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HYPERGEOMETRIC FUNCTIONS AND INFINITE DIVISIBILITY OF PROBABILITY DISTRIBUTIONS CONSISTING OF GAMMA FUNCTIONS

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Abstract: It is shown that a probability distribution with density function, $C|\Gamma(m+ix)|^2$, is an infinitely divisible probability distribution. Here m is a real positive constant and C equals $2^{2m}/\{2\pi\Gamma(2m)\}$.

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1. Introduction

A probability distribution function $F(x)$ is called an infinitely divisible probability distribution if for each integer $n > 1$ there is a probability distribution $F_n(x)$ such that the following relation holds,

$$F(x) = (F_n * \cdots * F_n)(x),$$

where * denotes the convolution. If a probability distribution function $F(x)$ is concentrated on the interval $[0, \infty)$ and an infinitely divisible probability distribution, and if we set

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$$\zeta(s) = \int_0^\infty e^{-sx} dF(x), \quad \zeta_n(s) = \int_0^\infty e^{-sx} dF_n(x),$$

the following relation

$$\zeta(s) = (\zeta_n(s))^n$$

holds. It is known that the Laplace-Stieltjes transform of an infinitely divisible probability distribution $F(x)$ which is concentrated on the interval $[0, \infty)$ can be written as follows:

$$\zeta(s) = \exp\{-as + \int_{+0}^\infty (e^{-sx} - 1)\frac{1}{x} dK(x)\},$$

where:

(c1) $K(x)$ is nondecreasing,

(c2) $K(-0) = 0$,

(c3) $\int_1^\infty 1/x \, dK(x) < \infty$.

Here, let us assume $a = 0$ in what follows. If an infinitely divisible probability distribution $F(x)$ which is concentrated on the interval $[0, \infty)$ and if the probability distribution function $F(x)$ has a density function $f(x)$, the density funcion $f(x)$ satisfies the following integral equation (cf. [6]):

$$xf(x) = \int_0^x f(x-t) dK(t), \quad x > 0.$$

It is known in [8] that the probability distribtion with density function consisting of normed conjugate product of gamma functions such as

$$\frac{2}{\pi} |\Gamma(1 + ix)|^2 = \frac{2}{\pi} \frac{1}{\Pi_{n=1}^\infty (1 + x^2/n^2)}, \tag{1}$$

is an infinitely divisible probability distribution. From the infinite divisibility of the above probability distribution this author guessed that a probability distribution with the following density function consisting of normed conjugate product of gamma functions, that is

$$c_0 \left| \frac{\Gamma(m + ix)}{\Gamma(m)} \right|^2 = \frac{c_0}{\Pi_{n=0}^\infty (1 + x^2/(m + n)^2)}, \tag{2}$$

where m is a positive constant and c_0 is a normalised constant, is an infinitely divisible probability distribution (cf. [1, 6.1.25]). In this case the hypergeometric function $F(-n, 2m; 2m + n + 1; z)$ appears and the proof is much more complicated than the case of (1). In this paper we will show that the hypergeometric series $F(-n, 2m; 2m + n + 1; z)$ has roots outside the unit disk and then we will show that the probability distribution with density function (2) is an infinitely divisible probability distribution. At last the author would like to emphasize that the methods developed in the proofs are important in the research of zeros of special functions.

2. The Hypergeometric Function

Let m be a positive constant number. In what follows, suppose that $a_1 = m$, $a_2 = m + 1,..., a_{n+1} = m + n$. Let us consider the following density function instead of (2)

$$f(x) = \frac{c}{\Pi_{j=1}^{n+1}(x^2 + a_j^2)}, \tag{3}$$

where c is a constant to be satisfied by the following

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

The probability density function $f(x)$ is an approximation of the above right hand side of (2) in the sense of weak limit (3). It holds that

$$f(x) = \frac{c}{\Pi_{j=1}^{n+1}(x^2 + a_j^2)} = c\sum_{j=1}^{n+1} \frac{1}{\Pi_{l=1,l\neq j}^{n+1}(-a_j^2 + a_l^2)(x^2 + a_j^2)}.$$

In fact, we have that for the case $n = 0$

$$f(x) = \frac{c}{(x^2 + a_1^2)},$$

and for the case $n = 1$

$$f(x) = \frac{c}{(x^2 + a_1^2)(x^2 + a_2^2)} = \frac{c}{(a_2^2 - a_1^2)}\left\{\frac{1}{(x^2 + a_1^2)} - \frac{1}{(x^2 + a_2^2)}\right\}.$$

We have that for the case $n = 2$

$$\begin{aligned} f(x) &= \frac{c}{(x^2 + a_1^2)(x^2 + a_2^2)(x^2 + a_3^2)} \\ &= \frac{c}{(a_3^2 - a_1^2)}\left\{\frac{1}{(x^2 + a_1^2)} - \frac{1}{(x^2 + a_3^2)}\right\}\frac{1}{(x^2 + a_2^2)} \\ &= \frac{c}{(a_3^2 - a_1^2)}\left\{\frac{1}{(x^2 + a_1^2)(x^2 + a_2^2)} - \frac{1}{(x^2 + a_2^2)(x^2 + a_3^2)}\right\}. \end{aligned}$$

By mathematical induction we obtain the above equality of the sum of fractions. Let us consider a characteristic function of the density fucntion. Since it holds that

$$\int_{-\infty}^{\infty} e^{itx}\frac{1}{(x^2+a_j^2)}dx = \pi\frac{\exp(-a_j|t|)}{a_j}\,,$$

we see that

$$\begin{aligned}\phi(t) &= \int_{-\infty}^{\infty} e^{itx}\frac{c}{\Pi_{j=1}^{n+1}(x^2+a_j^2)}dx \\ &= \pi c\sum_{j=1}^{n+1}\frac{\exp(-a_j|t|)}{a_j\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)}.\end{aligned} \tag{4}$$

If we set $x = \exp(-|t|)$ we obtain a function such as the following form

$$\phi(t) = \pi c\sum_{j=1}^{n+1}\frac{x^{a_j}}{a_j\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)}, \quad 0 \leq x \leq 1,$$

and we have a polynomial

$$\begin{aligned}&F(-n, 2m; 2m+n+1; z) \\ =\;& a_1\Pi_{l=2}^{n+1}(-a_1^2+a_l^2)\sum_{j=1}^{n+1}\frac{z^{a_j-m}}{a_j\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)}.\end{aligned}$$

