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NEWTON'S AND SECANT ITERATION OF QUOTIENTS OF FIBONACCI AND LUCAS NUMBERS

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Abstract: The forward and the backward Newton iterations on the elements $\mathbb{X}_{n+1}/\mathbb{X}_n$, where $\mathbb{X}_n = A\,\alpha^n + B\,\beta^n$, $\alpha := (1+\sqrt{5})/2$, $\beta := (1-\sqrt{5})/2$ are studied. Moreover, some relations for the sequence $G(\mathbb{X}_{n+1}/\mathbb{X}_n,\, \mathbb{Y}_{n+1}/\mathbb{Y}_n)$ are studied, where $G(x,y) := (xy+1)/(x+y-1)$, $\mathbb{X}_n$ and $\mathbb{Y}_n$ are recurrence sequences of the above-mentioned type. For example, it is proved that the identities $G\Big(\frac{F_n}{F_{n-1}}, \frac{F_m}{F_{m-1}}\Big) = G\Big(\frac{L_n}{L_{n-1}}, \frac{L_m}{L_{m-1}}\Big) = \frac{F_{n+m-1}}{F_{n+m-2}}$ and $G\Big(\frac{L_n}{L_{n-1}}, \frac{F_m}{F_{m-1}}\Big) = \frac{L_{n+m-1}}{L_{n+m-2}}$ are characteristic for Fibonacci and Lucas numbers.

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1. Introduction

The following two identities (see Grzymkowski et al [2]):

$$(1+\xi+\xi^4)^n = F_{n+1} + F_n(\xi+\xi^4) \tag{1.1}$$

and

$$(1+\xi^2+\xi^3)^n = F_{n+1} + F_n(\xi^2+\xi^3) \tag{1.2}$$

connect the Fibonacci numbers F_n with primitive roots of unity $\xi \in \mathbb{C}$ of the fifth order (i.e. $\xi^5 = 1$ and $\xi \neq 1$) form the basic of one of the three most well-known methods of generating identities for Fibonacci and Lucas numbers.

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The other two methods are as follows: the first one by applying the Binet's formulas:

$$F_n := \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad n \in \mathbb{Z}, \tag{1.3}$$

and

$$L_n = \alpha^n + \beta^n, \qquad n = 0, 1, 2, \dots, \tag{1.4}$$

where

$$\alpha := \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta := \frac{1-\sqrt{5}}{2} \tag{1.5}$$

and the second one consists of applying the generating function for the Fibonacci numbers: $f(y) = 1/(1 - y - y^2)$.

An attempt to generalize of equations (1.1) and (1.2) also seems to be interesting (Lemma 2.1 and identity (3.1)). It leads to generating two classical iterative methods (Newton's and Secant). The studying of Newton's and Secant Iteration of quotients of (general) Fibonacci and Lucas numbers is main aim of this paper.

2. The Backward Iteration Form

Let us start from the following result.

Lemma 2.1. *Let $\xi \in \mathbb{C}$, $\xi^5 = 1$ and $\xi \neq 1$. Let $k, l \in \mathbb{N}$, $k, l \leq 4$. If $2k \equiv l \pmod 5$, then for each $a \in \mathbb{C}$, $a \neq 1/2$, the following identity hold*

$$(1 + a\xi^k + \xi^l)^2 = (2a-1)\xi^k \left[1 + \frac{a^2+1}{2a-1}\xi^k + \xi^l\right]. \tag{2.1}$$

Proof. We have

$$\begin{aligned}
&\left(1 + a\,\xi^k + \xi^l\right)^2 = 1 + a^2\xi^{2k} + \xi^{2l} + 2a\xi^k + 2\xi^l + 2a\xi^{k+l} \\
&= (2a-1)\xi^k + (2a-1)\xi^{k+l} + (a^2+1)\xi^{2k} + \xi^k + \xi^{k+l} - \xi^{2k} + 1 + 2\xi^l + \xi^{2l}.
\end{aligned}$$

Accordingly;

$$\xi^{2k} = \xi^{2k-l}\xi^l = \xi^l, \quad \xi^{4k} = \left(\xi^{2k}\right)^2 = \xi^{2l}, \quad \xi^{3k} = \xi^k\xi^{2k} = \xi^k\xi^l = \xi^{k+l}.$$

So

$$\xi^k + \xi^{k+l} - \xi^{2k} + 1 + 2\xi^l + \xi^{2l}$$

$$= \xi^k + \xi^{3k} - \xi^{2k} + 1 + 2\xi^{2k} + \xi^{4k} = \sum_{r=0}^{4} \xi^{4r} = 0. \quad \square$$

Let us now consider the following recursive sequence connected with formula (2.1):

$$a_{n+1} = f(a_n), \qquad n \geq 1, \tag{2.2}$$

where

$$f(a) = \frac{a^2+1}{2a-1}.$$

In Newton's method we approximate to a root of the equation $h(x) = 0$ as follows: we choose initial approximant x_1 and compute the sequence $\{x_n\}$ from

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}, \qquad n \geq 1.$$

If we assume $h(x) = x^2 - x - 1$, then we obtain the formula (2.2). It may be observed, that if $\lim_{n\to\infty} a_n$ exists, then this limit fulfills the equation $x^2 - x - 1 = 0$, which is equal to the golden ratio or its reciprocal (see also Taher et al [8] and Villiers [9]).

We are now interested in describing the forward (Lemma 2.2) and backward (Lemmas 2.4, 2.6, 2.9 and 2.11) iterations of function f on elements $\mathbb{X}_{n+1}/\mathbb{X}_n$, where $\mathbb{X}_n := A\,\alpha^n + B\,\beta^n$, $|A| + |B| > 0$. We note that then $\mathbb{X}_{n+2} = \mathbb{X}_{n+1} + \mathbb{X}_n$, $n \in \mathbb{N}$. Only Corollary 2.3 is known fact from literature (see Phillips [6]).

