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VECTOR SUBSPACES OF SECTIONS AND STABLE COHERENT SYSTEMS ON CURVES

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Abstract: Let X be a smooth and connected projective curve, E a non-trivial rank n vector bundle on X and V an m-dimensional linear subspace of $H^0(X,E)$ spanning E. Assume that $\phi_{E,V} : X \to G(n,m)$ is unramified. Fix a general n-dimensional linear subspace of W. Then:

(a) for a general $(n-1)$-dimensional linear subspace A of W the evaluation map $u_A : \mathcal{O}_X \otimes W \to E$ is injective and with locally free cokernel;

(b) for every $(n-1)$-dimensional linear subspace B of W the evaluation map $u_B : \mathcal{O}_X \otimes W \to E$ is injective as a map of sheaves;

(c) there is a non-empty family $S(W)$ of the hyperplanes of W such that the evaluation map $u_B : \mathcal{O}_X \otimes W \to E$ is injective as a map of sheaves, but it has a non-locally free cokernel if and only if $B \in S(W)$; at each $P \in X$ the fiber at P of u_B has rank at least $n-2$; the saturation of $u_B(\mathcal{O}_X \otimes B)$ in E has degree one; $S(W)$ has pure dimension $n-2$.

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1. Introduction

For the general theory of coherent systems, see [2], [3], [4] and references therein. For any smooth and connected projective curve X, $\alpha \in \mathbb{R}$ and integers d, n, k let

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$G(X;\alpha;d,n,k)$ denote the moduli space of all α-stable coherent systems on X of type (d,n,k). Here we work over an algebraically closed field $\mathbb{K}$. For all integers $a>b>0$ let $G(b,a)$ denote the Grassmannian of all $(a-b)$-dimensional linear subspaces of $\mathbb{K}^{\oplus a}$. Set $N(b,a):=\binom{a}{b}-1$. Let $j_{b,a}:G(b,a)\to\mathbf{P}^{N(b,a)}$ denote the Plücker embedding. For any spanned rank n vector bundle on X and any m-dimensional linear subspace V of $H^0(X,E)$ spanning E, let $\phi_{E,V}:X\to G(n,m)$ denote the morphism associated to the pair (E,V). Write ϕ_E instead of $\phi_{E,V}$ if $V=H^0(X,E)$. Our aim is to prove the following result.

Theorem 1. *Let X be a smooth and connected projective curve, E a non-trivial rank n vector bundle on X and V an m-dimensional linear subspace of $H^0(X,E)$ spanning E. Let z be the maximal length of a a connected component of a fiber of the finite morphism $u\circ j_{b,a}\circ\phi_{E,V}:X\to G(n,m)$, where u is a general linear projection from $\mathbf{P}^{N(b,a)}$ into $\mathbf{P}^n$. Fix a general n-dimensional linear subspace of W. Then:*

(a) for a general $(n-1)$-dimensional linear subspace A of W the evaluation map $u_A:\mathcal{O}_X\otimes W\to E$ is injective and with locally free cokernel;

(b) for $(n-1)$-dimensional linear subspace B of W the evaluation map $u_B:\mathcal{O}_X\otimes W\to E$ is injective as a map of sheaves;

(c) there is a non-empty family $S(W)$ of the hyperplanes of W such that the evaluation mapu$_B:\mathcal{O}_X\otimes W\to E$ is injective as a map of sheaves, but it has a non-locally free cokernel if and only if $B\in S(W)$; at each $P\in X$ the fiber at P of u_B has rank at least $n-2$; the saturation of $u_B(\mathcal{O}_X\otimes W)$ in E has degree at most z; $S(W)$ has pure dimension $n-2$;

(d) for any integer t such that $1\le t\le n-2$ there is a non-empty family $S(W,t)$ of t-dimensional linear subspaces of W such that the evaluation map $u_B:\mathcal{O}_X\otimes W\to E$ is injective as a map of sheaves, but it has a non-locally free cokernel if and only $B\in S(W,t)$; the saturation of $u_B(\mathcal{O}_X\otimes W)$ in E has degree at most z; $S(W,t)$ has pure dimension $n-2+(t-1)(n-1)$.

An immediate consequence of Theorem 1 is the following result.

Corollary 1. *Let X be a smooth and connected projective curve, E a non-trivial rank n vector bundle on X and V an m-dimensional linear subspace of $H^0(X,E)$ spanning E. Assume that $\phi_{E,V}:X\to G(n,m)$ is unramified. Fix a general n-dimensional linear subspace of W. Then:*

(a) for a general $(n-1)$-dimensional linear subspace A of W the evaluation map $u_A:\mathcal{O}_X\otimes W\to E$ is injective and with locally free cokernel;

(b) for every $(n-1)$-dimensional linear subspace B of W the evaluation map $u_B:\mathcal{O}_X\otimes W\to E$ is injective as a map of sheaves;