They are concretely as follows. For the case $n = 0$ we have

$$F(0, 2m; 2m+1; z) = 1.$$

For the case $n = 1$ we have

$$\begin{aligned}&F(-1, 2m; 2m+2; z) \\ =\;& a_1\Pi_{l=2}^{2}(-a_1^2+a_l^2)\sum_{j=1}^{2}\frac{z^{a_j-m}}{a_j\Pi_{l=1,l\neq j}^{2}(-a_j^2+a_l^2)} \\ =\;& a_1(-a_1^2+a_2^2)\left\{\frac{1}{a_1(-a_1^2+a_2^2)}+\frac{z}{a_2(-a_2^2+a_1^2)}\right\} \\ =\;& 1-\frac{m}{m+1}z\,,\end{aligned}$$

and hence

$$F(-1, 2m; 2m+2; z) = 1+\frac{(-1)(2m)}{2m+2}z. \tag{5}$$

For the case $n = 2$ we have

$$\begin{aligned}
& F(-2, 2m; 2m+3; z) \\
= & \; a_1 \Pi_{l=2}^{3}(-a_1^2 + a_l^2) \sum_{j=1}^{3} \frac{z^{a_j - m}}{a_j \Pi_{l=1, l\neq j}^{3}(-a_j^2 + a_l^2)} \\
= & \; a_1(-a_1^2 + a_2^2)(-a_1^2 + a_3^2) \Big\{ \frac{1}{a_1(-a_1^2 + a_2^2)(-a_1^2 + a_3^2)} \\
+ & \; \frac{z}{a_2(-a_2^2 + a_1^2)(-a_2^2 + a_3^2)} + \frac{z^2}{a_3(-a_3^2 + a_1^2)(-a_3^2 + a_2^2)} \Big\} \\
= & \; 1 - \frac{a_1(-a_1^2 + a_3^2)}{a_2(-a_2^2 + a_3^2)} z + \frac{a_1(-a_1^2 + a_2^2)}{a_3(a_3^2 - a_2^2)} z^2 \\
= & \; 1 - \frac{m2(2m+2)}{(m+1)1(2m+3)} z + \frac{m1(2m+1)}{(m+2)1(2m+3)} z^2 \, ,
\end{aligned}$$

and hence

$$F(-2, 2m; 2m+3; z) = 1 + \frac{(-2)(2m)}{2m+3} z + \frac{(-2)(-1)(2m)(2m+1)}{(2m+3)(2m+4)} \frac{z^2}{2!}. \qquad (6)$$

For the case $n = 3$ we have

$$\begin{aligned}
F(-3, 2m; 2m+4; z) \; = & \; 1 + \frac{(-3)(2m)}{2m+4} z + \frac{(-3)(-2)(2m)(2m+1)}{(2m+4)(2m+5)} \frac{z^2}{2!} \\
+ & \; \frac{(-3)(-2)(-1)(2m)(2m+1)(2m+2)}{(2m+4)(2m+5)(2m+6)} \frac{z^3}{3!} \qquad (7)
\end{aligned}$$

and for the general case n we have

$$\begin{aligned}
& F(-n, 2m; 2m+n+1; z) \\
= & \; 1 + \frac{(-n)(2m)}{2m+n+1} z + \frac{(-n)(-n+1)(2m)(2m+1)}{(2m+n+1)(2m+n+2)} \frac{z^2}{2!} \\
+ & \; \frac{(-n)(-n+1)(-n+2)(2m)(2m+1)(2m+2)}{(2m+n+1)(2m+n+2)(2m+n+3)} \frac{z^3}{3!} + \cdots \\
+ & \; \frac{(-n)(-n+1)\cdots(-n+k-1)(2m)(2m+1)\cdots(2m+k-1)}{(2m+n+1)(2m+n+2)\cdots(2m+n+k)} \frac{z^k}{k!} \\
+ & \; \cdots \\
+ & \; \frac{(-n)(-n+1)\cdots(-2)(-1)(2m)(2m+1)\cdots(2m+n-1)}{(2m+n+1)(2m+n+2)\cdots(2m+2n)} \frac{z^n}{n!}. \qquad (8)
\end{aligned}$$

We write the above polynomial in the following form

$$F(-n, 2m; 2m+n+1; z) = \sum_{k=0}^{n} \frac{(-n)_k (2m)_k z^k}{(2m+n+1)_k k!}.$$

3. The Hypergeometric Function has Roots Outside the Unit Disk

If $m = 1$ it is known in [7] that the roots of $F(-n, 2; n+3; z)$ appear outside the unit disk. If $n = 1$ the root of $F(-1, 2m; 2m+2; z)$ is $z_1 = (m+1)/m$ and if $n = 2$ the roots of $F(-2, 2m; 2m+3; z)$ are

$$z_1 = \frac{2m+4}{2m+1} + i\frac{1}{2m+1}\sqrt{\frac{3(m+2)}{m}}, \quad z_2 = \frac{2m+4}{2m+1} - i\frac{1}{2m+1}\sqrt{\frac{3(m+2)}{m}}$$

for each positive number m. These roots are outside the unit disk. Concerning the roots of the Gauss hypergeometric series $F(-n, 2m; 2m+n+1; z)$ for n greater than 2 we obtain the following result.

Theorem 1. *If m is a positive constant and n is a natural number the Gauss hypergeometric function $F(-n, 2m; 2m+n+1; z)$ has roots outside the unit disk.*

Proof. We have $F(-n, 2m; 2m+n+1; 0) = 1$. Take and fix a positive number σ in the interval (0,1]. The curve of the function $F(-n, 2m; 2m+n+1; \sigma e^{it})$ of the variable t is symmetric with respect to the real axis in the complex plane. In order to show that all the roots of the Gauss hypergeometric function $F(-n, 2m; 2m+n+1; z)$ are outside the unit disk, it suffices to show that $|F(-n, 2m; 2m+n+1; \sigma e^{it})|^2$ is positive for all t in the interval $[0, \pi]$. We make use of Watson's formula

$$\begin{aligned} & F(-n, b; c; z) F(-n, b; c; Z) \\ = & \frac{(c-b)_n}{(c)_n} F_4[-n, b; c, 1-n+b-c; zZ, (1-z)(1-Z)] \end{aligned}$$

in Slater's book (cf. [5. (8.4.2)]). Let $b = 2m,\ c = 2m+n+1$. Then

$$\begin{aligned} & F_4[-n, 2m; 2m+n+1, -2n; zZ, (1-z)(1-Z)] \\ = & \sum_{r=0}^{n} \sum_{s=0}^{n} \frac{(-n)_{r+s} (2m)_{r+s} (zZ)^r [(1-z)(1-Z)]^s}{(2m+n+1)_r (-2n)_s r! s!}. \end{aligned}$$

Let $z = \sigma e^{it},\ Z = \sigma e^{-it}$. Then $zZ = \sigma^2$ and $(1-z)(1-Z) = 1 + \sigma^2 - 2\sigma\cos t$. Let $x = \sigma^2$ and $y = 1 + \sigma^2 - 2\sigma\cos t$. From the above F_4 we obtain

$$|F(-n, 2m; 2m+n+1; \sigma e^{it})|^2 = \frac{(n+1)_n}{(2m+n+1)_n}\sum_{r=0}^{n}\sum_{s=0}^{n}\frac{(-n)_{r+s}(2m)_{r+s}x^r y^s}{(2m+n+1)_r(-2n)_s r!s!}. \tag{9}$$

It holds that

$$|F(-n, 2m; 2m+n+1; \sigma e^{it})|^2 = \frac{(n+1)_n}{(2m+n+1)_n}\sum_{s=0}^{n}\frac{(-n)_s(2m)_s y^s}{(-2n)_s s!}\Big\{\sum_{r=0}^{n-s}\frac{(-n+s)_r(2m+s)_r x^r}{(2m+n+1)_r r!}\Big\}.$$

We see that the right hand side of the above equality is positive for all t in the interval $[0, \pi]$ since it holds that

$$\sum_{r=0}^{n-s}\frac{(-n+s)_r(2m+s)_r x^r}{(2m+n+1)_r r!} = \frac{\Gamma(2m+n+1)}{\Gamma(2m+s)\Gamma(n-s+1)}\int_0^1 \tau^{2m+s-1}(1-\tau)^{n-s}(1-x\tau)^{n-s}d\tau.$$

When $\sigma = 1$ we see by Vandermonde's formula that

$$|F(-n, 2m; 2m+n+1; e^{it})|^2 = \frac{(n+1)_n}{(2m+n+1)_n}\sum_{s=0}^{n}\frac{(-n)_s(n+1-s)_{n-s}(2m)_s y^s}{(-2n)_s(2m+n+1)_{n-s}s!}$$
$$= \sum_{s=0}^{n}\frac{(2m)_s}{(2m+n+1)_n(2m+n+1)_{n-s}}\binom{n}{s}\binom{2n-s}{n}(2(n-s))!y^s$$

where $y = 2(1-\cos t)$. □

It is often convenient for us to treat the function $z^m F(-n, 2m; 2m+n+1; z)$ instead of $F(-n, 2m; 2m+n+1; z)$. Let us take a principal branch of $\log z$ such that $z^m = \exp(m\log z)$ is positive if z is positive.

