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ON THE PARTIALLY DEFINED SOCIAL CHOICE PROBLEM

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Abstract: Let X be either a C^∞ paracompact differential manifold or a real analytic manifold. Then for any integer $n \ge 2$ there is a fundamental system of open neighborhoods $\{U_{n,\gamma}\}_{\gamma \in \Gamma_n}$ of the small diagonal $\Delta_{X,n}$ of X^n in X^n such that:

(i) each $U_{n,\gamma}$ is invariant for the action of S_n;

(ii) for each $n \ge 2$ and each $\gamma \in \Gamma_n$ there is an S_n-invariant differentiable submersion (or an S_n-invariant real analytic submersion) $u_{n,\gamma} : U_{n,\gamma} \to \Delta_{X,n}$ such that $u_{n,\gamma}|\Delta_{X,n} = \mathrm{Id}_{\Delta_{X,n}}$.

This is a partially defined solution for the so-called social choice problem introduced by G. Chichilnisky.

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1. Introduction

Fix a "set" X. Here as a "set" we may take either a usual set or a topological space or a C^∞ paracompact differentiable manifold or a real analytic manifold. In the first (resp. second, resp. third, resp. fourth) we will call "map" or

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"morphism" a usual (resp. continuous, resp. C^∞, resp. real analytic) map. For any integer $nge2$ let S_n denote the group permutations of the set $\{1, \dots, n\}$. Let $\delta_{X,n} : X \to X^n$ denote the diagonal map $x \mapsto (x, \dots, x)$ and $\Delta_{X,n} := \mathrm{Im}(\delta_{X,n}) \subset X^n$ the small diagonal. The finite group S_n acts on X^n. A regular n-invariant δ-projection is a map $\pi : X^n \to X$ such that $\pi \circ h = \pi$ for every $h \in S_n$ and $\pi \circ \delta_{X,n} = \mathrm{Id}_X$. The problem (but for all n simultaneously !) was considered in [2], [6] and [3] in the topological setting; see these papers for the relations of this topic with market equilibrium and the so-called social choice. Here we propose to find solutions (for one fixed n or for all $n \gg 0$) not everywhere defined, but only for a subset U_n of X^n containing the small diagonal $\Delta_{X,n}$ and invariant for the action of the permutation group. We forget the set-theoretic case and the topological case. In the remaining cases we will always assume that U_n is open in X^n equipped with the product topology. First, we will do the case in which U_n may be very small, but we can get it for all n and prove the following result.

Theorem 1. *Let X be either a C^∞ paracompact differential manifold or a real analytic manifold. Then for any integer $n \geq 2$ there is a fundamental system of open neighborhoods $\{U_{n,\gamma}\}_{\gamma \in \Gamma_n}$ of $\Delta_{X,n}$ such that:*

(i) each $U_{n,\gamma}$ is invariant for the action of S_n;

(ii) for each $n \geq 2$ and each $\gamma \in \Gamma_n$ there is an S_n-invariant differentiable submersion (or an S_n-invariant real analytic submersion) $u_{n,\gamma} : U_{n,\gamma} \to \Delta_{X,n}$ such that $u_{n,\gamma}|\Delta_{X,n} = \mathrm{Id}_{\Delta_{X,n}}$.

Remark 1. In the set-up of Theorem 1 let ${\delta_{X,n}}^{-1} : \Delta_{X,n} \to X$ denote the inverse of the diagonal map. Notice that ${\delta_{X,n}}^{-1} \circ u_{n,\gamma} : U_{n,\gamma} \to X$ is a social choice function defined only on the open subset $U_{n,\gamma}$ of X^n.

Remark 2. Theorem 1 is not true in the complex analytic category for compact complex manifolds: take $X = \mathbf{CP}^1$ and apply [1], Theorem 1.2.

Remark 3. In the category of algebraic varieties (or at least for projective varieties) we cannot take in Theorem 1 a Zariski open neighborhood of $\Delta_{X,n}$ in X^n (with the Zariski topology). For $X = \mathbf{P}^1$ this assertion is [1], Theorem 1. For higher genus smooth curves, we will state and prove it now.

Theorem 2. *Let X be a smooth and connected projective curve of genus $g > 0$ defined over an algebraically closed field $\mathbb{K}$. Fix an integer $n \geq 2$. There is no pair (U_n, h_n) such that U_n is a Zariski open S_n-invariant neighborhood of $\Delta_{X,n}$ in X^n and $h_n : U_n \to X$ is a regular S_n-invariant morphism such that $\delta_{X,n} \circ h_n = \mathrm{Id}_X$.*

Proof of Theorem 1. First, we will consider the category of C^∞-manifolds and C^∞-maps. Since X is assumed to be paracompact, it admits a Riemannian metric and any such metric induces a Riemannian metric on X^n. The finite group S_n acts on X^n leaving $\Delta_{X,n}$-pointwise fixed. Take the average with respect to the finite group S_n to get an S_n-invariant Riemannian metric on X^n inducing the given metric on $\Delta_{X,n}$. Then apply the usual proof for the existence of a tubular neighborhood of $\Delta_{X,n}$ in X^n ([5], pp. 109–111) to get the S_n-invariants tubulurar neighborhoods together with the required S_n-invariant retractions. For real analytic maps it is sufficient to check the existence of an S_n-invariant tubular neighborhood (with real analytic retraction) of $\Delta_{X,n}$. Just use that any $U_{n,\gamma}$ is a real analytic manifold and apply the average process, the existence of C^∞-retractions and the relative approximation theorem given in [4], Theorem 2.25 at p. 147. □

Proof of Theorem 2. Assume the existence of such a pair (U_n, h_n). Since $g > 0$, the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is an embedding. Since X^n is smooth and $\mathrm{Alb}(X)$ in an Abelian variety, the regular map $\alpha \circ h_n : U_n \to X$ is tre restriction of a regular map $\beta : X^n \to \mathrm{Alb}(X)$. Since h_n is S_n-invariant and U_n is dense in X_n, the morphism β is S_n-invariant. Since $\alpha(X)$ is closed in $\mathrm{Alb}(X)$, β induces an S_n-invariant morphism $\gamma : X^n \to X$ extending h_n and in particular such that $\delta_{X,n} \circ \gamma = \mathrm{Id}_X$, contradicting [1], Theorem 2. □

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References

[1] E. Ballico, On the algebraic and holomorphic social choice problem, *International Journal of Pure and Applied Math.*, **17**, No. 4 (2004), 557-560.

[2] G. Chichilnisky, Intersecting families of sets and the topology of cones in economics, *Bull. Amer. Math. Soc.*, **29**, No. 2 (1993), 189-207.

[3] B. Eckmann, Social choice and topology. A case of pure and applied mathematics, *Expo. Math.*, **22**, No. 4 (2004), 385-393.

[4] F. Guaraldo, P. Macrì, A. Tancredi, *Topics on real Analytic Spaces*, Friedr. Vieweg and Sohn, Braunchweig, Wiesbaden (1986).

[5] W. Hirsch, *Differential Topology*, Springer, New York (1976).

[6] C.D. Horvath, On the topological social choice, *Soc. Choice Welfare*, **18** (2001), 227-250.