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INTEGRAL OPERATORS BASIC IN RANDOM FIELDS ESTIMATION THEORY

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Abstract: The paper deals with the basic integral equation of random field estimation theory by the criterion of minimum of variance of the error estimate. This integral equation is of the first kind. The corresponding integral operator over a bounded domain Ω in $\mathbb{R}^n$ is weakly singular. This operator is an isomorphism between appropriate Sobolev spaces. This is proved by a reduction of the integral equation to an elliptic boundary value problem in the domain exterior to Ω. Extra difficulties arise due to the fact that the exterior boundary value problem should be solved in the Sobolev spaces of negative order.

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1. Introduction

Integral equations theory is well developed starting from the beginning of the last century. Of special interest are the classes of integral equations which can be solved in closed form or reduced to some boundary-value problems for differential equations. There are relatively few such classes of integral equations. They include equations with convolution kernels with domain of integration which is the whole space. These equations can be solved by applying the Fourier transform. The other class of integral equations solvable in closed form is the Wiener-Hopf equations. Yet another class consists of one-dimensional equations with special kernels (singular integral equations which are reducible to Riemann-Hilbert problems for analytic functions, equations with logarithmic kernels, etc.) (see e.g. [15], [2]). In [7] a new class of multidimensional integral equations is introduced. Equations of this class are solvable in closed form or reducible to a boundary-value problem for elliptic equations. This class consists of equations (3) (see below), whose kernels $R(x, y)$ are kernels of positive rational functions of an arbitrary selfadjoint elliptic operator in $L^2(\mathbb{R}^n)$, where $n \geq 1$. In [8] this theory is generalized to the class of kernels $R(x, y)$ which solve problem $QR = P\delta(x - y)$, where $\delta(x)$ is the delta-function, Q and P are elliptic differential operators, and $x \in \mathbb{R}^1$. Ellipticity in this case means that the coefficient in front of the senior derivative does not vanish. In [8] integral equations (3) with the kernels of the above class are solved in closed form by reducing them to a boundary-value problem for ODE. The aim of our paper is to generalize the approach proposed in [8] to the multidimensional equations (3) whose kernel solves equation $QR = P\delta(x - y)$ in $\mathbb{R}^n$, where $n > 1$. This is not only of theoretical interest, but also of great practical interest, because, as shown in [7], equations (3) are basic equations of random fields estimation theory. Thus, solving such equations with larger class of kernels amounts to solving estimation problems for larger class of random fields. The kernel $R(x, y)$ is the covariance function of a random field (see [7]). The class of kernels R, which solve equation $QR = P\delta(x - y)$ in $\mathbb{R}^n$, contains the class of kernels introduced and studied in [7].

Our theory is not only basic in random fields estimation theory, but can be considered as a contribution to the theory of integral equations. Any new class of integral equations, which can be solved analytically or reduced to some boundary-value problems is certainly of interest, and potentially can be used in many applied areas.

For convenience of the reader, the notations and auxiliary material, used in this paper, including the definitions of the Sobolev spaces, etc. are put in

Appendix.

Let P be a differential operator in $\mathbb{R}^n$ of order μ,

$$P := P(x, D) := \sum_{|\alpha| \leq \mu} a_\alpha(x) D^\alpha,$$

where $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$.

The polynomials

$$p(x, \xi) := \sum_{|\alpha| \leq \mu} a_\alpha(x) \xi^\alpha \quad \text{and} \quad p_0(x, \xi) := \sum_{|\alpha| = \mu} a_\alpha(x) \xi^\alpha$$

are called respectively symbol and principal symbol of P.