Consider $z = e^{it}\ \ (0 \le t \le 2\pi)$. Let

$$u(m, n; t) = \Re\, e^{imt}F(-n, 2m; 2m+n+1; e^{it}),$$

$$v(m, n; t) = \Im\, e^{imt}F(-n, 2m; 2m+n+1; e^{it}).$$

We have

$$u(m,n;t)=\sum_{k=0}^{n}\frac{(-n)_k(2m)_k}{(2m+n+1)_k}\frac{\cos(m+k)t}{k!} \tag{10}$$

and

$$v(m,n;t)=\sum_{k=0}^{n}\frac{(-n)_k(2m)_k}{(2m+n+1)_k}\frac{\sin(m+k)t}{k!}. \tag{11}$$

We note that the curve of $F(-n,2m;2m+n+1;e^{it})$ in the complex plane do not always make a Jordan curve when t moves on the interval $[0,2\pi]$. It is known in [1] that the Gauss hypergeometric series $F(-n,2m;2m+n+1;z)$ is a solution of the hypergeometric equation, namely,

$$\begin{aligned} & z(1-z)\frac{d^2}{dz^2}F(-n,2m;2m+n+1;z)\\ + & (2m+n+1-(2m-n+1)z)\frac{d}{dz}F(-n,2m;2m+n+1;z)\\ + & 2mn\ F(-n,2m;2m+n+1;z)=0. \end{aligned} \tag{12}$$

Lemma 2. *When m is a positive constant and n is a natural number the functions $u(m,n;t)$ and $v(m,n;t)$ are solutions of the following differential equation*

$$\sin\frac{t}{2}\,x''(t)-n\cos\frac{t}{2}\,x'(t)+m(m+n)\sin\frac{t}{2}\,x(t)=0. \tag{13}$$

Proof. Let $h(z)=z^mF(-n,2m;2m+n+1;z)$. Then we obtain the following differential equation

$$z^2(1-z)h''(z)+(n+1+(n-1)z)zh'(z)-m(m+n)(1-z)h(z)=0. \tag{14}$$

If $z=e^{it}$ we obtain the following equation

$$(1-e^{it})\frac{d^2h(e^{it})}{dt^2}\ +n(1+e^{it})i\frac{dh(e^{it})}{dt}+m(m+n)(1-e^{it})h(e^{it})=0 \tag{15}$$

and from the real part and imaginary part of the above equation we have two equations,

$$\begin{aligned} & (1-\cos t)u''(m,n;t)-n\sin t\cdot u'(m,n;t)\\ & +m(m+n)(1-\cos t)u(m,n;t)\\ = & -\sin t\cdot v''(m,n;t)+n(1+\cos t)v'(m,n;t)\\ & -m(m+n)\sin t\cdot v(m,n;t) \end{aligned} \tag{16}$$

and

$$\begin{aligned} & \sin t \cdot u''(m,n;t) - n(1+\cos t)u'(m,n;t) \\ & +m(m+n)\sin t \cdot u(m,n;t) \\ = \;& (1-\cos t)v''(m,n;t) - n\sin t \cdot v'(m,n;t) \\ & +m(m+n)(1-\cos t)v(m,n;t). \end{aligned} \tag{17}$$

Then we have

$$\begin{aligned} & \sin\frac{t}{2}\big\{\sin\frac{t}{2}\cdot u''(m,n;t) - n\cos\frac{t}{2}\cdot u'(m,n;t) \\ & +m(m+n)\sin\frac{t}{2}\cdot u(m,n;t)\big\} \\ = \;& -\cos\frac{t}{2}\big\{\sin\frac{t}{2}\cdot v''(m,n;t) - n\cos\frac{t}{2}\cdot v'(m,n;t) \\ & +m(m+n)\sin\frac{t}{2}\cdot v(m,n;t)\big\} \end{aligned} \tag{18}$$

and

$$\begin{aligned} & \cos\frac{t}{2}\big\{\sin\frac{t}{2}\cdot u''(m,n;t) - n\cos\frac{t}{2}\cdot u'(m,n;t) \\ & +m(m+n)\sin\frac{t}{2}\cdot u(m,n;t)\big\} \\ = \;& \sin\frac{t}{2}\big\{\sin\frac{t}{2}\cdot v''(m,n;t) - n\cos\frac{t}{2}\cdot v'(m,n;t) \\ & +m(m+n)\sin\frac{t}{2}\cdot v(m,n;t)\big\}. \end{aligned} \tag{19}$$

From the above equations we obtain

$$\sin\frac{t}{2}u''(m,n;t) - n\cos\frac{t}{2}u'(m,n;t) + m(m+n)\sin\frac{t}{2}u(m,n;t) = 0$$

and

$$\sin\frac{t}{2}v''(m,n;t) - n\cos\frac{t}{2}v'(m,n;t) + m(m+n)\sin\frac{t}{2}v(m,n;t) = 0.$$

□

Theorem 3. *If m is a positive constant and n is a natural number the Gauss hypergeometric function $F(-n, 2m; 2m+n+1; z)$ does not have roots on the unit circle.*

Proof. In order to show that the Gauss hypergeometric function $F(-n, 2m; 2m+n+1; z)$ does not have roots on the unit circle we will show that the following relation

$$W(t) = u(m,n;t)v'(m,n;t) - u'(m,n;t)v(m,n;t) = c(1-\cos t)^n \tag{20}$$

holds, where c is a positive constant not depending on the variable t. If $t_0 = 0$ or 2π then $W(t_0) = 0$, while we have

$$\begin{aligned} u(m,n;t_0) &= \frac{(n+1)_n}{(2m+n+1)_n}\cos mt_0, \\ v(m,n;t_0) &= \frac{(n+1)_n}{(2m+n+1)_n}\sin mt_0, \end{aligned}$$

by Vandermonde's formula and see that

$$u(m,n;t_0)^2 + v(m,n;t_0)^2 = \left\{\frac{(n+1)_n}{(2m+n+1)_n}\right\}^2 \neq 0.$$

Let

$$\alpha(t) = 2^{-2n}\sin^{-2n}(\frac{t}{2})$$

and

$$\beta(t) = 2^{-2n}m(m+n)\sin^{-2n}(\frac{t}{2}).$$

Then the differential equation (13) can be written in the following form,

$$\{\alpha(t)x'(t)\}' + \beta(t)x(t) = 0$$

and the Wronskian $W(t)$ can be expressed as the following form,

$$\alpha(t)W(t) = c_1 \ (const).$$

Therefore we obtain

$$W(t) = c_1\, 2^{2n}\sin^{2n}(\frac{t}{2}).$$

To determine c we take $t = \pi$. Then it implies

$$W(\pi) = u(m,n;\pi)v'(m,n;\pi) - u'(m,n;\pi)v(m,n;\pi) = c_1\, 2^{2n} = c2^n.$$

We can see that

$$W(\pi)$$

$$
\begin{aligned}
&= \{\sum_{k=0}^{n} \frac{(-n)_k(2m)_k(-1)^k}{(2m+n+1)_k k!}\}\{\sum_{k=0}^{n} \frac{(-n)_k(2m)_k(m+k)(-1)^k}{(2m+n+1)_k k!}\} \\
&= \frac{(2m)_{n+1}2^{2n-1}}{(2m+n+1)_n}. \qquad (21)
\end{aligned}
$$

Therefore we obtain $c = (2m)_{n+1}2^{n-1}/(2m+n+1)_n$ and c is the positive constant. □

4. The Infinite Divisibility

We will show that the probability distribution with density function (3) is infinitely divisible. We obtain the following main result.

