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GEOMETRIC PROPERTIES (STRONG t-SPANNEDNESS) OF GENERIC α-STABLE COHERENT SYSTEMS ON SMOOTH CURVES

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Abstract: Here we study the geometric properties of "general" α-stable coherent systems (E, V) on curves of genus 0 and 1 (using as a key tool two papers of Lange and Newstead) and on curves of genus $g \geq 2$ when E is stable and general in its moduli space.

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1. Introduction

Let X be a smooth and connected projective curve of genus g. A coherent system on X is a pair (E, V) such that E is a vector bundle on X and $V \subseteq H^0(X, E)$ is a linear subspace. The coherent system (E, V) is of type (n, d, k) if $\text{rank}(E) = n$, $\deg(E) = d$ and $\dim(V) = k$. Hence (E, V) is uniquely determined by E and by a map of $\mathcal{O}_X$-sheaves $e_{E,V} : {\mathcal{O}_X}^{\oplus k} \to E$ which induces

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an injection of global sections. Fix $\alpha \in \mathbb{R}$. Let $\mu(E) := d/n$ denote the slope of E. Set $\mu_\alpha(E,V) := \mu(E) + \alpha k/n$. The real number μ_α is called the α-slope of the pair (E,V). A coherent subsystem $(F,W) \subseteq (E,V)$ is a coherent system such that $F \subseteq E$ and $W \subseteq V \cap H^0(X,F)$. The pair (E,V) is said to be α-stable (resp. α-semistable) if $\mu_\alpha(F,W) < \mu_\alpha(E,V)$ (resp. $\mu_\alpha(F,W) \le \mu_\alpha(E,V)$) for all coherent subsystems (F,W) of (E,V). See [2] for the general theory in the case X smooth. See [4] for the general theory for non-integral curves. See [7] for the general theory for higher dimensional projective varieties. See [5] and [6] for results on coherent systems respectively on $\mathbf{P}^1$ and on elliptic curves. See [3] for other results linked to the "spannedness theme".

Definition 1. Let X be a smooth and connected projective curve and (E,V) a coherent system on X such that $V \neq \{0\}$. Fix an integer $t \ge 0$. We will say that (E,V) is generically strongly t-spanned if $\dim(V \cap H^0(X,E(-(t+1)P))) = \dim(V) - (t+1)\cdot \mathrm{rank}(E)$ for a general $P \in X$. We will say that (E,V) is strongly t-spanned if $\dim(V \cap H^0(X,E(-Z))) = \dim(V) - (t+1)\cdot \mathrm{rank}(E)$ for every effective divisor Z of X such that $\mathrm{length}(Z) = t+1$. We will say that E is generically strongly t-spread if $h^0(X,E(-P_0 - \cdots - P_t)) = h^0(X,E) - (t+1)\mathrm{rank}(E)$ for a general $(P_0,\dots,P_t) \in X^{t+1}$.

Obviously, without changing the definition of strongly t-spannedness we could write "$\mathrm{length}(Z) \le t+1$" instead of "$\mathrm{length}(Z) = t+1$" in the previous definition. If $X \cong \mathbf{P}^1$, then "strongly t-spanned" is equivalent to "generically strongly t-spanned".

We work over an algebraically closed field $\mathbb{K}$.

Notation 1. Fix integers $n > 0$, d, $k \ge 0$, $\alpha \in \mathbb{R}$, $\alpha \ge 0$, and a smooth and connected projective curve X. Let $G(X;\alpha,n,d,k)$ denote the moduli space of all α-stable coherent systems of type (n,d,k) on X.

Notation 2. For any finite-dimensional vector space W and any integer k such that $0 \le k \le \dim(W)$, let $\mathrm{Grass}(k,W)$ denote the Grassmannian of all k-dimensional linear subspaces of W.

Now we list our results.