Suppose that the symbol $p(x, \xi)$ belongs to the class $SG^{(\mu,0)}(\mathbb{R}^n)$ consisting of all C^∞ functions $p(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$, such that for any multiindices α, β there exists a constant $C_{\alpha,\beta}$ such that

$$\left| D_x^\alpha D_\xi^\beta p(x, \xi) \right| \leq C_{\alpha,\beta} \langle\xi\rangle^{\mu - |\beta|} \langle x \rangle^{-|\alpha|}, \ x, \xi \in \mathbb{R}^n, \ \langle\xi\rangle := (1 + |\xi|^2)^{1/2}. \qquad (1)$$

It is known (cf. [14, Proposition 7.2]) that the map $P(x, D) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of smooth rapidly decaying functions. Let $H^s(\mathbb{R}^n)$ $(s \in \mathbb{R})$ be the usual Sobolev space. It is known that the operator $P(x, D)$ acts naturally on the Sobolev spaces, that is, the operator $P(x, D)$ is (cf. [14, Section 7.6]) a bounded operator: $H^s(\mathbb{R}^n) \to H^{s-\mu}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

$P(x, D)$ is called elliptic, if $p_0(x, \xi) \neq 0$ for any $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus \{0\}$.

Let $P(x, D)$ and $Q(x, D)$ be both elliptic differential operators of even orders μ and ν respectively, $0 \leq \mu < \nu$, with symbols satisfying (1) (for $Q(x, D)$ we replace p and μ in (1) respectively by q and ν). The case $\mu \geq \nu$ is a simpler case which leads to an elliptic operator perturbed by a compact integral operator in a bounded domain.

We assume also that $P(x, D)$ and $Q(x, D)$ are invertible operators, that is, there exist the inverse bounded operators $P^{-1}(x, D) : H^{s-\mu}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ and $Q^{-1}(x, D) : H^{s-\nu}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

Let $R := Q^{-1}(x, D) P(x, D)$. The invertibility of $P(x, D)$ and $Q(x, D)$ imply that R is an invertible pseudodifferential operator of negative order $\mu - \nu$ acting from $H^s(\mathbb{R}^n)$ onto $H^{s+\nu-\mu}(\mathbb{R}^n)$ $(s \in \mathbb{R})$.

Since P and Q are elliptic, their orders μ and ν are even for $n > 2$. If $n = 2$, we assume that μ and ν are even numbers. Therefore, the number $a := (\nu - \mu)/2 > 0$ is an integer.

Let Ω denote a bounded connected open set in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ (C^∞-class surface) and $\overline{\Omega}$ its closure in $L^2(\Omega)$, $\overline{\Omega} = \Omega \cup \partial\Omega$. The smoothness restriction on the domain can be weakened, but we do not go into detail.

The restriction R_Ω of the operator R to the domain $\Omega \subset \mathbb{R}^n$ is defined as

$$R_\Omega := r_\Omega R e_{\Omega_-}, \tag{2}$$

where e_{Ω_-} is the extension by zero to $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ and r_Ω is the restriction to Ω.

It is known (cf. [3, Theorem 3.11, p. 312]) that the operator R_Ω defines a continuous mapping

$$R_\Omega : H^s(\Omega) \to H^{s+\nu-\mu}(\Omega) \qquad (s > -1/2),$$

where $H^s(\Omega)$ is the space of restrictions of elements of $H^s(\mathbb{R}^n)$ to Ω with the usual infimum norm (see Appendix).

The pseudodifferential operator R of negative order $\mu - \nu$ and its restriction R_Ω can be represented as integral operators with kernel $R(x,y)$:

$$Rh = \int_{\mathbb{R}^n} R(x,y)\, h(y)\, dy, \quad R_\Omega h = \int_\Omega R(x,y)\, h(y)\, dy \ \ (x \in \Omega),$$

where $R(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \mathrm{Diag})$, Diag is the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$, Moreover, $R(x,y)$ has a weak singularity:

$$|R(x,y)| \le C\,|x-y|^{-\sigma}, \quad n + \mu - \nu \le \sigma < n.$$

For $n + \mu - \nu < 0$, $R(x,y)$ is continuous.

Let $\gamma := n + \mu - \nu$ and $r_{xy} := |x-y| \to 0$. Then $R(x,y) = O(r_{xy}^{-\gamma})$ if n is odd or if n is even and $\nu < n$, and $R(x,y) = O(r_{xy}^{-\gamma} \log r_{xy})$ if n is even and $\nu > n$.