Theorem 4. *The probability distribution with density function (3) is infinitely divisible for every positive integer n and the probability distribution with density function*

$$c_0 \left|\frac{\Gamma(m+ix)}{\Gamma(m)}\right|^2 = \frac{c_0}{\Pi_{j=0}^{\infty}(1+x^2/(m+j)^2)}$$

is an infinitely divisible probability distribution. Here the normalised constant c_0 is equal to $2^{2m}\{\Gamma(m)\}^2/\{2\pi\Gamma(2m)\}$.

Proof. From the relation

$$
\begin{aligned}
\frac{1}{x^2+a_j^2} &= \int_0^\infty e^{-t(x^2+a_j^2)}dt \\
&= \int_0^\infty \frac{1}{\sqrt{\pi v}} e^{-x^2/v}\sqrt{\pi}\, e^{-a_j^2/v} v^{-3/2} dv,
\end{aligned}
$$

we have

$$
\begin{aligned}
f(x) &= \frac{c}{\Pi_{j=1}^{n+1}(x^2+a_j^2)} \\
&= \int_0^\infty \frac{1}{\sqrt{\pi v}} e^{-x^2/v} \sum_{j=1}^{n+1} \frac{c\sqrt{\pi}}{\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)}\, e^{-a_j^2/v} v^{-3/2}\, dv. \qquad (22)
\end{aligned}
$$

Let

$$g(v) = c\sqrt{\pi}\sum_{j=1}^{n+1} \frac{1}{\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)}\, e^{-a_j^2/v} v^{-3/2},\ (v>0).$$

When $n = 0$ we have

$$g(v) = c\sqrt{\pi}e^{-a_1^2/v}v^{-3/2}, \ (v > 0).$$

When $n = 1$ we have

$$g(v) = c\sqrt{\pi}\Big(\frac{1}{(-a_1^2 + a_2^2)}\ e^{-a_1^2/v} + \frac{1}{(-a_2^2 + a_1^2)}e^{-a_2^2/v}\Big)v^{-3/2}, \ (v > 0).$$

When $n = 2$ we have

$$\begin{aligned} g(v) &= c\sqrt{\pi}\Big(\frac{1}{(-a_1^2 + a_2^2)(-a_1^2 + a_3^2)}\ e^{-a_1^2/v} \\ &+ \frac{1}{(-a_2^2 + a_1^2)(-a_2^2 + a_3^2)}e^{-a_2^2/v} \\ &+ \frac{1}{(-a_3^2 + a_1^2)(-a_3^2 + a_2^2)}e^{-a_3^2/v}\Big)v^{-3/2}, \ (v > 0). \end{aligned}$$

The mixing density function $g(v)$ is positive for $v > 0$ and a probability density. For the case $n = 2$ we show this fact. By a change of the variables $u_1 + u_2 + u_3 = t, \ u_2 = v_1, \ u_3 = v_2$ we see that

$$\begin{aligned} f(x) &= \frac{c}{\Pi_{j=1}^3(x^2 + a_j^2)} \\ &= c\int_0^\infty\int_0^\infty\int_0^\infty e^{-u_1(x^2+a_1^2)-u_2(x^2+a_2^2)-u_3(x^2+a_3^2)}du_1du_2du_3 \\ &= c\int_0^\infty e^{-tx^2}\Big\{\int\int_{v_1\geq 0, v_2\geq 0, v_1+v_2\leq t} \\ &\qquad e^{-a_1^2t-(a_2^2-a_1^2)v_1-(a_3^2-a_1^2)v_2}dv_1dv_2\Big\}dt. \end{aligned} \tag{23}$$

On the other hand, we have

$$\begin{aligned} &f(x) \\ &= c\ \int_0^\infty e^{-tx^2}\sum_{j=1}^3\frac{1}{\Pi_{l=1,l\neq j}^3(-a_j^2 + a_l^2)}\ e^{-a_j^2t}\ dt \\ &= \int_0^\infty \frac{1}{\sqrt{\pi v}}e^{-x^2/v}\sum_{j=1}^3\frac{c\sqrt{\pi}}{\Pi_{l=1,l\neq j}^3(-a_j^2 + a_l^2)}\ e^{-a_j^2/v}v^{-3/2}\ dv. \end{aligned} \tag{24}$$

Since the double integrals can be written as follows,

$$\int\int_{v_1\geq 0, v_2\geq 0, v_1+v_2\leq t}\ e^{-a_1^2t-(a_2^2-a_1^2)v_1-(a_3^2-a_1^2)v_2}dv_1dv_2$$

$$= \sum_{j=1}^{3} \frac{1}{\Pi_{l=1,l\neq j}^{3}(-a_j^2+a_l^2)} e^{-a_j^2 t},$$

we see that the right hand side of the above is positive. Let us show that the density function $g(v)$ is an infinitely divisible density for every positive integer n. To show the infinite divisibility of the distribution with density $g(v)$, it suffices to show that the following relation

$$-\zeta'(s) = \zeta(s) \int_0^\infty e^{-sx} k(x) dx$$

holds, where $k(x)$ is a nonnegative function and satisfies the conditions (c1), (c2), (c3) imposed on an infinitely divisible probability distribution (cf. [6]). This is an expression of the integral equation mentioned in the introduction in terms of the Laplace transform. In this case the Lebesgue-Stieltjes measure $dK(x)$ is supposed to be absolutely continuous, i.e., $dK(x) = k(x)dx$. We take the Laplace transform of $g(v)$. Since it holds that

$$\int_0^\infty e^{-sv}\, e^{-a_j^2/v} v^{-3/2} dv = \frac{\sqrt{\pi}}{a_j} e^{-2a_j\sqrt{s}},$$

then we obtain

$$\begin{aligned}
\zeta(s) &= \int_0^\infty e^{-sv} g(v) dv \\
&= c\sqrt{\pi} \sum_{j=1}^{n+1} \frac{1}{\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)} \int_0^\infty e^{-sv}\, e^{-a_j^2/v} v^{-3/2} dv \\
&= c\sqrt{\pi} \sum_{j=1}^{n+1} \frac{1}{\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)} \frac{\sqrt{\pi}}{a_j} e^{-2a_j\sqrt{s}} \\
&= c\pi \sum_{j=1}^{n+1} \frac{1}{a_j \Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)} e^{-2a_j\sqrt{s}}. \qquad (25)
\end{aligned}$$

Making use of the hypergeometric function, we obtain

$$\zeta(s) = \Big[c\pi/\big(n!(2m)_n\big)\Big] z^m F(-n, 2m; 2m+n+1; z),$$

where let $z = e^{-2\sqrt{s}}$ and $\Re\{\sqrt{s}\} \geq 0$, and hence $|z| \leq 1$. Besides we should note that $F(-n, 2m; 2m+n+1; z) \neq 0$.