Lemma 2.2. *If $n \in \mathbb{N}$ and $a = \mathbb{X}_{n+1}/\mathbb{X}_n$, then*

$$f(a) = \begin{cases} F_{2n+1}/F_{2n}, & \text{when } A^2 = B^2, \\ L_{2n}/L_{2n-1}, & \text{when } \alpha^2 A^2 = B^2 \\ & \text{(i.e. } \big(\alpha^2+1\big)A^2 = \big(\beta^2+1\big)B^2\text{)}. \end{cases}$$

Proof. We have

$$f(a) = \frac{\mathbb{X}_{n+1}^2 + \mathbb{X}_n^2}{\mathbb{X}_n\big(2\mathbb{X}_{n+1} - \mathbb{X}_n\big)}$$

$$= \frac{\big(A^2\alpha^{2n+2} + B^2\beta^{2n+2}\big) + \big(A^2\alpha^{2n} + B^2\beta^{2n}\big)}{\big(2\,A^2\alpha^{2n+1} + 2\,B^2\beta^{2n+1}\big) - \big(A^2\alpha^{2n} + B^2\beta^{2n}\big)}$$

$$\Big(\text{when } A^2 = B^2\Big) = \frac{L_{2n+2} + L_{2n}}{2\,L_{2n+1} - L_{2n}} = \frac{L_{2n+2} + L_{2n}}{L_{2n+1} - L_{2n-1}} = \frac{F_{2n+1}}{F_{2n}},$$

$$\Big(\text{when } \alpha^2 A^2 = B^2\Big) = \frac{A^2(\alpha^2+1)L_{2n}}{A^2(2\,\alpha-1)\alpha^{2n} - B^2(1-2\,\beta)\beta^{2n}}$$
$$= \frac{(\alpha^2+1)L_{2n}}{\sqrt{5}\left(\alpha^{2n} - \frac{\alpha^2+1}{\beta^2+1}\beta^{2n}\right)} = \frac{(\alpha^2+1)(\beta^2+1)L_{2n}}{\sqrt{5}\left(\alpha^{2n-2}+\alpha^{2n}-\beta^{2n-2}-\beta^{2n}\right)}$$
$$= \frac{L_{2n}}{F_{2n-2}+F_{2n}} = \frac{L_{2n}}{L_{2n-1}}. \quad \square$$

Corollary 2.3. *If $a_1 = F_{k+1}/F_k$ then $a_{n+1} = F_{2^n k+1}/F_{2^n k}$ for $n \in \mathbb{N}$.*

Also immediately from Lemma 2.1 and Lemma 2.2 two following identities can be derived

$$(1 + a_1\xi^k + \xi^l)^{2^n} = \xi^{k(2^n-1)}(1 + a_{n+1}\xi^k + \xi^l)\prod_{k=1}^{n}(2a_k - 1)^{2^{n-k}} \tag{2.3}$$

and, for $a_1 = F_{r+1}/F_r$:

$$(1 + a_1\xi^k + \xi^l)^{2^n}$$
$$= \xi^{k(2^n-1)}\left(1 + \frac{F_{2^n r+1}}{F_{2^n r}}\,\xi^k + \xi^l\right)\prod_{k=1}^{n}\left(\frac{L_{2^{k-1}r}}{F_{2^{k-1}r}}\right)^{2^{n-k}}. \tag{2.4}$$

Lemma 2.4. *There exists $a \in \mathbb{R}$ such that: $f(a) = \mathbb{X}_{n+1}/\mathbb{X}_n$ for some $n \in \mathbb{N}$ iff $(-1)^n A\,B < 0$. If this condition is satisfied, then*

$$a = \frac{\mathbb{X}_{n+1} \pm \sqrt{5\,(-1)^{n-1}A\,B}}{\mathbb{X}_n}.$$

Proof. We have

$$f(a) = y \;\Leftrightarrow\; a^2 + 1 = y(2a-1) \;\Leftrightarrow\; (a-y)^2 = y^2 - y - 1. \tag{2.5}$$

Hence, if $y = \mathbb{X}_{n+1}/\mathbb{X}_n$, then

$$f(a) = y \;\Leftrightarrow\; \left(a - \frac{\mathbb{X}_{n+1}}{\mathbb{X}_n}\right)^2 = \frac{\mathbb{X}_{n+1}^2 - \mathbb{X}_n\mathbb{X}_{n+1} - \mathbb{X}_n^2}{\mathbb{X}_n^2}$$
$$= \frac{\mathbb{X}_{n-1}\mathbb{X}_{n+1} - \mathbb{X}_n^2}{\mathbb{X}_n^2} = \frac{5\,(-1)^{n-1}A\,B}{\mathbb{X}_n^2}. \quad \square$$

Corollary 2.5. *If $a \in \mathbb{R}$ and $f(a) = F_{n+1}/F_n$, then $n \in 2\mathbb{N}$ and*

$$a = \frac{F_{n+1} \pm 1}{F_n}.$$

Lemma 2.6. *Let $n \in \mathbb{N}$ and $(-1)^n A\,B < 0$. Then there exists $a \in \mathbb{R}$ such that*

$$f(a) = y := \big(\mathbb{X}_n\big)^{-1}\Big(\mathbb{X}_{n+1} \pm \sqrt{5\,(-1)^{n-1}A\,B}\Big)$$

iff

$$2\,\sqrt{5\,(-1)^{n-1}A\,B} \pm \big(\mathbb{X}_{n+1} + \mathbb{X}_{n-1}\big) \geq 0$$

(respectively to the sign "$\pm$"). Moreover, the appropriate values of a are equal to

$$a = y + \big(\mathbb{X}_n\big)^{-1}\sqrt{10(-1)^n A\,B \pm \sqrt{5\,(-1)^{n-1}A\,B}\big(\mathbb{X}_{n+1} + \mathbb{X}_{n-1}\big)}$$

and

$$a = y - \big(\mathbb{X}_n\big)^{-1}\sqrt{10(-1)^n A\,B \pm \sqrt{5\,(-1)^{n-1}A\,B}\big(\mathbb{X}_{n+1} + \mathbb{X}_{n-1}\big)}.$$

Proof. Suppose that there exists $a \in \mathbb{R}$, so that $f(a) = y$. Then, by (2.5) we obtain

$$\begin{aligned}(a-y)^2 = y^2 - y - 1 = \frac{1}{\mathbb{X}_n^2}\bigg(&\Big(\mathbb{X}_{n+1} \pm \sqrt{5\,(-1)^{n-1}A\,B}\Big)^2 \\ &- \mathbb{X}_n\Big(\mathbb{X}_{n+1} \pm \sqrt{5\,(-1)^{n-1}A\,B}\Big) - \mathbb{X}_n^2\bigg) \\ = \frac{1}{\mathbb{X}_n^2}\Big(&(\mathbb{X}_{n+1}^2 - \mathbb{X}_n\mathbb{X}_{n+1} - \mathbb{X}_n^2) \pm 2\sqrt{5\,(-1)^{n-1}A\,B}\,\mathbb{X}_{n+1} \\ &+ 5\,(-1)^{n-1}A\,B \mp \sqrt{5\,(-1)^{n-1}A\,B}\,\mathbb{X}_n\Big)\end{aligned}$$