In [7], the equation

$$R_\Omega h = f \in H^a(\Omega), \quad h \in H_0^{-a}(\Omega), \quad a = \frac{\nu - \mu}{2}, \tag{3}$$

is derived as a necessary and sufficient condition for the optimal estimate of random fields by the criterion of minimum of variance of the error of the estimate. The kernel $R(x,y)$ is a known covariance function, and $h(x,y)$ is the distributional kernel of the operator of optimal filter. The kernel $h(x,y)$ should be of minimal order of singularity, because only in this case this kernel solves the estimation problem: the variance of the error of the estimate is infinite for the solutions to equation (3), which do not have minimal order of singularity (see [7]). In [7], equation (3) was studied under the assumption that P and Q are polynomial functions of a selfadjoint elliptic operator defined in the whole space. In [8] and [9], some generalizations of this theory are given. In particular, the operators P and Q are not necessarily selfadjoint and commuting.

In this paper the authors present an extension to multidimensional integral equations of some results from [8].

The purpose of this paper is to prove that, under some natural assumptions, the operator R_Ω is an isomorphism of the space $H_0^{-a}(\Omega)$ onto $H^a(\Omega)$, where $a = (\nu - \mu)/2 > 0$, and $H_0^s(\Omega)$, $s \in \mathbb{R}$, denotes the subspace of $H^s(\mathbb{R}^n)$ that consists of the elements supported in $\overline{\Omega}$.

To prove the isomorphism property, we reduce the integral equation (3) to an equivalent elliptic exterior boundary-value problem. Since we look for a solution u belonging to the space $H^a(\Omega_-) = H^{(\nu-\mu)/2}(\Omega_-)$, and the differential operator Q is of order ν, then Qu should belong to some Sobolev space of negative order. This means that we need results on the solvability of equation (3) in Sobolev spaces of negative order. Such spaces as well as solvability in them of elliptic differential boundary value problems in *bounded* domains have been investigated in [10] and later in [6]. The case of pseudodifferential boundary value problems has been studied in [5]. In [1] and in [12] the solvability of elliptic differential and pseudodifferential boundary value problems for unbounded manifolds, and in particular for exterior domains, has been established. These solvability results have been obtained in weighted Sobolev spaces of positive order s. To obtain the isomorphism property, we need similar solvability results for exterior domain in the weighted Sobolev spaces of *negative* order. One can find in Appendix the definition of these spaces (cf. [10]).

2. Reduction of the Basic Integral Equation to a Boundary-Value Problem

In Theorem 1 the differentiation along the normal to the boundary $D^j_{\mathbf{n}}$ is used. This operator is defined in Appendix.

Theorem 1. *Integral equation* (3) *is equivalent to the following system* (4), (5), (6):

$$\begin{cases} Qu = 0 & \text{in } \Omega_- \\ D^j_{\mathbf{n}} u = D^j_{\mathbf{n}} f & \text{on } \partial\Omega, \quad 0 \le j \le a-1, \end{cases} \tag{4}$$

$$Ph = QF, \qquad h \in H_0^{-a}(\Omega), \tag{5}$$

where $u \in H^a(\Omega_-)$ *is an extension of* f:

$$F \in H^a(\mathbb{R}^n), \quad F := \begin{cases} f \in H^a(\Omega) & in\ \Omega, \\ u \in H^a(\Omega_-) & in\ \Omega_-. \end{cases} \tag{6}$$

Proof. Let $h \in H_0^{-a}(\Omega)$ solve equation (3), $R_\Omega h = f \in H^a(\Omega)$. Let us define $F := Q^{-1}Ph$. Since $h \in H_0^{-a}(\Omega)$, it follows that $Ph \in H^{-a-\mu}(\mathbb{R}^n)$ and $F = Q^{-1}Ph \in H^{-a+\nu-\mu}(\mathbb{R}^n) = H^a(\mathbb{R}^n)$. We have $f = R_\Omega h = r_\Omega Q^{-1}Ph = r_\Omega F$, so F is an extension of f. Therefore, F can be represented in the form (6). Furthermore, since $F = Q^{-1}Ph$, then $Ph = QF$, that is, h solves (5). Since $h \in H_0^{-a}(\Omega)$, then $QF = Ph \in H_0^{-a-\nu}(\Omega)$. It follows, that $Qu = 0$ in Ω_-. Since $F \in H^a(\mathbb{R}^n)$, we get $D_{\mathbf{n}}^j u = D_{\mathbf{n}}^j f$ on $\partial\Omega_-$, $0 \le j \le a-1$. This means that $u \in H^a(\Omega_-)$ solves the boundary-value problem (4). Thus, it is proved that any solution to (3) solves problem (4), see (5).