When $n=0$ we have

$$\zeta(s)=c\pi\frac{1}{a_1}e^{-2a_1\sqrt{s}}.$$

When $n=1$ we have

$$\zeta(s)=c\pi\Big(\frac{1}{a_1(-a_1^2+a_2^2)}e^{-2a_1\sqrt{s}}+\frac{1}{a_2(-a_2^2+a_1^2)}e^{-2a_2\sqrt{s}}\Big).$$

When $n=2$ we have

$$\begin{aligned}\zeta(s) &= c\pi\Big(\frac{1}{a_1(-a_1^2+a_2^2)(-a_1^2+a_3^2)}\,e^{-2a_1\sqrt{s}}\\ &+ \frac{1}{a_2(-a_2^2+a_1^2)(-a_2^2+a_3^2)}e^{-2a_2\sqrt{s}}\\ &+ \frac{1}{a_3(-a_3^2+a_1^2)(-a_3^2+a_2^2)}e^{-2a_3\sqrt{s}}\Big).\end{aligned}$$

We have the derivative of $\zeta(s)$, namely,

$$\zeta'(s)=-c\pi\sum_{j=1}^{n+1}\frac{1}{\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)}\frac{e^{-2a_j\sqrt{s}}}{\sqrt{s}}. \tag{26}$$

When $n=0$ we have

$$\zeta'(s)=-\frac{c\pi}{\sqrt{s}}e^{-2a_1\sqrt{s}}.$$

When $n=1$ we have

$$\zeta'(s)=-\frac{c\pi}{\sqrt{s}}\Big(\frac{1}{(-a_1^2+a_2^2)}e^{-2a_1\sqrt{s}}+\frac{1}{(-a_2^2+a_1^2)}e^{-2a_2\sqrt{s}}\Big).$$

When $n=2$ we have

$$\begin{aligned}\zeta'(s)&=-\frac{c\pi}{\sqrt{s}}\Big(\frac{1}{(-a_1^2+a_2^2)(-a_1^2+a_3^2)}\,e^{-2a_1\sqrt{s}}\\ &+\frac{1}{(-a_2^2+a_1^2)(-a_2^2+a_3^2)}e^{-2a_2\sqrt{s}}+\frac{1}{(-a_3^2+a_1^2)(-a_3^2+a_2^2)}e^{-2a_3\sqrt{s}}\Big).\end{aligned}$$

We obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \Big[\ \sum_{j=1}^{n+1} e^{-2a_j\sqrt{s}}/\Big(\sqrt{s}\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)\Big)\Big]$$

$$\cdot \quad /\Big[\sum_{j=1}^{n+1} e^{-2a_j\sqrt{s}}/\Big(a_j\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)\Big)\Big]. \tag{27}$$

The denominator of (27) does not vanish in the whole complex plane except for the origin from Theorem 1. Let us write the above fraction in the following form

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{N}{D}.$$

When $n = 0$ and $a_1 = m$ we have

$$D = \frac{1}{m}e^{-m2\sqrt{s}}, \quad N = \frac{1}{\sqrt{s}}e^{-m2\sqrt{s}}.$$

When $n = 1$ and $a_1 = m,\ a_2 = m+1$ we have

$$\begin{aligned} D &= \frac{1}{a_1(-a_1^2+a_2^2)}e^{-2a_1\sqrt{s}} + \frac{1}{a_2(-a_2^2+a_1^2)}e^{-2a_2\sqrt{s}} \\ &= \frac{1}{m(2m+1)}e^{-m2\sqrt{s}}\Big\{1 + \frac{(-1)2m}{2m+2}e^{-2\sqrt{s}}\Big\} \end{aligned}$$

and

$$\begin{aligned} N &= \frac{1}{\sqrt{s}}\Big\{\frac{1}{(-a_1^2+a_2^2)}e^{-2a_1\sqrt{s}} + \frac{1}{(-a_2^2+a_1^2)}e^{-2a_2\sqrt{s}}\Big\} \\ &= \frac{1}{m(2m+1)\sqrt{s}}e^{-m2\sqrt{s}}\Big\{m + \frac{(-1)2m}{2m+2}(m+1)e^{-2\sqrt{s}}\Big\}. \end{aligned}$$

When $n = 2$ and $a_1 = m,\ a_2 = m+1,\ a_3 = m+2$ we have

$$\begin{aligned} D &= \frac{1}{a_1(-a_1^2+a_2^2)(-a_1^2+a_3^2)}\ e^{-2a_1\sqrt{s}} \\ &+ \frac{1}{a_2(-a_2^2+a_1^2)(-a_2^2+a_3^2)}e^{-2a_2\sqrt{s}} \\ &+ \frac{1}{a_3(-a_3^2+a_1^2)(-a_3^2+a_2^2)}e^{-2a_3\sqrt{s}} \\ &= \frac{1}{m(2m+1)(2m+2)2!}e^{-m2\sqrt{s}}\Big\{1 + \frac{(-2)2m}{(2m+3)}e^{-2\sqrt{s}} \\ &+ \frac{(-2)(-1)(2m)(2m+1)}{(2m+3)(2m+4)2!}e^{-2\cdot 2\sqrt{s}}\Big\} \end{aligned}$$

and

$$N = \frac{1}{\sqrt{s}}\Big\{\frac{1}{(-a_1^2+a_2^2)(-a_1^2+a_3^2)}\ e^{-2a_1\sqrt{s}}$$

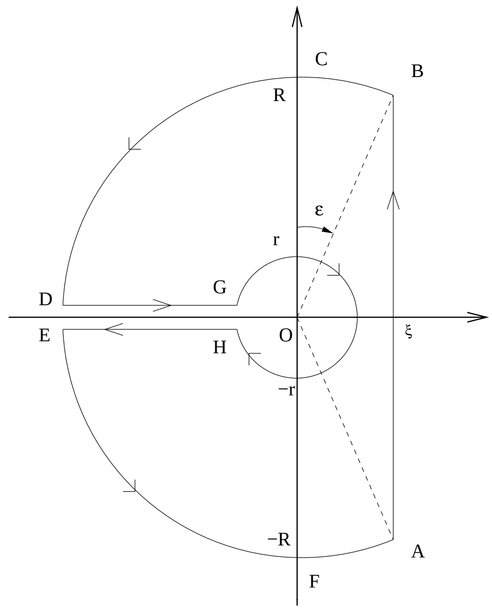

Figure 1:

$$
\begin{aligned}
&+ \quad \frac{1}{(-a_2^2+a_1^2)(-a_2^2+a_3^2)} e^{-2a_2\sqrt{s}} \\
&+ \quad \frac{1}{(-a_3^2+a_1^2)(-a_3^2+a_2^2)} e^{-2a_3\sqrt{s}} \Big\} \\
&= \quad \frac{1}{m(2m+1)(2m+2)2!\sqrt{s}} e^{-m2\sqrt{s}} \Big\{ m + \frac{(-2)(2m)}{(2m+3)}(m+1)e^{-2\sqrt{s}} \\
&+ \quad \frac{(-2)(-1)(2m)(2m+1)}{(2m+3)(2m+4)2!}(m+2)e^{-2\cdot 2\sqrt{s}} \Big\}.
\end{aligned}
$$