Theorem 1. *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Fix integers* $d \ge k > n > 0$ $t \ge 0$, $\alpha \in \mathbb{R}$, $\alpha \ge 0$, *and a smooth and connected elliptic curve* X. *Assume* $G(X;\alpha,n,d,k) \neq \emptyset$ *and take a general* $(E,V) \in G(X;\alpha,n,d,k)$. *Then* (E,V) *is generically strongly* t*-spanned if and only if* $k \ge (t+1)n$. *If* $d > k \ge n(t+1)$, *then* (E,V) *is strongly* t*-spanned.*

Theorem 2. *Fix integers* $n > \rho > 0$, $t \ge 0$, $k \ge (t+1)n$. *Let* X *be a smooth and connected curve,* E *a rank* n *generically strongly* t*-spread vector bundle on* X *such that* $h^0(X,E) \ge k$ *and* V *a general element of*

Grass$(k, H^0(X,E))$. *Then for all rank ρ subsheaves F of E we have* $\dim(H^0(X,F) \cap V) \leq k-(t+1)(n-\rho)$.

Theorem 3. *Fix integers $n > \rho > 0$, $t \geq 0$, $k < (t+1)n$. Let X be a smooth and connected curve, E a rank n generically strongly t-spread vector bundle on X such that $h^0(X,E) \geq k$ and V a general element of* Grass$(k, H^0(X,E))$. *Then for all rank ρ subsheaves F of E we have* $\dim(H^0(X,F) \cap V) \leq (t+1)\rho$.

Theorem 4. *Let X be a smooth and projective curve of genus $g \geq 2$. Fix integers $k > n \geq 2$, $t \geq 0$, $d \geq k + n(g-1)$ and take a general $E \in M(X;n,d)$. Fix a general $V \in$* Grass$(k, H^0(X,E))$. *Let F be a subsheaf of E such that $1 \leq \rho :=$* rank$(F) \leq n-1$. *Then* $\dim(V \cap H^0(X,F)) \leq \max\{(t+1)\rho, k-(t+1)(n-\rho)\}$.

Theorem 5. *Let X be a smooth and projective curve of genus $g \geq 2$. Fix integers $n \geq 2$, $t \geq 0$, $d \geq n(g+t)$ and take a general $E \in M(X;n,d)$. Fix a general $V \in$* Grass$(n(t+1), H^0(X,E))$. *Then (E,V) is α-stable for all $\alpha \geq 0$.*

2. The Proofs

Notation 3. let X be a curve, E a vector bundle on X and $V \subseteq H^0(X,E)$ a linear subspace. Let $e_{E,V} : V \otimes \mathcal{O}_X \to E$ denote the evaluation map. We will often write e_E instead of $e_{E,V}$ when $V = H^0(X,E)$.

Remark 1. Here we assume that X has genus $g \geq 2$. For all integers r, d such that $r > 0$ let $M(X;r,d)$ denote the moduli space of all stable vector bundles on X with rank r and degree d. The scheme $M(X;r,d)$ is non-empty, irreducible and $\dim(M(X;r,d)) = r^2(g-1)+1$. Fix a general $E \in M(X;r,d)$. We have $h^0(X,E) = 0$ and $h^1(X,E) = r(g-1)-d$ if $d \leq r(g-1)$. We have $h^0(X,E) = d + r(1-g)$ and $h^1(C,E) = 0$ if $d \geq r(g-1)$. If $r(g-1)+1 \leq d \leq r(g-1)+r-1$, then the evaluation map e_E is injective and with a locally free cokernel. If $d = rg$, then e_E is injective and hence E is generically strongly 0-spanned. If $d \geq rg+1$, then e_E is surjective. Notice that for general $P \in X$ when E moves in $M(X;r,d)$ the vector bundle $E(-P)$ may be considered as a general element of $M(X;r,d)$. Hence if $d \geq r(g-1+t)$ for some integer $t \geq 0$, we see that E is generically strongly t-spanned, while if $d \geq r(g-1+t)-1$, then E is strongly t-spanned.