(c) there is a non-empty family $S(W)$ of the hyperplanes of W such that

the evaluation map $u_B : \mathcal{O}_X \otimes W \to E$ *is injective as a map of sheaves, but it has a non-locally free cokernel if and only if* $B \in S(W)$*; at each* $P \in X$ *the fiber at* P *of* u_B *has rank at least* $n-2$*; the saturation of* $u_B(\mathcal{O}_X \otimes B)$ *in* E *has degree one;* $S(W)$ *has pure dimension* $n-2$*;*

(d) for any integer t *such that* $1 \le t \le n-2$ *there is a non-empty family* $S(W,t)$ *of* t*-dimensional linear subspaces of* W *such that the evaluation map* $u_B : \mathcal{O}_X \otimes W \to E$ *is injective as a map of sheaves, but it has a non-locally free cokernel if and only* $B \in S(W,t)$*; the saturation of* $u_B(\mathcal{O}_X \otimes W)$ *in* E *has degree one;* $S(W,t)$ *has pure dimension* $n-2+(t-1)(n-1)$*.*

Proof of Theorem 1. Let N be a general $(n-1)$-dimensional linear subspace of V. Since V spans E, $\dim(N) + \dim(X) \le \mathrm{rank}(E)$, the evaluation map $u_N : \mathcal{O}_X \otimes N \to E$ is injective and its cokernel is locally free. Since a general W contains such a general N, we get part (a). Let M be a general $(n+1)$-dimensional linear subspace of V. By the generality of M we may assume that a general cofimension two linear subspace of M satisfies part (a), i.e. its evaluation map is injective and with Since V spans E and $\dim(X)+\mathrm{rank}(E) \le \dim(M)$, M spans E. By the very definition of the Plücker embedding and the generality of M, the morphism $u \circ j_{b,a} \circ \phi_{E,V}$ is in induced by the spanned pair $(\det(E), \wedge^n(M))$. Fix a general hyperplane W of M. Since M spans E and $\dim(X) = 1$, we see that the evaluation map u_W is injective as maps of sheaves and its restriction at each $P \in X$ has rank at least $n-1$, i.e. kernel at most one-dimensional. This gives the corresponding condition in parts (c) and (d). Set $T := \mathrm{Coker}(u_W)$. Hence T is a skyscraper sheaf on X and $h^0(X,T) = \deg(E)$. Since E is not trivial, $T \ne \emptyset$ and this gives $S(W) \ne \emptyset$ in part (c) and $S(W,t) \ne \emptyset$ in part (d). By the generality of W each connected component of T has length at most z, proving part (c). Part (d) follows from the proof of parts (a) and (c) (in particular the rank condition at the bad fibers) and the equality $\dim(G(t-1,n-1)) = (t-1)(n-t)$. □

Now we will see that the assumptions made in the statements of Corollary 1 are satisfied in many important cases.

Lemma 1. *Fix integers* $d \ge n+2 \ge 3$*. Let* X *be a smooth elliptic curve and* E *a semistable vector bundle on* C *with rank* n *and degree* d*. Then* $h^1(X,E) = 0$*,* $h^0(X,E) = d$ *and the morphism* $\phi_E : X \to G(n,d)$ *is an embedding.*

Proof. Every semistable vector bundle is a direct sum of indecomposable vector bundles with the same slope. Hence we may assume E indecomposable. By [1], Lemma 15, we have $h^1(X,E) = 0$ and, $h^0(X,E) = d$. Fix $P \in X$. Since $E(-P)$ is indecomposable of degree $d-n > 0$, we have $h^0(X,E(-P)) =$

$h^0(X,E)-n$. Hence E is spanned. Fix any length two closed subscheme $Z \subset X$ and $P \in Z_{red}$. Since $d \geq n+2$ and $E(-Z)$ is indecomposable of degree $d-2n$, we have $h^0(X,E(-Z)) < h^0(X,E(-P))$. Taking $Z = 2P$, we see that ϕ_E is unramified. Taking $Z = \{P,Q\}$ with $Q \neq P$, we see that ϕ_E is injective. Hence ϕ_E is an embedding. □

The proof of the following well-known and easy result is similar and hence it is omitted.

Lemma 2. *Let E be a rigid vector bundle on $\mathbf{P}^1$ with rank n and degree $d \geq n+1$. Then $h^1(\mathbf{P}^1,E) = 0$, $h^0(\mathbf{P}^1,E) = d+n$ and the morphism $\phi_E : X \to G(n,d+n)$ is an embedding.*

Remark 1. By Lemma 1, semicontinuity and [4], Theorem 4.4, we see that the assummptions of Corollary 1 are satisfied (with $k := m$) when X is an elliptic curve, $k > n$ and (E,V) is a general α-stable coherent system on X of type (d,n,k), $d \geq n+2$.

Remark 2. By Lemma 2, semicontinuity and [3], Theorem 3.2, we see that the assummptions of Corollary 1 are satisfied (with $k := m$) when $X = \mathbf{P}^1$, $d > k > n$ and (E,V) is a general α-stable coherent system on $\mathbf{P}^1$ of type (d,n,k), $d \geq n+2$.

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