By the contour integration we can calculate the inverse Laplace transform of the following formula (cf. [1]).

$$
k(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\xi-iR_1}^{\xi+iR_1} e^{ts} \frac{N}{D} ds \quad (\xi > 0,\ t > 0,\ R_1 = R\cos\epsilon) \qquad (28)
$$

We will show the calculation of the inverse Laplace transform for the case $n = 1$. We have

$$
\frac{N}{D} \quad = \quad \frac{1}{\sqrt{s}} e^{-m2\sqrt{s}} \Big\{ m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-2\sqrt{s}} \Big\}
$$

$$/e^{-m2\sqrt{s}}\Big\{1 + \frac{(-1)2m}{(2m+2)}e^{-2\sqrt{s}}\Big\}.$$

(A) Contour integral along a small circle with the center at O.
From $s = re^{i\theta}$, $\sqrt{s} = \sqrt{r}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})$, for $-\pi < \theta < \pi$, we see that

$$\begin{aligned}
&\oint e^{st}\frac{N}{D}ds \\
= &\int_{\pi}^{-\pi} e^{st}\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-2\sqrt{s}}\Big\}re^{i\theta}i \\
&/\Big[\sqrt{r}e^{i\theta/2}\Big\{1 + \frac{(-1)2m}{(2m+2)}e^{-2\sqrt{s}}\Big\}\Big]d\theta \\
= &\int_{\pi}^{-\pi} e^{re^{i\theta}t}\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-2\sqrt{r}e^{i\theta/2}}\Big\}\sqrt{r}e^{i\theta/2}i \\
&/\Big\{1 + \frac{(-1)2m}{(2m+2)}e^{-2\sqrt{r}e^{i\theta/2}}\Big\}d\theta. \qquad (29)
\end{aligned}$$

Since it holds that

$$F(-1, 2m; 2m+2; e^{-2\sqrt{r}e^{i\theta/2}}) \neq 0 \qquad (30)$$

we have

$$\oint e^{st}\frac{N}{D}ds = (29) \to 0 \text{ as } r \to +0.$$

(B) Integral along $B \frown D$.
From $s = Re^{i\theta}$, $\sqrt{s} = \sqrt{R}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})$ and we see that

$$\begin{aligned}
&\int_{B\frown D} e^{st}\frac{N}{D}ds \\
= &\int_{\frac{\pi}{2}-\epsilon}^{\pi} e^{Re^{i\theta}t}\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-2\sqrt{R}e^{i\theta/2}}\Big\}Re^{i\theta}i \\
&/\Big[\sqrt{R}e^{i\theta/2}\Big\{1 + \frac{(-1)2m}{(2m+2)}e^{-2\sqrt{R}e^{i\theta/2}}\Big\}\Big]d\theta \\
= &\; i\int_{\frac{\pi}{2}}^{\pi} \sqrt{R}e^{i\theta/2}e^{Re^{i\theta}t}\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-2\sqrt{R}e^{i\theta/2}}\Big\} \\
&/\Big\{1 + \frac{(-1)2m}{(2m+2)}e^{-2\sqrt{R}e^{i\theta/2}}\Big\}d\theta \\
+ &\; i\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \sqrt{R}e^{i\theta/2}e^{Re^{i\theta}t}\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-2\sqrt{R}e^{i\theta/2}}\Big\}
\end{aligned}$$

$$/\Big\{1 + \frac{(-1)2m}{(2m+2)} e^{-2\sqrt{R}e^{i\theta/2}}\Big\} d\theta. \tag{31}$$

We see that

$$\begin{aligned}
&\int_{\frac{\pi}{2}}^{\pi} \sqrt{R}|e^{i\theta/2}e^{Re^{i\theta}t}|d\theta = \int_{\frac{\pi}{2}}^{\pi} \sqrt{R}e^{tR\cos\theta}d\theta \\
= &\int_0^{\frac{\pi}{2}} \sqrt{R}e^{-tR\sin\phi}d\phi \\
\leq &\int_0^{\frac{\pi}{2}} \sqrt{R}e^{-2tR\phi/\pi}d\phi = \sqrt{R}[-\frac{\pi}{2tR}e^{-2tR\phi/\pi}]_0^{\frac{\pi}{2}} \\
= &\sqrt{R}\{\frac{\pi}{2tR}(-e^{-tR}+1)\} \to 0
\end{aligned} \tag{32}$$

as $R \to +\infty$. Next, we show that

$$\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} |\sqrt{R}e^{i\theta/2}e^{Re^{i\theta}t}|d\theta \to 0 \tag{33}$$

as $R \to \infty$. From the fact that

$$\begin{aligned}
&\cos\theta = \cos(\phi + \frac{\pi}{2}) = -\sin\phi, \quad -\epsilon \leq \phi \leq 0, \\
&\sin\epsilon = \frac{\xi}{R} \geq -\sin\phi \geq 0,
\end{aligned}$$

we see that

$$\begin{aligned}
&\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \sqrt{R}|e^{i\theta/2}e^{Re^{i\theta}t}|d\theta = \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \sqrt{R}e^{tR\cos\theta}d\theta \\
= &\int_{-\epsilon}^{0} \sqrt{R}e^{tR\cos(\phi+\pi/2)}d\phi \leq \int_{-\epsilon}^{0} \sqrt{R}e^{tR\xi/R}d\phi = \sqrt{R}e^{t\xi}\epsilon \\
= &e^{t\xi}(\sqrt{R}\sin\epsilon)\frac{\epsilon}{\sin\epsilon} = e^{t\xi}(\sqrt{R}\frac{\xi}{R})\frac{\epsilon}{\sin\epsilon} \to 0
\end{aligned} \tag{34}$$

as $R \to \infty$.

(C) Integrals along $D \to G$ and $H \to E$.

From $s = \rho e^{i\pi},\ r \leq\ \rho \leq R$ on $D \to G$ and from $\sqrt{s} = \sqrt{\rho}e^{i\pi/2} = i\sqrt{\rho}$, we see that

$$\int_{D\to G} e^{st}\frac{N}{D}ds \tag{35}$$

$$
\begin{aligned}
&= \int_{D\to G} e^{\rho e^{i\pi} t}\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-2\sqrt{\rho}e^{i\pi/2}}\Big\} \\
&\quad /\Big[\sqrt{\rho}e^{i\pi/2}\Big\{1 \ + \frac{(-1)2m}{(2m+2)}e^{-2\sqrt{\rho}e^{i\pi/2}}\Big\}\Big]e^{i\pi}d\rho \\
&= \int_R^r e^{-\rho t}\Big[\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-i2\sqrt{\rho}}\Big\} \\
&\quad /\Big\{1 \ + \frac{(-1)2m}{(2m+2)}e^{-i2\sqrt{\rho}}\Big\}\Big]\frac{d\rho}{\sqrt{\rho}i} \qquad (36) \\
&= \int_r^R e^{-\rho t}\Big[\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-i2\sqrt{\rho}}\Big\} \\
&\quad /\Big\{1 \ + \frac{(-1)2m}{(2m+2)}e^{-i2\sqrt{\rho}}\Big\}\Big]\frac{d\rho}{\sqrt{\rho}i}. \qquad (37)
\end{aligned}
$$