(by the proof of Lemma 2.4)

$$= \frac{1}{\mathbb{X}_n^2}\Big(10\,(-1)^{n-1}A\,B \pm \sqrt{5\,(-1)^{n-1}A\,B}\,\big(2\,\mathbb{X}_{n+1} - \mathbb{X}_n\big)\Big)\,,$$

which implies the assertion of the lemma. □

Corollary 2.7. *If, additionally, we suppose that: $A\,\alpha^2 = B$, i.e. $A(\alpha^2 + 1) = B(\beta^2 + 1)$, then $n \in 2\mathbb{N} + 1$,*

$$f(a) = \frac{1}{\mathbb{X}_n}\Big(\mathbb{X}_{n+1} + \sqrt{5\,(-1)^{n-1}A\,B}\Big) = \frac{1}{\mathbb{X}_n}\Big(\mathbb{X}_{n+1} + \sqrt{5}\,\alpha\,A\Big)$$

and

$$a = \begin{cases} y \pm \sqrt{5}\,\alpha\,A\,\dfrac{L_k}{\mathbb{X}_n}, & \textit{if} \quad n = 2k+1 \ \wedge \ k \in 2\mathbb{N}, \\[2ex] y \pm \sqrt{5}\,\alpha\,A\,\dfrac{\sqrt{L_k^2 + 4}}{\mathbb{X}_n}, & \textit{if} \quad n = 2k+1 \ \wedge \ k \in 2\mathbb{N} - 1. \end{cases}$$

Proof. We have $(-1)^n A\,B < 0 \iff n \in 2\mathbb{N}+1$ and

$$\begin{aligned}10(-1)^{n-1}A\,B \pm \sqrt{5\,(-1)^{n-1}A\,B}\big(\mathbb{X}_{n+1}+\mathbb{X}_{n-1}\big)&\\ = 10\,A^2\alpha^2 \pm \sqrt{5}\,A^2\alpha\,(\alpha+1)\big(\alpha^{n-1}+\beta^{n-1}\big)&\\ = 10\,A^2\alpha^2 \pm 5\,A^2\alpha^2 L_{n-1} = 5\,A^2\alpha^2\big(2 \pm L_{n-1}\big)&\end{aligned}$$

(the case with the minus sign is rejected)

$$= \begin{cases} 5\,A^2\alpha^2 L_k^2, & \text{if} \quad n = 2k+1 \ \wedge\ k \in 2\mathbb{N}, \\ 5\,A^2\alpha^2\big(L_k^2+4\big), & \text{if} \quad n = 2k+1 \ \wedge\ k \in 2\mathbb{N}-1. \end{cases} \qquad \square$$

Corollary 2.8. *If $a \in \mathbb{R}$ and $f(a) = \dfrac{F_{2k+1} \pm 1}{F_{2k}}$ (see Corollary 2.5), then $f(a) = \dfrac{F_{2k+1}+1}{F_{2k}}$ and*

$$a = \begin{cases} \dfrac{F_{2k+1}+1 \pm L_k}{F_{2k}}, & k \in 2\mathbb{N}, \\ \dfrac{F_{2k+1}+1 \pm \sqrt{L_k^2+4}}{F_{2k}}, & k \in 2\mathbb{N}-1. \end{cases}$$

Lemma 2.9. *If $a \in \mathbb{R}$ and $f(a) = \dfrac{F_{4r+1}+1 \pm L_{2r}}{F_{4r}}$, then*

$$f(a) = \frac{F_{4r+1}+1+L_{2r}}{F_{4r}}$$

and

$$a = \begin{cases} \dfrac{F_{8l+1}+1+L_{4l} \pm L_{4l}L_{2l}}{F_{8l}} & \text{for} \quad r = 2l, \\ \dfrac{F_{8l-3}+1+L_{4l-2} \pm L_{4l}\sqrt{4+L_{2l-1}^2}}{F_{8l-4}} & \text{for} \quad r = 2l-1. \end{cases}$$

Proof. Let $f(a) = y = (F_{4r+1}+1 \pm L_{2r})/F_{4r}$. Then, by (2.5) the following formula is derived

$$\begin{aligned}(a-y)^2 &= y^2 - y - 1\\ &= \frac{1}{F_{4r}^2}\big((F_{4r+1}+1 \pm L_{2r})^2 - F_{4r}(F_{4r+1}+1 \pm L_{2r}) - F_{4r}^2\big)\\ &= \frac{1}{F_{4r}^2}\big(F_{4r+1}^2 + 1 + L_{2r}^2 + 2F_{4r+1} \pm 2F_{4r+1}L_{2r}\end{aligned}$$

$$
\begin{aligned}
&\pm 2L_{2r} - F_{4r}F_{4r+1} - F_{4r} \mp F_{4r}L_{2r} - F_{4r}^2\big)\\
&= \frac{1}{F_{4r}^2}\big((F_{4r+1}^2 - F_{4r}F_{4r+1} - F_{4r}^2) \pm L_{2r}(2F_{4r+1} - F_{4r})\\
&\quad + 1 + L_{2r}^2 + (2F_{4r+1} - F_{4r}) \pm 2L_{2r}\big)\\
&= \frac{1}{F_{4r}^2}\big(2 \pm L_{2r}L_{4r} + L_{2r}^2 + L_{4r} \pm 2L_{2r}\big)\\
&= \frac{1}{F_{4r}^2}\big(2 \pm L_{2r}(L_{4r} + 2) + L_{2r}^2 + L_{2r}^2 - 2\big)\\
&= \frac{1}{F_{4r}^2}\big(\pm L_{2r}^3 + 2L_{2r}^2\big) = \left(\frac{L_{2r}}{F_{4r}}\right)^2(2 \pm L_{2r})
\end{aligned}
$$

(only the case with plus sign shall be considered)

$$
= \left(\frac{L_{2r}}{F_{4r}}\right)^2(2 + L_{2r}) = \left(\frac{L_{2r}}{F_{4r}}\right)^2\big(2 - 2(-1)^r + L_r^2\big)
$$

$$
= \begin{cases} \left(\dfrac{L_{2r}L_r}{F_{4r}}\right)^2 & \text{for} \quad r \in 2\mathbb{N},\\ \left(\dfrac{L_{2r}}{F_{4r}}\right)^2(4 + L_r^2) & \text{for} \quad r \in 2\mathbb{N} - 1. \end{cases} \qquad \square
$$

Now, it is possible to generalize Corollary 2.5, Corollary 2.8 and Lemma 2.9. Let us start with some auxiliary result.