Conversely, let a pair $(u,h) \in H^a(\Omega_-) \times H_0^{-a}(\Omega)$ solves the system (4), (5), (6). Since $Ph = QF$, then $Rh = Q^{-1}Ph = F$. It follows from (6) that $R_\Omega h = Rh|_\Omega = F|_\Omega = f$, i.e. h solves (3). □

Remark. If $\mu > 0$, the boundary value problem (4) is underdetermined because Q is an elliptic operator of order ν which needs $\nu/2$ boundary conditions, but we have only a $(a < \nu/2)$ conditions in (4). Therefore, the next step is a transformation of equation (5) into $\mu/2$ extra boundary conditions to the boundary value problem (4). This will be done in Theorem 2.

Let us define $\kappa(\xi',\lambda) := \left(1 + |\xi'|^2 + \lambda^2\right)^{1/2}$. Choose a function $\rho(\tau) \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp} \mathcal{F}^{-1}\rho \subset \overline{\mathbb{R}}_-$ and $\rho(0) = 1$. Let $a > 2\sup|\partial_\tau \rho(\tau)|$. Let $\Xi_{+,\lambda}^t$ denote a family $(\lambda \in \mathbb{R}_+,\ t \in \mathbb{Z})$ of order-reducing pseudodifferential operators $\Xi_{+,\lambda}^t := \mathcal{F}^{-1}\chi_+(\xi,\lambda)\mathcal{F}$, where $\chi_+(\xi,\lambda) := \left(\kappa(\xi',\lambda)\rho\left(\frac{\xi_n}{a\kappa(\xi',\lambda)}\right) + i\xi_n\right)^t$ are their symbols. It has been proved in [4, Section 2.5] that the operator $\Xi_{+,\lambda}^t$ maps the space $\mathcal{S}_0\left(\overline{\mathbb{R}}_+^n\right) := \left\{u \in \mathcal{S}(\mathbb{R}^n) : \operatorname{supp} u \subset \overline{\mathbb{R}}_+^n\right\}$ onto itself and has the following isomorphism properties for $s \in \mathbb{R}$:

$$\Xi_{+,\lambda}^t : H^s(\mathbb{R}^n) \simeq H^{s-t}(\mathbb{R}^n), \tag{7}$$

$$\Xi_{+,\lambda}^t : H_0^s\left(\mathbb{R}_+^n\right) \simeq H_0^{s-t}\left(\mathbb{R}_+^n\right). \tag{8}$$

It is known ([4], [12]) that using $\Xi_{+,\lambda}^t$ and an appropriate partition of unity one can obtain, for sufficiently large λ, the operator Λ_+^t which is an isomorphism:

$$\Lambda_+^t : H^s(\mathbb{R}^n) \simeq H^{s-t}(\mathbb{R}^n), \quad \forall s \in \mathbb{R},$$

and

$$\Lambda_{+}^{t} : H_0^s(\Omega) \simeq H_0^{s-t}(\Omega), \quad \forall s \in \mathbb{R}. \tag{9}$$

Lemma 1. *Let $P(x,D)$ be an invertible differential operator of order μ, that is, there exists the inverse operator $P^{-1}(x,D)$ which is bounded: $P^{-1}(x,D)H^{s-\mu}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$. Then a solution h to the equation:*

$$P(x,D)\,h = g, \quad g \in H_0^{-a-\mu}(\Omega)$$

belongs to the space $H_0^{-a}(\Omega)$ if and only if g satisfies the following $\mu/2$ boundary conditions:

$$r_{\partial\Omega} D_{\mathbf{n}}^{j} \Lambda_{+}^{-a-\mu/2} P^{-1}(x,D)\, g = 0 \quad (j = 0, ..., \mu/2 - 1).$$

Proof. *Necessity*. Let $h = P^{-1}(x,D)\,g$, $h \in H_0^{-a}(\Omega)$ solve the equation $P(x,D)\,h = g$, $g \in H_0^{-a-\mu}(\Omega)$. By (9), we have $\Lambda_{+}^{-a-\mu/2} h \in H_0^{\mu/2}(\Omega)$. Therefore, $r_{\partial\Omega} D_{\mathbf{n}}^{j} \Lambda_{+}^{-a-\mu/2} h = 0 \quad (j = 0, ..., \mu/2 - 1)$.