From $s = \rho e^{-i\pi} = -\rho, \ \ r \le \ \rho \le R$ on $H \to E$ and from $\sqrt{s} = -i\sqrt{\rho}$ we see that

$$
\begin{aligned}
&\int_{H\to E} e^{st}\frac{N}{D}ds \\
&= \int_{H\to E} e^{\rho e^{-i\pi} t}\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-2\sqrt{\rho}e^{-i\pi/2}}\Big\} \\
&\quad /\Big[\sqrt{\rho}e^{-i\pi/2}\Big\{1 \ + \frac{(-1)2m}{(2m+2)}e^{-2\sqrt{\rho}e^{-i\pi/2}}\Big\}\Big]e^{-i\pi}d\rho \\
&= \int_r^R e^{-\rho t}\Big[\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{+i2\sqrt{\rho}}\Big\} \\
&\quad /\Big\{1 \ + \frac{(-1)2m}{(2m+2)}e^{+i2\sqrt{\rho}}\Big\}\Big]\frac{d\rho}{\sqrt{\rho}i}. \qquad (38)
\end{aligned}
$$

Therefore we see that

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_r^R e^{-\rho t}\Big[\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-i2\sqrt{\rho}}\Big\} \\
&\quad /\Big\{1 \ + \frac{(-1)2m}{(2m+2)}e^{-i2\sqrt{\rho}}\Big\} \\
&+ \quad \Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{+i2\sqrt{\rho}}\Big\} \\
&\quad /\Big\{1 \ + \frac{(-1)2m}{(2m+2)}e^{+i2\sqrt{\rho}}\Big\}\Big]\frac{d\rho}{\sqrt{\rho}i} \\
&\to \quad -\frac{1}{2\pi}\int_0^\infty e^{-\rho t}\Big[\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-i2\sqrt{\rho}}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
& /\Big\{1 \ + \frac{(-1)2m}{(2m+2)}e^{-i2\sqrt{\rho}}\Big\} \\
+ \quad & \Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{+i2\sqrt{\rho}}\Big\} \\
& /\Big\{1 \ + \frac{(-1)2m}{(2m+2)}e^{+i2\sqrt{\rho}}\Big\}\Big]\frac{d\rho}{\sqrt{\rho}}\,, \qquad (39)
\end{aligned}
$$

as $r \to 0$ and $R \to \infty$. From the Cauchy theorem we see that

$$
\begin{aligned}
& \frac{1}{2\pi i}\int_{A\to B} e^{st}\frac{N}{D}ds = \frac{1}{2\pi i}\int_{\xi-iR_1}^{\xi+iR_1} e^{st}\frac{N}{D}ds \\
\to \quad & \frac{1}{2\pi}\int_0^\infty e^{-\rho t}\Big[\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-i2\sqrt{\rho}}\Big\} \\
& /\Big\{1 \ + \frac{(-1)2m}{(2m+2)}e^{-i2\sqrt{\rho}}\Big\} \\
+ \quad & \Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{+i2\sqrt{\rho}}\Big\} \\
& /\Big\{1 \ + \frac{(-1)2m}{(2m+2)}e^{+i2\sqrt{\rho}}\Big\}\Big]\frac{d\rho}{\sqrt{\rho}}\,, \qquad (40)
\end{aligned}
$$

as $R \to \infty$. By (40) and by a change of the variable $\sqrt{\rho} = y$, we obtain

$$
\begin{aligned}
k(t) \quad = \quad & \frac{1}{\pi}\int_0^\infty e^{-ty^2}\Big[\Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{-i2y}\Big\} \\
& /\Big\{1 \ + \frac{(-1)2m}{(2m+2)}e^{-i2y}\Big\} \\
+ \quad & \Big\{m + \frac{(-1)(2m)}{(2m+2)}(m+1)e^{+i2y}\Big\} \\
& /\Big\{1 \ + \frac{(-1)2m}{(2m+2)}e^{+i2y}\Big\}\Big]dy\,. \qquad (41)
\end{aligned}
$$

For the general case n we obtain

$$
\begin{aligned}
& k(t) \\
= \quad & \frac{1}{\pi}\int_0^\infty e^{-ty^2}\Big\{\sum_{j=1}^{n+1}\frac{e^{-ia_j 2y}}{\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)}\Big/\sum_{j=1}^{n+1}\frac{e^{-ia_j 2y}}{a_j\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)} \\
+ \quad & \sum_{j=1}^{n+1}\frac{e^{ia_j 2y}}{\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)}\Big/\sum_{j=1}^{n+1}\frac{e^{ia_j 2y}}{a_j\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)}\Big\}dy. \qquad (42)
\end{aligned}
$$

To show the infinite divisibility it is necessary to show that the following function

$$\Re\Big\{\sum_{j=1}^{n+1} \frac{e^{-ia_j 2y}}{\Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)} \cdot \sum_{j=1}^{n+1} \frac{e^{+ia_j 2y}}{a_j \Pi_{l=1,l\neq j}^{n+1}(-a_j^2+a_l^2)}\Big\} \tag{43}$$

is nonnegative for $y \geq 0$. Let $2y = \theta$ in the above (43) and let

$$\begin{aligned} A &= \Re\Big\{\sum_{j=0}^{n} \frac{e^{-i(m+j)\theta}}{\Pi_{l=m,l\neq m+j}^{m+n}(-(m+j)^2+l^2)} \\ &\times \sum_{j=0}^{n} \frac{e^{+i(m+j)\theta}}{(m+j)\,\Pi_{l=m,l\neq m+j}^{m+n}(-(m+j)^2+l^2)}\Big\}. \end{aligned} \tag{44}$$

We see that

$$\begin{aligned} &\sum_{j=0}^{n} \frac{e^{-i(m+j)\theta}}{\Pi_{l=m,l\neq m+j}^{m+n}(-(m+j)^2+l^2)} \\ &= \frac{1}{m(2m+1)_n n!}\Big\{\sum_{j=0}^{n} \frac{(-n)_j(2m)_j(m+j)}{(2m+n+1)_j j!}\cos(m+j)\theta \\ &- i\sum_{j=0}^{n} \frac{(-n)_j(2m)_j(m+j)}{(2m+n+1)_j j!}\sin(m+j)\theta\Big\} \end{aligned}$$

and

$$\begin{aligned} &\sum_{j=0}^{n} \frac{e^{i(m+j)\theta}}{(m+j)\,\Pi_{l=m,l\neq m+j}^{m+n}(-(m+j)^2+l^2)} \\ &= \frac{1}{m(2m+1)_n n!}\Big\{\sum_{j=0}^{n} \frac{(-n)_j(2m)_j}{(2m+n+1)_j j!}\cos(m+j)\theta \\ &+ i\sum_{j=0}^{n} \frac{(-n)_j(2m)_j}{(2m+n+1)_j j!}\sin(m+j)\theta\Big\}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} A &= \frac{1}{\{m(2m+1)_n n!\}^2}\Big[\Big\{\sum_{j=0}^{n} \frac{(-n)_j(2m)_j}{(2m+n+1)_j j!}\cos(m+j)\theta\Big\} \\ &\times\Big\{\sum_{j=0}^{n} \frac{(-n)_j(2m)_j(m+j)}{(2m+n+1)_j j!}\cos(m+j)\theta\Big\} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \Big\{ \sum_{j=0}^{n} \frac{(-n)_j (2m)_j (m+j)}{(2m+n+1)_j j!} \sin(m+j)\theta \Big\} \\
&\quad \times \Big\{ \sum_{j=0}^{n} \frac{(-n)_j (2m)_j}{(2m+n+1)_j j!} \sin(m+j)\theta \Big\} \Big] \\
&= \quad \frac{1}{\{m(2m+1)_n n!\}^2} W(\theta). \qquad (45)
\end{aligned}
$$