Lemma 2.10. *The following identities and inequalities hold:*

a)

$$
\sum_{l=1}^{n}\prod_{k=l}^{n} L_{2^k r} = \sum_{k=1}^{2^n-1} L_{2kr}
$$

and

$$
\Big(\sum_{l=2}^{n}\prod_{k=l}^{n} L_{2^k r}\Big) - \prod_{k=1}^{n} L_{2^k r} = -L_{2r} + \sum_{k=2}^{2^{n-2}-1} (L_{4kr} - L_{(4k+2)r});
$$

b)

$$
1 + \sum_{k=1}^{2^{n-1}-1} L_{2kr} = \frac{L_{(2^n-2)r} - L_{2^n r}}{2 - L_{2r}};
$$

c)

$$F_{2^n r} < F_{2^n r-1} + 1 + \Big(\sum_{l=2}^{n-1}\prod_{k=l}^{n-1} L_{2^k r}\Big) - \prod_{k=1}^{n} L_{2^k r} < \frac{1+\sqrt{5}}{2}F_{2^n r};$$

d)

$$\prod_{k=0}^{n-1} L_{2^k r} = \sum_{k=1}^{2^{n-1}} L_{2k-1} = \frac{L_{(2^n-1)r} - L_{(2^n+1)r}}{2 - L_{2r}};$$

e)

$$(L_{k-l} - L_{k+l})^2 - (2 - L_{2l})^2 = (L_{k-2l} - L_k)(L_k - L_{k+2l}) + \\ + \big((-1)^k - 1\big)L_{4l} + \big(4 - 2(-1)^{k-l} - 2(-1)^k\big)L_{2l}.$$

In the sequel, if k, l are even positive integers ($k > 2l$), thus the following identity hold

$$(L_{k-l} - L_{k+l})^2 = (2 - L_{2l})^2 + (L_{k-2l} - L_k)(L_k - L_{k+2l}).$$

Proof. a), b), d) The identities could be verified by immediate application of Binet's formula for Lucas number.

e) We have

$$\begin{aligned}(L_{k-l} - L_{k+l})^2 - (2 - L_{2l})^2 &= L_{k-l}^2 - 2L_{k-l}L_{k+l} + L_{k+l}^2 - 4 + 4L_{2l} - L_{2l}^2 = \\ &= L_{2k-2l} + 2 - 2(L_{2k} + (-1)^{k-l}L_{2l}) + L_{2k+2l} + 2 - 4 + 4L_{2l} - L_{4l} - 2 \\ &= L_{2k-2l} + L_{2k+2l} - 2 + (4 - 2(-1)^{k-l})L_{2l} - L_{4l} - 2L_{2k}\end{aligned}$$

and

$$\begin{aligned}(L_{k-2l} - L_k)(L_k - L_{k+2l}) &= L_k(L_{k-2l} + L_{k+2l}) - L_{k-2l}L_{k+2l} - L_k^2 \\ &= L_{2k-2l} + 2(-1)^k L_{2l} + L_{2k+2l} - 2L_{2k} - (-1)^k L_{4l} - 2.\end{aligned}$$

c) We have (by a)):

$$\begin{aligned}&\frac{1}{F_{2^n r}}\Big(F_{2^n r+1} + 1 + \Big(\sum_{l=2}^{n-1}\prod_{k=l}^{n-1} L_{2^k r}\Big) - \prod_{k=1}^{n} L_{2^k r}\Big) \\ &\overset{r\geq 1}{=} \frac{1}{F_{2^n r}}\big(F_{2^n r} + 1 + (F_{2^n r-1} - L_{(2^n-2)r}) \\ &\qquad + (L_{(2^n-4)r} - L_{(2^n-6)r}) + \ldots + (L_{4r} - L_{2r})\big)\end{aligned}$$

$$> \frac{1}{F_{2^n r}}(F_{2^n r}+1) > 1.$$

On the other hand, we have

$$\begin{aligned}
F_{2^n r+1} + 1 + \Big(\sum_{l=2}^{n-1}\prod_{k=l}^{n-1} L_{2^k r}\Big) - \prod_{k=1}^{n} L_{2^k r} & \\
< F_{2^n r} + 1 + F_{2^n r-2} + L_{(2^n-4)r} + L_{(2^n-8)r} + \ldots + L_{4r} & \\
= F_{2^n r} + 1 + F_{2^n r-2} + \frac{L_{2^n r} - L_{(2^n-4)r} - L_{4r} + 2}{L_{4r}-2} & \\
= F_{2^n r} + F_{2^n r-2} + \frac{L_{2^n r-} + L_{2^n r-3}}{L_{4r}-2}. &
\end{aligned}$$

Next

$$\begin{aligned}
\frac{L_{2^n r-1} + L_{2^n r-3}}{L_{4r}-2} < \frac{2L_{2^n r-1}}{L_{4r}-2} < \frac{2}{1-\frac{2}{L_8}}\cdot\frac{L_{2^n r-1}}{L_{4r}} & \\
< 2.15\,\frac{L_{2^n r-1}}{L_{4r}} < 2.15\,L_{(2^n-4)r-1}\,, &
\end{aligned}$$

thus, we get

$$F_{2^n r} + F_{2^n r-2} + \frac{L_{2^n r-1} + L_{2^n r-3}}{L_{4r}-2} < F_{2^n r} + F_{2^n r-2} + 2.15\,L_{(2^n-4)r-1}$$

$$= \frac{\sqrt{5}+1}{2}F_{2^n r} + \frac{1-\sqrt{5}}{2}F_{2^n r} + F_{2^n r-2} + 2.15\,(F_{(2^n-4)r} + F_{(2^n-4)r-2})$$

$$\begin{aligned}
= \frac{\sqrt{5}+1}{2}F_{2^n r} + (2-\sqrt{5})F_{2^n r-2} + \frac{1-\sqrt{5}}{2}F_{2^n r-3} & \\
+ 2.15\,(F_{2^n r-4r} + F_{2^n r-4r-2}) &
\end{aligned}$$

$$\begin{aligned}
\le \frac{\sqrt{5}+1}{2}F_{2^n r} + (2-\sqrt{5})F_{2^n r-2} & \\
+ \frac{1-\sqrt{5}}{2}F_{2^n r-3} + 2.15\,(F_{2^n r-8} + F_{2^n r-10}) &
\end{aligned}$$

(since $F_{2^n r-3} = 8\,F_{2^n r-8} + 5\,F_{2^n r-9}$)

$$= \frac{\sqrt{5}+1}{2} F_{2^n r} + (2-\sqrt{5}) F_{2^n r-2} + \frac{1-\sqrt{5}}{2}(8\,F_{2^n r-8} + 5\,F_{2^n r-9}) + 2.15\,(F_{2^n r-8} + F_{2^n r-10})$$

$$\leq \frac{\sqrt{5}+1}{2} F_{2^n r} + (2-\sqrt{5}) F_{2^n r-2} - 2.7\,F_{2^n r-8} - 3\,F_{2^n r-9} + 2.15\,F_{2^n r-10} < \frac{\sqrt{5}+1}{2} F_{2^n r}.$$

Now let $r = 1$. Then we have

$$\frac{1}{F_{2^n}}\Big(F_{2^n+1} + 1 - L_2 + \sum_{k=2}^{2^{n-2}-1}(L_{4k} - L_{4k+2})\Big)$$

$$= \frac{1}{F_{2^n}}\Big(F_{2^n+1} + 1 - L_2 - \sum_{k=2}^{2^{n-2}-1} L_{4k+1}\Big)$$

$$= \frac{1}{F_{2^n}}\Big(F_{2^n+1} - 2 - \frac{L_{2^n+1} - L_{2^n-3} - 65}{5}\Big)$$

$$= \frac{1}{F_{2^n}}\Big(F_{2^n+1} - \frac{1}{5}(L_{2^n+1} - L_{2^n-3}) + 11\Big)$$

(by the identity: $5\,F_n = L_{n+2} - L_{n-2}$)

$$= \frac{1}{F_{2^n}}(11 + F_{2^n}) < \frac{1+\sqrt{5}}{2},$$

for every $n \geq 3$. □

Now, the announced generalization of Lemma 2.9 will be presented. It is described "the full" backward iteration of $f(a)$ for $a = F_{n+1}/F_n$.

