilde{f}_t) = \deg(f_t) > 0$, condition (vii) implies that $\tilde{f}_t$ is not étale and that it has ramification order $\deg(\tilde{f}_t)$ at the counterimages of all singular points. Since $p_a(Y_{s(t)} = p_a(Y_t)$, we get $Y_t \cong \mathbf{P}^1$ for all t. Since $\tilde{f}_t$ induces a degree $\deg(f_t)$ étale and surjective map $u_{s(t)}^{-1}((X_{s(t)})_{reg}) \to (X_t)_{reg}$, we get $\sharp(u_t^{-1}(\mathrm{Sing}(X_t))) \geq 2$.

Proof of Proposition 1. We use the notation introduced in Remark 8. By Remark 8 we have $Y_t \cong \mathbf{P}^1$. By Remark 8 $\tilde{f}_t$ has exactly z ramification points and all of them are total ramification points. Since either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > x$, we may apply the usual Riemann-Hurwitz formula and obtain $-2 = -2x + z(x-1)$, i.e. $z = 2$. Hence either X_t has two singular points, both of them being unibranch, or it has a unique singular point with two branches. The converse part is proved in [1], Theorem 0.2: we may even take as T just one point. □

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