Sufficiency. Assume that the equalities $r_{\partial\Omega} D_{\mathbf{n}}^{j} \Lambda_{+}^{-a-\mu/2} h = 0$ $(j = 0, ..., \mu/2 - 1)$ hold. Since $g \in H_0^{-a-\mu}(\Omega) \subset H^{-a-\mu}(\mathbb{R}^n)$, we have $h = P^{-1}(x,D)\,g \in H^{-a}(\mathbb{R}^n)$. Therefore, $\Psi := \Lambda_{+}^{-a-\mu/2} h \in H^{\mu/2}(\mathbb{R}^n)$. Since $r_{\partial\Omega} D_{\mathbf{n}}^{j} \Psi = 0 \quad (j = 0, ..., \mu/2 - 1)$, we have $\Psi = \Psi_{+} + \Psi_{-}$, where $\Psi_{+} := e_{\Omega_{-}} r_{\overline{\Omega}} \Psi \in H_0^{\mu/2}(\Omega)$ and $\Psi_{-} := e_{\Omega} r_{\Omega_{-}} \Psi \in H_0^{\mu/2}(\Omega_{-})$. Since $\Lambda_{+}^{\nu/2} : H_0^{\mu/2}(\Omega) \simeq H_0^{-a}(\Omega)$, it follows that $\Lambda_{+}^{\nu/2} \Psi_{+} \in H_0^{-a}(\Omega)$. Moreover, $\Lambda_{+}^{\nu/2}$ is a differential operator with respect to the variable x_n, hence $\mathrm{supp}\Psi_{-} \subset \overline{\Omega}_{-}$ implies $\mathrm{supp}\Lambda_{+}^{\nu/2}\Psi_{-} \subset \overline{\Omega}_{-}$. Since P is a differential operator,

$$\mathrm{supp}\left(P\Lambda_{+}^{\nu/2}\right)\Psi_{-} \subset \mathrm{supp}\Lambda_{+}^{\nu/2}\Psi_{-} \subset \overline{\Omega}_{-}.$$

On the other hand, we have

$$\Phi := \left(P\Lambda_{+}^{\nu/2}\right)\Psi = \left(P\Lambda_{+}^{\nu/2}\right)(\Psi_{+} + \Psi_{-}) = \left(P\Lambda_{+}^{\nu/2}\right)\Psi_{+} + \left(P\Lambda_{+}^{\nu/2}\right)\Psi_{-}.$$

For any $\varphi \in C_0^{\infty}(\Omega_{-})$ one has

$$0 = \langle \Phi, \varphi \rangle = \left\langle \left(P\Lambda_{+}^{\nu/2}\right)\Psi_{+}, \varphi \right\rangle + \left\langle \left(P\Lambda_{+}^{\nu/2}\right)\Psi_{-}, \varphi \right\rangle = \left\langle \left(P\Lambda_{+}^{\nu/2}\right)\Psi_{-}, \varphi \right\rangle.$$

Thus $\mathrm{supp}\left(P\Lambda_{+}^{\nu/2}\right)\Psi_{-} \subset \overline{\Omega}$. It follows that $\mathrm{supp}\left(P\Lambda_{+}^{\nu/2}\right)\Psi_{-} \subset \partial\Omega$. For any $\Psi_{-} \in C_0^{\infty}(\Omega_{-})$, we have $\left(P\Lambda_{+}^{\nu/2}\right)\Psi_{-} \in C^{\infty}(\mathbb{R}^n)$ and