Lemma 2.11. *If $x \in \mathbb{R}$, and*

$$f(x) = \frac{1}{F_{2^n r}}(F_{2^n r+1} + 1 + L_{2^{n-1}r} + L_{2^{n-1}r}L_{2^{n-2}r} \ldots + L_{2^{n-1}r}L_{2^{n-2}r}\ldots L_{2^2 r} \pm L_{2^{n-1}r}L_{2^{n-2}r}\ldots L_{2^2 r}L_{2r}),$$

then

$$f(x) = \frac{1}{F_{2^n r}}\Big(F_{2^n r+1} + \frac{L_{(2^n-2)r} - L_{2^n r}}{2 - L_{2r}}\Big)$$

and

$$x = \frac{1}{F_{2^n r}}\Big(F_{2^n r+1} + \frac{L_{(2^n-2)r} - L_{2^n r} \pm (L_{(2^n-1)r} - L_{(2^n+1)r})}{2 - L_{2r}}\Big)$$

(by Lemma 2.10 a))

$$= \frac{1}{F_{2^n r}}(F_{2^n r+1} + 1 + L_{2^{n-1}r} + L_{2^{n-1}r}L_{2^{n-2}r} + \dots$$
$$\dots + L_{2^{n-1}r}L_{2^{n-2}r}\dots L_{2r} \pm L_{2^{n-1}r}L_{2^{n-2}r}\dots L_{2r}L_r).$$

Proof. Let $f(x) = y = \frac{1}{F_{2^n r}}\Big(F_{2^n r+1} + \frac{L_{(2^n-2)r} - L_{2^n r}}{2 - L_{2r}}\Big)$. Then, by (2.5) we get

$$\begin{aligned}(x-y)^2 &= y^2 - y - 1\\
&= \frac{1}{F^2_{2^n r}}\Big[\Big(F_{2^n r+1} + \frac{L_{(2^n-2)r} - L_{2^n r}}{2 - L_{2r}}\Big)^2\\
&\quad - F_{2^n r}\Big(F_{2^n r+1} + \frac{L_{(2^n-2)r} - L_{2^n r}}{2 - L_{2r}}\Big) - F^2_{2^n r}\Big]\\
&= \frac{1}{F^2_{2^n r}}\Big[\Big(F_{2^n r-1} + \frac{L_{(2^n-2)r} - L_{2^n r}}{2 - L_{2r}}\Big)\\
&\quad \times \Big(F_{2^n r+1} + \frac{L_{(2^n-2)r} - L_{2^n r}}{2 - L_{2r}}\Big) - F^2_{2^n r}\Big]\end{aligned}$$

$$\begin{aligned}&= \frac{1}{F^2_{2^n r}}\Big[(F_{2^n r-1}F_{2^n r+1} - F^2_{2^n r})\\
&\quad + (F_{2^n r-1} + F_{2^n r+1})\frac{L_{(2^n-2)r} - L_{2^n r}}{2 - L_{2r}} + \Big(\frac{L_{(2^n-2)r} - L_{2^n r}}{2 - L_{2r}}\Big)^2\Big]\\
&= \frac{1}{F^2_{2^n r}}\Big[1 + L_{2^n r}\frac{L_{(2^n-2)r} - L_{2^n r}}{2 - L_{2r}} + \Big(\frac{L_{(2^n-2)r} - L_{2^n r}}{2 - L_{2r}}\Big)^2\Big]\\
&= \frac{1}{F^2_{2^n r}}\Big[1 + \frac{L_{(2^n-2)r} - L_{2^n r}}{(2 - L_{2r})^2}(2L_{2^n r} - L_{2^n r}L_{2r} + L_{(2^n-2)r} - L_{2^n r})\Big]\end{aligned}$$

(by Lemma 2.10 e))

$$= \frac{1}{F^2_{2^n r}} \cdot \frac{(L_{(2^n-1)r} - L_{(2^n+1)r})^2}{(2 - L_{2r})^2}.$$

On the other hand, if

$$y = f(x) = \frac{1}{F_{2^n r}}(F_{2^n r+1} + 1 + L_{2^{n-1}r} + L_{2^{n-1}r}L_{2^{n-2}r} + \dots$$
$$\dots + L_{2^{n-1}r}L_{2^{n-2}r}\dots L_{2^2 r} - L_{2^{n-1}r}L_{2^{n-2}r}\dots L_{2^2 r}L_{2r}),$$

then, by Lemma 2.10 a) we have: $y \in (0, \frac{\sqrt{5}+1}{2})$, which implies: $y^2 - y - 1 < 0$. □

3. Newton's and Secant Iteration of Quotients of Fibonacci and Lucas Numbers

A generalization of identities (1.1) and (1.2) other than (2.1) may be started on the following identity

$$(1 + a\xi + \xi^2)(1 + b\xi + \xi^2) = \xi\big((b + a - 1) + (ab + 1)\xi + (b + a - 1)\xi^2\big). \quad (3.1)$$

Let us consider the sequence $\{a_n\}$ connected with this identity in the following way

$$a_{n+1} = \frac{a_n b_n + 1}{a_n + b_n - 1}, \qquad n \in \mathbb{N}, \quad (3.2)$$

where $\{b_n\}$ is a given sequence of complex numbers.

In Secant Method we approximate to a root of the equation $h(x) = 0$ as follows: we choose two initial approximants x_1 and x_2 and compute the sequence $\{x_n\}$ from

$$x_{n+1} = x_n - \frac{h(x_n)}{h(x_n) - h(x_{n-1})}\big(x_n - x_{n-1}\big), \qquad n \geq 2.$$

If we assume $h(x) = x^2 - x - 1$, then we obtain the formula (3.2), for $a_n = x_n$ and $b_n = x_{n-1}$.