Remark 2. Let X be an elliptic curve and E a semistable vector bundle on X with rank r and degree d. By Atiyah's classification of vector bundles on an elliptic curve ([1], Part II) we have $h^0(X,E) = 0$ and $h^1(X,E) = -d$ if $d < 0$, $0 \leq h^0(X,E) = h^1(X,E) \leq r$ if $d = 0$, $h^0(X,E) = d$ and $h^1(X,E) = 0$

if $d > 0$. Furthermore, if $d = 0$, then $h^0(X, E \otimes M) = h^1(X, E \otimes M) = 0$ for a general $M \in \mathrm{Pic}^0(X)$. Hence we immediately get that if $d \geq r(t+1)$ for some integer $t \geq 0$, then E is generically strongly t-spanned (and hence generically strongly t-spread), while if $d \geq r(t+1)+1$, then E is strongly t-spanned.

Remark 3. Let E a rigid vector bundle on $\mathbf{P}^1$ with rank r and degree d. Obviously, E is strongly t-spanned for some integer $t \geq 0$ if and only if $d \geq (t+1)r$. Fix integers n, d, k and $\alpha \in \mathbb{R}$ such that $G(\mathbf{P}^1; \alpha, n, d, k) \neq \emptyset$. By [5], Theorem 3.2, $G(\mathbf{P}^1; \alpha, n, d, k)$ is irreducible and for a general $(F, V) \in G(\mathbf{P}^1; \alpha, n, d, k)$ the vector bundle F is rigid. Hence we may apply to F the first part of this remark.

Remark 4. Fix $\alpha \in \mathbb{R}$ such that $\alpha > 0$ and an α-stable (resp. α-semistable) coherent system (E, V) on X of type (n, d, k). By the openness of stability (resp. α-semistability) the coherent system (E, W) is α-stable (resp. α-semistable) for a general k-dimensional linear subspace W of $H^0(X, E)$.

Lemma 1. *Let t be a non-negative integer, X a smooth and connected projective curve and (E, V) a generically t-spanned coherent system of type (n, d, k). Then the coherent system (E, W) is generically t-spanned for a general k-dimensional linear subspace W of $H^0(X, E)$.*

Proof. Fix any k-dimensional linear subspace A of $H^0(X, E)$. Notice that (E, A) is generically t-spanned if and only if for a general $P \in Z$ the restriction map $\rho_{(t+1)P,A} : A \to H^0((t+1)P, E|(t+1)P) \cong \mathbb{K}^{\oplus n(t+1)}$ is surjective. This is obviously an open condition on A. □

The same proof gives the following result.

Lemma 2. *Let t be a non-negative integer, X a smooth and connected projective curve and (E, V) a generically t-spread coherent system of type (n, d, k). Then the coherent system (E, W) is generically t-spanned for a general k-dimensional linear subspace W of $H^0(X, E)$.*

Lemma 3. *Fix integers $n > 0$ and $t \geq 0$. Let X be a smooth and connected projective curve and E a rank n strongly t-spanned vector bundle on X such that $h^0(X, E) > (t+1)n$. Fix any integer k such that $n(t+1)+1 \leq k \leq h^0(X, E)$ and take a general $V \in \mathrm{Grass}(k, H^0(X, E))$. Then (E, V) is strongly t-spanned.*

Proof. If $k = h^0(X, E)$, then the lemma is obviously true. Hence we may assume $k < h^0(X, E)$. Fix $Q \in X$ and let $A(Q)$ be the set of all $W \in \mathrm{Grass}(k, H^0(X, E))$ such that the restriction map $W \to H^0((t+1)Q, E|(t+$