$\operatorname{supp}\left(P\Lambda_{+}^{\nu/2}\right)\Psi_{-}\subset\partial\Omega$. Therefore, $\left(P\Lambda_{+}^{\nu/2}\right)\Psi_{-}=0$. Since P is invertible, $\Lambda_{+}^{\nu/2}\Psi_{-}=0$ for $\Psi_{-}\in C_{0}^{\infty}\left(\Omega_{-}\right)$. Since $C_{0}^{\infty}\left(\Omega_{-}\right)$ is dense in $H_{0}^{\mu/2}\left(\Omega_{-}\right)$, one gets $\Lambda_{+}^{\nu/2}\Psi_{-}=0$ for $\Psi_{-}\in H_{0}^{\mu/2}\left(\Omega_{-}\right)$. It follows, that

$$h=\Lambda_{+}^{\nu/2}\Psi=\Lambda_{+}^{\nu/2}\Psi_{+}+\Lambda_{+}^{\nu/2}\Psi_{-}=\Lambda_{+}^{\nu/2}\Psi_{+}\in H_{0}^{-a}\left(\Omega\right).$$

Lemma 1 is proved. □

Let $F\in C^{\infty}\left(\Omega\right)\cap\mathcal{S}\left(\overline{\Omega}_{-}\right)$. Assume that F has finite jumps F_k of the normal derivative of order k $(k=0,1,...)$ on $\partial\Omega$. For $x'\in\partial\Omega$, we will use the following notation:

$$F_{0}\left(x'\right):=\left[F\right]_{\partial\Omega}\left(x'\right):=\lim_{\varepsilon\to+0}\left(F\left(x'+\varepsilon\mathbf{n}\right)-F\left(x'-\varepsilon\mathbf{n}\right)\right),$$
$$F_{k}\left(x'\right):=\left[D_{\mathbf{n}}^{k}F\right]_{\partial\Omega}\left(x'\right).$$

Let $f\in C^{\infty}\left(\overline{\Omega}\right)$ and $u\in\mathcal{S}\left(\overline{\Omega}_{-}\right)$, and define $\gamma_{k}f\left(x'\right):=r_{\partial\Omega}D_{\mathbf{n}}^{k}f\left(x'\right)$, $\gamma_{k}u\left(x'\right):=r_{\partial\Omega}D_{\mathbf{n}}^{k}u\left(x'\right)$.

Let $\delta_{\partial\Omega}$ denotes the Dirac measure supported on $\partial\Omega$, that is, a distribution acting as

$$\left(\delta_{\partial\Omega},\varphi\right):=\int_{\partial\Omega}\overline{\varphi}\left(x\right)dS,\quad\varphi\left(x\right)\in C_{0}^{\infty}\left(\mathbb{R}^{n}\right).$$

It is known that for any differential operator Q of order ν there exists a representation $Q=\sum_{j=0}^{\nu}Q_{j}D_{\mathbf{n}}^{j}$, where Q_j is a tangential differential operator of order $\nu-j$ (cf. Appendix). We denote by $\{D^{\alpha}F\left(x\right)\}$ the classical derivative at the points, where it exists.

The following Lemma 2 is essentially known but for convenience of the reader a short proof of this lemma is given.

Lemma 2. *The following equality holds for the distribution QF:*

$$QF=\{QF\}-i\sum_{j=0}^{\nu}Q_{j}\sum_{k=0}^{j-1}D_{\mathbf{n}}^{k}\left(F_{j-1-k}\delta_{\partial\Omega}\right).\tag{10}$$

Proof. Let $\cos\left(\mathbf{n}x_{j}\right)$ denote cosine of the angle between the exterior unit normal vector $\mathbf{n}$ to the boundary $\partial\Omega$ of Ω and the x_j-axis.

We use the known formulas

$$\int_{\Omega}\frac{\partial u}{\partial x_{j}}dx=\int_{\partial\Omega}u\left(x\right)\cos\left(\mathbf{n}x_{j}\right)d\sigma,\quad u\left(x\right)\in C^{\infty}\left(\overline{\Omega}\right),\quad j=1,...,n,$$

$$\int\limits_{\Omega_-} \frac{\partial v}{\partial x_j} dx = -\int\limits_{\partial\Omega} v(x)\cos(\mathbf{n}x_j)\, d\sigma, \quad v(x) \in C_0^\infty\left(\overline{\Omega}_-\right), \quad j = 1, ..., n\,,$$