If $a_n \to g$ and $b_n \to b$, then by the continuity of the function

$$G(x, y) = \frac{xy + 1}{x + y - 1}\,,$$

we get

$$g(g + b - 1) = gb + 1\,,$$

i.e.

$$g^2 - g - 1 = 0.$$

Thus, the limit g is equal to the golden ratio or its reciprocal. The convergence of the sequences $\{a_n\}$ of the form (3.2) is described in the following lemma:

Lemma 3.1. *Let $b_n \to b$. If $b < \frac{1}{2}$ ($|b - \alpha| > |b - \beta|$, i.e. $\mathcal{R}e(b) < \frac{1}{2}$ in the complex case) then*

$$a_n \to \alpha \qquad \textit{iff} \qquad \exists N_\alpha \in \mathbb{N} : a_n = \alpha, \quad n \geq N_\alpha.$$

If $b > \alpha$ and the sequence $\{a_n\}$ is almost positive, then the sequence $\{a_n\}$ is almost constant or it is almost increasing to α or almost decreasing to α. If $b \in (\frac{\alpha}{2} + \frac{1}{4}, \alpha)$ and the sequence $\{a_n\}$ is almost positive then $a_n \to \alpha$. If $b_n > 1$ for every $n \in \mathbb{N}$ and $\{a_n\} \subset (0, \alpha)$ then $b_n > \alpha$ for every $n \in \mathbb{N}$, sequence $\{a_n\}$ is increasing and $\lim a_n = \alpha$.

Proof. We first note that the following auxiliary identities hold:

$$\begin{aligned} a_{n+1} - \alpha &= \frac{a_n b_n + 1 - a_n\alpha - b_n\alpha + \alpha}{a_n + b_n - 1} \\ &= \frac{(a_n - \alpha)(b_n - \alpha)}{a_n + b_n - 1}, \end{aligned} \tag{3.3}$$

$$a_{n+1} - a_n = \frac{1 + a_n - a_n^2}{a_n + b_n - 1} = \frac{(\alpha - a_n)(a_n - \beta)}{a_n + b_n - 1}, \tag{3.4}$$

$$a_{n+1} - 1 = \frac{a_n(b_n - 1) + 2 - b_n}{a_n + b_n - 1}, \tag{3.5}$$

$$\frac{\alpha - b_n}{a_n + b_n - 1} - 1 = \frac{(\alpha - a_n) - 2(b_n - \frac{1}{2})}{a_n + b_n - 1}. \tag{3.6}$$

If $|\frac{b-\alpha}{\alpha+b-1}| > 1$, i.e. $|\frac{b-\alpha}{b-\beta}| > 1$ then $\Re(b) < \frac{1}{2}(\alpha+\beta) = \frac{1}{2}$ and by (3.3), we get $a_n \to \alpha$ iff the sequence $\{a_n\}$ is almost constant. Now, if $a_n > 0$ and $b_n > \alpha$ for $n \geq n_0$, then also

$$0 < \frac{b_n - \alpha}{b_n + a_n - 1} \leq \frac{b_n - \alpha}{b_n - 1} \leq \sup\left\{\frac{b_n - \alpha}{b_n - 1} : n \geq n_0\right\} < 1.$$

So, by (3.3), the sequence $\{a_n\}$ is turns to be almost constance or is almost increasing to α or is almost decreasing to α.

Let now $b \in (\frac{\alpha}{2} + \frac{1}{4}, \alpha)$, and let us assume that $a_n \geq 0$ and $b_n \in (\frac{\alpha}{2} + \frac{1}{4}, \alpha)$ for $n \geq n_0$. Then $2 > b_n > 1$ for $n \geq n_0$ and $a_n + b_n - 1 > 0$ for $n \geq n_0$ which, by (3.5) implies $a_{n+1} > 1$ for $n \geq n_0$. Hence, we get the following estimation

$$\alpha - a_n - 2b_n + 1 < \alpha - 1 - 2\left(\frac{\alpha}{2} + \frac{1}{4}\right) + 1 < 0\,,$$

so, by (3.6)

$$0 < \frac{\alpha - b_n}{a_n + b_n - 1} < 1$$

and by (3.3) we easy deduce that

$$|a_{n+1} - \alpha| < |a_n - \alpha| \qquad \text{for every} \qquad n \geq n_0.$$

Consequently, we derive

$$\sup\left\{\frac{\alpha-b_n}{a_n+b_n-1} : n\ge n_0\right\}<1$$

and again by (3.3) we obtain $a_n\to\alpha$.

If $b_n>1$ for every $n\in\mathbb{N}$ and $\{a_n\}\subset(0,\alpha)$, then by (3.4) the sequence $\{a_n\}$ is increasing and by (3.3) $b_n>\alpha$ for every $n\in\mathbb{N}$ and

$$\sup\left\{\frac{b_n-\alpha}{a_n+b_n-1} : n\in\mathbb{N}\right\}<1$$

which, again by (3.3) implies that $a_n\to\alpha$. □

The following identities on Fibonacci and Lucas numbers (see for example Koshy [4] or Vorobjov [10]) will be used in the next lemmas:

$$F_{k+l}=F_{k+1}F_l+F_kF_{l-1}, \tag{3.7}$$

$$5\,F_kF_l=L_{k+l}+(-1)^{l+1}L_{k-l}, \tag{3.8}$$

$$L_kL_l=L_{k+l}+(-1)^{l}L_{k-l}, \tag{3.9}$$

$$L_kF_l=F_{k+l}+(-1)^{l+1}F_{k-l}. \tag{3.10}$$

Lemma 3.2. *The following identities are satisfied*

$$G\Big(\frac{F_n}{F_{n-1}},\frac{F_m}{F_{m-1}}\Big)=G\Big(\frac{L_n}{L_{n-1}},\frac{L_m}{L_{m-1}}\Big)=\frac{F_{n+m-1}}{F_{n+m-2}} \tag{3.11}$$

and

$$G\Big(\frac{L_n}{L_{n-1}},\frac{F_m}{F_{m-1}}\Big)=\frac{L_{n+m-1}}{L_{n+m-2}}. \tag{3.12}$$

Proof. We have

$$G\left(\frac{F_n}{F_{n-1}},\frac{F_m}{F_{m-1}}\right)=\frac{F_nF_m+F_{n-1}F_{m-1}}{F_nF_{m-1}+F_mF_{n-1}-F_{n-1}F_{m-1}}$$

$$=\frac{F_nF_m+F_{n-1}F_{m-1}}{F_nF_{m-1}+F_{n-1}F_{m-2}}$$

$$\text{(first, by (3.7))}=\frac{F_{n+m-1}}{F_{n+m-2}} \tag{3.13}$$

$$\text{(next, by (3.8))}=\frac{L_{n+m}+L_{n+m-2}}{L_{n+m-1}+L_{n+m-3}}. \tag{3.14}$$

Similarly, by (3.9) we obtain the identity

$$G\left(\frac{L_n}{L_{n-1}}, \frac{L_m}{L_{m-1}}\right) = \frac{L_{n+m} + L_{n+m-2}}{L_{n+m-1} + L_{n+m-3}}.$$