If $n = 0$ we obtain $A = (1/m^2)(\cos\theta \cdot m\cos\theta + m\sin\theta \cdot \sin\theta) = 1/m$.
If $n = 1$ we obtain

$$
\begin{aligned}
A \quad &= \quad \frac{1}{\{m(2m+1)\}^2} \Big[\Big\{ \sum_{j=0}^{1} \frac{(-1)_j (2m)_j}{(2m+2)_j j!} \cos(m+j)\theta \Big\} \\
&\times \quad \Big\{ \sum_{j=0}^{1} \frac{(-1)_j (2m)_j (m+j)}{(2m+2)_j j!} \cos(m+j)\theta \Big\} \\
&+ \quad \Big\{ \sum_{j=0}^{1} \frac{(-1)_j (2m)_j (m+j)}{(2m+2)_j j!} \sin(m+j)\theta \Big\} \\
&\times \quad \Big\{ \sum_{j=0}^{1} \frac{(-1)_j (2m)_j}{(2m+2)_j j!} \sin(m+j)\theta \Big\} \Big] \\
&= \quad \frac{1}{\{m(2m+1)\}^2} \frac{2m(2m+1)}{2m+2} (1-\cos\theta). \qquad (46)
\end{aligned}
$$

If $n = 2$ we obtain

$$
\begin{aligned}
A \quad &= \quad \frac{1}{\{m(2m+1)_2 2!\}^2} \Big[\Big\{ \sum_{j=0}^{2} \frac{(-2)_j (2m)_j}{(2m+3)_j j!} \cos(m+j)\theta \Big\} \\
&\times \quad \Big\{ \sum_{j=0}^{2} \frac{(-2)_j (2m)_j (m+j)}{(2m+3)_j j!} \cos(m+j)\theta \Big\} \\
&+ \quad \Big\{ \sum_{j=0}^{2} \frac{(-2)_j (2m)_j (m+j)}{(2m+3)_j j!} \sin(m+j)\theta \Big\} \\
&\times \quad \Big\{ \sum_{j=0}^{2} \frac{(-2)_j (2m)_j}{(2m+3)_j j!} \sin(m+j)\theta \Big\} \Big] \\
&= \quad \frac{1}{\{m(2m+1)_2 2!\}^2} \frac{2(2m)_3}{(2m+3)_2} (1-\cos\theta)^2. \qquad (47)
\end{aligned}
$$

By the formula (20) we have

$$W(\theta) = \frac{2^{n-1}(2m)_{n+1}}{(2m+n+1)_n}(1-\cos\theta)^n.$$

Therefore we obtain

$$A = \frac{1}{\{m(2m+1)_n n!\}^2}\frac{2^{n-1}(2m)_{n+1}}{(2m+n+1)_n}(1-\cos\theta)^n. \tag{48}$$

Let

$$B = |\sum_{j=0}^{n} \frac{e^{i(m+j)\theta}}{(m+j)\,\Pi_{l=m,l\neq m+j}^{m+n}(-(m+j)^2+l^2)}|^2.$$

Then we have

$$B = \frac{1}{\{m(2m+1)_n n!\}^2}|F(-n,2m;2m+n+1;e^{i\theta})|^2.$$

From the fact that A is nonnegative for $\theta = 2y \geq 0$, we see that the function

$$k(t) = \frac{1}{\pi}\int_0^\infty e^{-ty^2}\frac{2A}{B}dy$$

is positive for $t > 0$ and we obtain

$$\frac{2A}{B} = \frac{2^n(2m)_{n+1}}{(2m+n+1)_n}\frac{(1-\cos\theta)^n}{|F(-n,2m;2m+n+1;e^{i\theta})|^2}.$$

Since A is nonnegative and bounded, and B does not vanish and bounded, $k(t)$ satisfies the conditions (c1), (c2) and (c3). Therefore the density function $g(v)$ is an infinitely divisible density, and the probability distribution with density function (3) is infinitely divisible since it is a mixture density of the normal distribution. The probability distribution with density function (3) converges in the sense of weak limit. In fact we obtain

$$\frac{1}{c} = \int_{-\infty}^\infty \frac{1}{\Pi_{j=1}^{n+1}(x^2+a_j^2)}dx = \frac{\pi(n+1)_n}{m(2m+1)_n n!(2m+n+1)_n}$$

and by the Stirling formula we obtain

$$\begin{aligned}\frac{c}{\Pi_{j=0}^{n}(m+j)^2} &= \frac{\{\Gamma(m)\}^2}{\{\Gamma(m+n+1)\}^2}\times\frac{m(2m+1)_n n!(2m+n+1)_n}{\pi(n+1)_n} \\ &= \frac{\{\Gamma(m)\}^2}{2\pi\Gamma(2m)}\frac{\{n!\}^2}{\{\Gamma(m+n+1)\}^2}\frac{\Gamma(2m+2n+1)}{(2n)!}\end{aligned}$$

$$\to \quad c_0 = \frac{2^{2m}\{\Gamma(m)\}^2}{2\pi\Gamma(2m)}, \tag{49}$$

as $n \to \infty$. In fact, we see that

$$\begin{aligned}
& \frac{\{n!\}^2}{\{\Gamma(m+n+1)\}^2}\frac{\Gamma(2m+2n+1)}{(2n)!} \\
\approx \quad & \frac{\{e^{-n-1}(n+1)^{n+1}(2\pi)^{1/2}\}^2}{\{e^{-(m+n+1)}(m+n+1)^{m+n+1-1/2}(2\pi)^{1/2}\}^2} \\
\times \quad & \frac{e^{-(2m+2n+1)}(2m+2n+1)^{2m+2n+1-1/2}(2\pi)^{1/2}}{e^{-2n-1}(2n+1)^{2n+1/2}(2\pi)^{1/2}} \\
= \quad & \frac{(n+1)^{2n+1}}{(m+n+1)^{2m+2n+1}} \times \frac{(2m+2n+1)^{2m+2n+1/2}}{(2n+1)^{2n+1/2}} \\
= \quad & \frac{1}{\{(1+m/(n+1))^{(n+1)/m}\}^{2m}}\Big\{(1+\frac{2m}{2n+1})^{(2n+1)/2m}\Big\}^{2m} \\
\times \quad & (\frac{m}{n+1}+1)\frac{1}{(1+2m/(2n+1))^{1/2}}\Big(\frac{2m+2n+1}{m+n+1}\Big)^{2m} \\
\to \quad & \frac{1}{e^{2m}}e^{2m}2^{2m} = 2^{2m},
\end{aligned} \tag{50}$$

as $n \to \infty$. Then we have

$$\begin{aligned}
f(x) \quad = \quad & \frac{c}{\Pi_{j=1}^{n+1}(x^2+a_j^2)} = \frac{c/(\Pi_{j=0}^{n}(m+j)^2)}{\Pi_{j=0}^{n}(1+x^2/(m+j)^2)} \\
\to \quad & \frac{c_0}{\Pi_{j=0}^{\infty}(1+x^2/(m+j)^2)},
\end{aligned} \tag{51}$$

as $n \to \infty$ in the pointwise convergence and since the infinite divisibility is closed under the weak convergence, therefore the probability distribution with density function (2) is infinitely divisible. □

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