$1)Q) \cong \mathbb{K}^{\oplus n(t+1)}$ is not surjective. See $H^0(X, E(-(t+1)Q))$ as a linear subspace of $H^0(X, E)$. Since E is strongly t-spanned, $H^0(X, E(-(t+1)Q))$ has codimension $n(t+1)$ in $H^0(X, E)$. Hence $A(Q) = \{W \in \mathrm{Grass}(k, H^0(X, E)) : \dim(W \cap H^0(X, E(-(t+1)Q)) > k - t(n+1))\}$. Since $k > n(t+1)$ the Schubert cell $A(Q)$ has codimension at least two in $\mathrm{Grass}(k, H^0(X, E))$. Since $\dim(X) = 1$, a general $V \in \mathrm{Grass}(k, H^0(X, E))$ is not contained in $\bigcup_{Q \in X} A(Q)$ and hence we are done. □

Proof of Theorem 1. By [6], Theorem 3.3, $G(X; \alpha, n, d, k)$ is irreducible and hence it make sense to consider its general element. By [6], Theorem 3.3, E is polystable and in particular it is semistable. By Remark 4 we may assume that V is a general k-dimensional linear subspace of $H^0(X, E)$. Since $d \geq k \geq (t+1)n$, the vector bundle E is generically strongly t-spanned (Remark 2). Since $k \geq t+1$, (E, V) is generically strongly t-spanned by Lemma 1. Hence the proof of the first part is over. The last assertion follows from Lemma 3. □

Remark 5. By Remark 3 both Theorem 1 and its proof are true with only trivial modification if we take $\mathbf{P}^1$ instead of an elliptic curve. The only missing tool to extend it to the case of curves with genus $g \geq 2$ is a stability theorem for coherent systems (E, V) with E general in some $M(X; n, d)$.

Proof of Theorem 2. Fix any rank ρ subsheaf F of E and then take a general $(P_0, \ldots, P_t) \in X^{t+1}$. Since (E, V) is generically strongly t-spread (Lemma 2), we have $\dim(V \cap H^0(X, E(-P_0 - \cdots - P_t))) = k - (t+1)n$. Since $\mathrm{rank}(F) = \rho$, the vector space $H^0(X, F(-P_0 - \cdots - P_t))$ has codimension at most $(t+1)\rho$ in $H^0(X, F)$. Hence $\dim(V \cap H^0(X, F(-P_0 - \cdots - P_t))) \geq \dim(V \cap H^0(X, F)) - (t+1)\rho$. Since $\dim(V \cap H^0(X, F(-P_0 - \cdots - P_t))) \leq \dim(V \cap H^0(X, E(-P_0 - \cdots - P_t)))$, we are done. □

Proof of Theorem 3. Fix any rank ρ subsheaf F of E and then take a general $(P_0, \ldots, P_t) \in X^{t+1}$. By assumption $H^0(X, E(-P_0 - \cdots - P_t))$ has codimension $(t+1)n$ in $H^0(X, E)$. Since $k < (t+1)n$ and V is general in $\mathrm{Grass}(k, H^0(X, E))$, we have $V \cap H^0(X, E(-P_0 - \cdots - P_t)) = \{0\}$. Hence $V \cap H^0(X, F(-P_0 - \cdots - P_t)) = \{0\}$. Since $\mathrm{rank}(F) = \rho$, the vector space $H^0(X, F(-P_0 - \cdots - P_t))$ has codimension at most $(t+1)\rho$ in $H^0(X, F)$. Hence $\dim(H^0(X, F) \cap V) \leq (t+1)\rho$. □

Proof of Theorem 4. Notice that t is a non-negative integer. By Remark 1 the vector bundle E is generically t-spread. Apply Theorem 2 and Theorem 3. □

Proof of Theorem 5. Fix a real number $\alpha \geq 0$, an integer ρ such that $1 \leq \rho \leq n-1$ and a rank ρ subsheaf F of E. Since E is stable, we have $\mu(F) < \mu(E)$. By Theorem 4 we have $\dim(V \cap H^0(X, F)) \leq (t+1)\rho$. Hence

$\mu_\alpha(F, V \cap H^0(X,F)) \le \alpha \cdot (t+1) + \mu(F) < \alpha \cdot (t+1) + \mu(E) = \mu_\alpha(E,V)$, concluding the proof. □

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