Next, by (3.10) we get

$$G\left(\frac{L_n}{L_{n-1}}, \frac{F_m}{F_{m-1}}\right) = \frac{L_n F_m + L_{n-1}F_{m-1}}{L_n F_{m-1} + L_{n-1}F_m - L_{n-1}F_{m-1}}$$
$$= \frac{L_n F_m + L_{n-1}F_{m-1}}{L_n F_{m-1} + L_{n-1}F_{m-2}} = \frac{F_{n+m} + F_{n+m-2}}{F_{n+m-1} + F_{n+m-3}} = \frac{L_{n+m-1}}{L_{n+m-2}}. \quad \square$$

Lemma 3.3. *Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Substituting*

$$l_1 = G\left(\frac{L_{r_1}}{L_{r_1-1}}, \frac{L_{r_2}}{L_{r_2-1}}\right)$$

and

$$l_s := G\left(\frac{L_{r_{s+1}}}{L_{r_{s+1}-1}}, l_{s-1}\right), \quad s = 2, 3, \dots .$$

Then

$$l_s = \left(\sum_{k=0}^{s}\binom{s}{k}L_{R-2k}\right)\Big/\left(\sum_{k=0}^{s}\binom{s}{k}L_{R-2k-1}\right)$$
$$= \left(\sum_{k=0}^{s-1}\binom{s-1}{k}F_{R-2k-1}\right)\Big/\left(\sum_{k=0}^{s-1}\binom{s-1}{k}F_{R-2k-2}\right)$$
$$= \begin{cases} F_{R-s}/F_{R-s-1} & \text{if} \quad s \in 2\mathbb{N}-1, \\ L_{R-s}/L_{R-s-1} & \text{if} \quad s \in 2\mathbb{N}, \end{cases}$$

where $R = R_s := r_1 + r_2 + \ldots + r_s + r_{s+1}$, $s \in \mathbb{N}$.

Proof. By induction on s follows. $\square$

Remark 3.4. The following two identities connecting Fibonacci and Lucas numbers are known

$$F_{r-1} + F_{r+1} = L_r, \tag{3.15}$$
$$L_{r-1} + L_{r+1} = 5F_r. \tag{3.16}$$

Hence for every pair sequences $\{\mathbb{X}_r\}$ and $\{\mathbb{Y}_r\}$ of the form $\mathbb{X}_r \equiv F_r$, $\mathbb{Y}_r \equiv L_r$ or $\mathbb{X}_r \equiv L_r$, $\mathbb{Y}_r \equiv F_r$ we get:

$$\frac{\mathbb{X}_{r-2}}{\mathbb{X}_{r-3}} = \frac{\mathbb{Y}_{r-1} + \mathbb{Y}_{r-3}}{\mathbb{Y}_{r-2} + \mathbb{Y}_{r-4}} = \frac{\mathbb{X}_r + 2\mathbb{X}_{r-2} + \mathbb{X}_{r-4}}{\mathbb{X}_{r-1} + 2\mathbb{X}_{r-3} + \mathbb{X}_{r-5}} = \frac{\mathbb{Y}_{r+1} + 3\mathbb{Y}_{r-1} + 3\mathbb{Y}_{r-3} + \mathbb{Y}_{r-5}}{\mathbb{Y}_r + 3\mathbb{Y}_{r-2} + 3\mathbb{Y}_{r-4} + \mathbb{Y}_{r-6}}$$

(by induction)

$$= \left(\sum_{k=0}^{2s} \binom{2s}{k} \mathbb{X}_{r+2s-2k-2}\right) \Big/ \left(\sum_{k=0}^{2s} \binom{2s}{k} \mathbb{X}_{r+2s-2k-3}\right) = \left(\sum_{k=0}^{2s-1} \binom{2s-1}{k} \mathbb{Y}_{r+2s-2k-3}\right) \Big/ \left(\sum_{k=0}^{2s-1} \binom{2s-1}{k} \mathbb{Y}_{r+2s-2k-4}\right).$$

Corollary 3.5. *If $r_i > 2$ for an infinite number of indices $i \in \mathbb{N}$, then by Binet's formula we get*

$$\lim_{s\to\infty} l_s = \alpha.$$

Moreover, for every $s \in \mathbb{N}$ we obtain

$$\lim_{R_s\to\infty} l_s = \alpha.$$

The following result is a generalization of Lemma 3.2. In particular, it shows that identities (3.11) and (3.12) are characteristic indeed for Fibonacci and Lucas sequences among sequences defined by the recursive equations (for different initial conditions):

$$\mathbb{X}_{n+2} - \mathbb{X}_{n+1} - \mathbb{X}_n = 0.$$

Lemma 3.6. *a) Let $\mathbb{X}_n = A\alpha^n + B\beta^n$. Then we have*

$$\mathbb{X}_{n+m-2}\big(\mathbb{X}_n\mathbb{X}_{m-1} + \mathbb{X}_{n-1}\mathbb{X}_m - \mathbb{X}_{n-1}\mathbb{X}_{m-1}\big) \times \left(G\Big(\frac{\mathbb{X}_n}{\mathbb{X}_{n-1}}, \frac{\mathbb{X}_m}{\mathbb{X}_{m-1}}\Big) - \frac{\mathbb{X}_{n+m-1}}{\mathbb{X}_{n+m-2}}\right) = 3AB(A+B)(-1)^{n+m}$$

for $m, n \in \mathbb{N}$.

b) Let $\mathbb{X}_n = A\alpha^n + B\beta^n$ and $\mathbb{Y}_n = C\alpha^n + D\beta^n$. If $A + B = 0$, then

$$G\Big(\frac{\mathbb{X}_{n+1}}{\mathbb{X}_n}, \frac{\mathbb{Y}_{m+1}}{\mathbb{Y}_m}\Big) = \frac{\mathbb{Y}_{m+n+1}}{\mathbb{Y}_{m+n}}.$$

On the other hand, if $A^2D^2 = B^2C^2$ then

$$G\Big(\frac{\mathbb{X}_{n+1}}{\mathbb{X}_n}, \frac{\mathbb{X}_{m+1}}{\mathbb{X}_m}\Big) = G\Big(\frac{\mathbb{Y}_{n+1}}{\mathbb{Y}_n}, \frac{\mathbb{Y}_{m+1}}{\mathbb{Y}_m}\Big)$$

for every pair m, n of positive integers (such that four denominators $\mathbb{X}_n$, $\mathbb{X}_m$, $\mathbb{Y}_n$, $\mathbb{Y}_m$ are all different from zero).