where $d\sigma$ is the surface measure on $\partial\Omega$. Applying these formulas to the products $u(x)\varphi(x)$ and $v(x)\varphi(x)$, where $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$, $u(x) \in C^\infty\left(\overline{\Omega}\right)$, $v(x) \in C_0^\infty\left(\overline{\Omega}_-\right)$, we get

$$\int\limits_{\Omega} \frac{\partial u}{\partial x_j}\varphi(x)\, dx = -\int\limits_{\Omega} u(x)\frac{\partial\varphi}{\partial x_j} dx + \int\limits_{\partial\Omega} u(x)\varphi(x)\cos(\mathbf{n}x_j)\, d\sigma, \qquad j = 1, ..., n\,, \quad (11)$$

$$\int\limits_{\Omega_-} \frac{\partial v}{\partial x_j}\varphi(x)\, dx = -\int\limits_{\Omega_-} v(x)\frac{\partial\varphi}{\partial x_j} dx - \int\limits_{\partial\Omega} v(x)\varphi(x)\cos(\mathbf{n}x_j)\, d\sigma, \qquad j = 1, ..., n\,. \quad (12)$$

By (11), (12), we have

$$\begin{aligned}\left(\frac{\partial F}{\partial x_j}, \varphi\right) &= -\left(F, \frac{\partial\varphi}{\partial x_j}\right) = -\int_{\mathbb{R}^n} F(x)\frac{\partial\overline{\varphi}(x)}{\partial x_j} dx \\ &= \int_{\mathbb{R}^n}\left\{\frac{\partial F(x)}{\partial x_j}\right\}\overline{\varphi}(x)\, dx + \int_{\partial\Omega}[F]_{\partial\Omega}(x)\cos(\mathbf{n}x_j)\,\overline{\varphi}(x)\, dS \\ &= \left(\left\{\frac{\partial F}{\partial x_j}\right\} + [F]_{\partial\Omega}\cos(\mathbf{n}x_j)\,\delta_{\partial\Omega},\ \varphi(x)\right), \quad \varphi(x) \in C_0^\infty(\mathbb{R}^n)\,.\end{aligned}$$

This means,

$$\frac{\partial F}{\partial x_j} = \left\{\frac{\partial F}{\partial x_j}\right\} + [F]_{\partial\Omega}\cos(\mathbf{n}x_j)\,\delta_{\partial\Omega}, \quad j = 1, ..., n.$$

It follows, $D_{\mathbf{n}}F = \{D_{\mathbf{n}}F\} - iF_0\delta_{\partial\Omega}$. Furthermore, using the last formula we have $D_{\mathbf{n}}^2F = D_{\mathbf{n}}\{D_{\mathbf{n}}F\} - iD_{\mathbf{n}}(F_0\delta_{\partial\Omega}) = \left\{D_{\mathbf{n}}^2F\right\} - iF_1\delta_{\partial\Omega} - iD_{\mathbf{n}}(F_0\delta_{\partial\Omega})$ and so on. By induction one gets:

$$D_{\mathbf{n}}^jF = \left\{D_{\mathbf{n}}^jF\right\} - i\sum_{k=0}^{j-1} D_{\mathbf{n}}^k\left(F_{j-1-k}\delta_{\partial\Omega}\right) \quad (j = 1, 2, ...)\,.$$

Substituting this formula for $D_{\mathbf{n}}^jF$ into the representation $Q = \sum_{j=0}^{\nu} Q_jD_{\mathbf{n}}^j$, we get (10). Lemma 2 is proved. □