Proof. a) We note that

$$\begin{aligned}
&\big(\mathbb{X}_n\mathbb{X}_m + \mathbb{X}_{n-1}\mathbb{X}_{m-1}\big)\mathbb{X}_{n+m-2}\\
&= \big(\mathbb{X}_n\mathbb{X}_{m-1} + \mathbb{X}_{n-1}\mathbb{X}_m - \mathbb{X}_{n-1}\mathbb{X}_{m-1}\big)\mathbb{X}_{n+m-1}\\
&\Leftrightarrow \Big(A^2\big(\alpha^{n+m} + \alpha^{n+m-2}\big) + B^2\big(\beta^{n+m} + \beta^{n+m-2}\big)\\
&\quad + AB\big(\alpha^n\beta^m + \alpha^m\beta^n + \alpha^{n-1}\beta^{m-1} + \alpha^{m-1}\beta^{n-1}\big)\Big)\\
&\quad \times \big(A\alpha^{n+m-2} + B\beta^{n+m-2}\big)\\
&\quad = A^2\big(2\alpha^{n+m-1} - \alpha^{n+m-2}\big) + B^2\big(2\beta^{m+m-1} - \beta^{n+m-2}\big)\\
&\quad + AB\big(\alpha^n\beta^{m-1} + \alpha^{m-1}\beta^n + \alpha^{n-1}\beta^m + \alpha^m\beta^{n-1} - \alpha^{n-1}\beta^{m-1}\\
&\quad - \alpha^{m-1}\beta^{n-1}\big)\big(A\alpha^{n+m-1} + B\beta^{n+m-1}\big)\\
&\Leftrightarrow 3AB(A+B)(-1)^{n+m} = 0.
\end{aligned}$$

b) Since $\alpha^2 - \alpha - 1 = \beta^2 - \beta - 1 = 0$, so we get

$$\begin{aligned}
&\mathbb{Y}_{m+n}\big(\mathbb{X}_{n+1}\mathbb{Y}_m + \mathbb{Y}_{m+1}\mathbb{X}_n - \mathbb{X}_n\mathbb{Y}_m\big)\\
&\qquad \times \left(G\Big(\frac{\mathbb{X}_{n+1}}{\mathbb{X}_n}, \frac{\mathbb{Y}_{m+1}}{\mathbb{Y}_m}\Big) - \frac{\mathbb{Y}_{m+n+1}}{\mathbb{Y}_{m+n}}\right)\\
&\qquad = \big(\mathbb{X}_{n+1}\mathbb{Y}_{m+1} + \mathbb{X}_n\mathbb{Y}_m\big)\mathbb{Y}_{m+n}\\
&\quad - \big(\mathbb{X}_{n+1}\mathbb{Y}_m + \mathbb{X}_n\mathbb{Y}_{m+1} - \mathbb{X}_n\mathbb{Y}_m\big)\mathbb{Y}_{m+n+1} = 5\,(-1)^{n+m}\,CD(A+B)
\end{aligned}$$

and

$$\begin{aligned}
&G\Big(\frac{\mathbb{X}_n}{\mathbb{X}_{n-1}}, \frac{\mathbb{X}_m}{\mathbb{X}_{m-1}}\Big) = G\Big(\frac{\mathbb{Y}_n}{\mathbb{Y}_{n-1}}, \frac{\mathbb{Y}_m}{\mathbb{Y}_{m-1}}\Big)\\
&\Leftrightarrow \big(\mathbb{X}_n\mathbb{X}_m + \mathbb{X}_{n-1}\mathbb{X}_{m-1}\big)\big(\mathbb{Y}_n\mathbb{Y}_{m-1} + \mathbb{Y}_m\mathbb{Y}_{n-1} - \mathbb{Y}_{n-1}\mathbb{Y}_{m-1}\big)\\
&\quad = \big(\mathbb{Y}_n\mathbb{Y}_m + \mathbb{Y}_{n-1}\mathbb{Y}_{m-1}\big)\big(\mathbb{X}_n\mathbb{X}_{m-1} + \mathbb{X}_m\mathbb{X}_{n-1} - \mathbb{X}_{n-1}\mathbb{X}_{m-1}\big)\\
&\Leftrightarrow \Big(A^2\big(\alpha^{n+m} + \alpha^{n+m-2}\big) + B^2\big(\beta^{n+m} + \beta^{n+m-2}\big)\\
&\quad + AB\big(\alpha^n\beta^m + \alpha^m\beta^n + \alpha^{n-1}\beta^{m-1} + \alpha^{m-1}\beta^{n-1}\big)\Big)\\
&\quad \times\Big(C^2\big(2\alpha^{n+m-1} - \alpha^{n+m-2}\big) + D^2\big(2\beta^{n+m-1} - \beta^{n+m-2}\big)\\
&\quad + CD\big(\alpha^n\beta^{m-1} + \alpha^{m-1}\beta^n + \alpha^{n-1}\beta^m + \alpha^m\beta^{n-1}\\
&\quad - \alpha^{n-1}\beta^{m-1} - \alpha^{m-1}\beta^{n-1}\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \Big(C^2\big(\alpha^{n+m} + \alpha^{n+m-2}\big) + D^2\big(\beta^{n+m} + \beta^{n+m-2}\big)\\
&\quad + CD\big(\alpha^n\beta^m + \alpha^m\beta^n + \alpha^{n-1}\beta^{m-1} + \alpha^{m-1}\beta^{n-1}\big)\Big)\\
&\quad \times \Big(A^2\big(2\alpha^{n+m-1} - \alpha^{n+m-2}\big) + B^2\big(2\beta^{n+m-1} - \beta^{n+m-2}\big)\\
&\quad + AB\big(\alpha^n\beta^{m-1}\alpha^{m-1}\beta^n\\
&\quad + \alpha^{n-1}\beta^m + \alpha^m\beta^{n-1} - \alpha^{n-1}\beta^{m-1} - \alpha^{m-1}\beta^{n-1}\big)\Big)\\
&\Leftrightarrow 5\big(A^2D^2 - B^2C^2\big)\big(\beta - \alpha\big) = 0. \qquad \square
\end{aligned}
$$

Remark 3.7. In papers: Alexander [1], McCabe et al [5], Phillips [6] and Taher et al [8] the following phenomenon is discovered.

If Aitken extrapolation with any step length $k < n$ is applied to the sequence

$$
x_1 = \frac{A_1}{B_1}, \qquad x_{n+1} = \frac{A_1\,x_n + A_1'}{B_1\,x_n + B_1'}, \qquad n \geq 1,
$$

where $B_1 \neq 0$ and $A_1\,B_1' - A_1'\,B_1 \neq 0$, then the following relation is produced

$$
x_{2n} = \frac{x_{n+k}\,x_{n-k} - x_n^2}{x_{n+k} - 2x_n + x_{n-k}}.
$$

In the sequel, this relation is satisfied for $x_n = F_{n+1}/F_n$, $A_1 = B_1 = A_1' = 1$, $B_1' = 0$.

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