Denoting in the sequel the extensions by zero to $\mathbb{R}^n$ of functions $f(x) \in C^{\infty}(\overline{\Omega})$, $u(x) \in \mathcal{S}(\overline{\Omega}_-)$, as f^0 and u^0, and using Lemma 2, we obtain the following formulas:

$$(Qf)^0 = Q(f^0) - i\sum_{j=1}^{\nu} Q_j \sum_{k=0}^{j-1} D_{\mathbf{n}}^k \left(\left(D_{\mathbf{n}}^{j-1-k} f \right)\Big|_{\partial\Omega} \delta_{\partial\Omega} \right) \quad \left(f \in C^{\infty}(\overline{\Omega}) \right), \tag{13}$$

$$(Qu)^0 = Q(u^0) + i\sum_{j=1}^{\nu} Q_j \sum_{k=0}^{j-1} D_{\mathbf{n}}^k \left(\left(D_{\mathbf{n}}^{j-1-k} u \right)\Big|_{\partial\Omega} \delta_{\partial\Omega} \right) \quad \left(u \in \mathcal{S}(\overline{\Omega}_-) \right), \tag{14}$$

where $\left(D_{\mathbf{n}}^j f \right)\Big|_{\partial\Omega} := r_{\partial\Omega} D_{\mathbf{n}}^j f$. Using these formulas one can define the action of the operator Q upon the elements of the spaces $\mathfrak{H}^{s,\nu}(\Omega)$ and $\mathfrak{H}^{s,\nu}(\Omega_-)$ $(s \in \mathbb{R})$ (defined in Appendix) as follows (cf. [6, Section 3.2.], [10, Section 2.4]):

$$\left(Q(f, \underline{\psi}) \right)^0 := Q(f^0) - i\sum_{j=1}^{\nu} Q_j \sum_{k=0}^{j-1} D_{\mathbf{n}}^k (\psi_{j-k} \delta_{\partial\Omega}) \quad \left((f, \underline{\psi}) \in \mathfrak{H}^{s,\nu}(\Omega) \right), \tag{15}$$

$$\left(Q(u, \underline{\phi}) \right)^0 := Q(u^0) + i\sum_{j=1}^{\nu} Q_j \sum_{k=0}^{j-1} D_{\mathbf{n}}^k (\phi_{j-k} \delta_{\partial\Omega}) \quad \left((u, \underline{\phi}) \in \mathfrak{H}^{s,\nu}(\Omega_-) \right). \tag{16}$$

It is known ([10], [6], [5]) that Q, defined respectively in (15) and (16), is a bounded mapping

$$Q : \mathfrak{H}^{s,\nu}(\Omega) \to \mathcal{H}^{s-\nu}(\Omega) \text{ and } Q : \mathfrak{H}^{s,\nu}(\Omega_-) \to \mathcal{H}^{s-\nu}(\Omega_-).$$

Moreover, Q is respectively the closure of the mapping $f \to Q(x, D) f \quad \left(f \in C^{\infty}(\overline{\Omega}) \right)$ or $u \to Q(x, D) u \quad \left(u \in \mathcal{S}(\overline{\Omega}_-) \right)$ between the corresponding spaces.

Let $W_{m\ell}$ $(m = 1, ..., \mu/2, \quad \ell = a+1, ..., \nu)$ be the operator acting as follows:

$$W_{m\ell}(\phi) := i\gamma_{m-1} \Lambda_+^{-a-\mu/2} P^{-1} \sum_{\ell=a+1}^{\nu} \sum_{j=\ell}^{\nu} Q_j D_{\mathbf{n}}^{j-\ell} (\phi \delta_{\partial\Omega}), \ \phi \in C^{\infty}(\partial\Omega), \tag{17}$$

where γ_k is the restriction to $\partial\Omega$ of the $D_{\mathbf{n}}^k$ (cf. Appendix).

The mapping $W_{m\ell}$ is a pseudodifferential operator of order $m - \mu + \nu/2 - 1 - \ell$. Therefore, for any real s, this mapping is a bounded operator:

$$W_{m\ell} : H^s(\partial\Omega) \to H^{s-m+\mu-\nu/2+1+\ell}(\partial\Omega).$$

For $\left(f,\underline{\psi}\right)\in\mathfrak{H}^{a,\nu}(\Omega)$, we have $g:=Q\left(f,\underline{\psi}\right)\in H_0^{a-\nu}(\Omega)$, and we set

$$w_{a+m}:=-\gamma_{m-1}\Lambda_+^{-a-\mu/2}P^{-1}g^0\quad (m=1,...,\mu/2)\,, \tag{18}$$

where the operator $\gamma_{m-1}\Lambda_+^{-a-\mu/2}P^{-1}(x,D)$ is a trace operator of order $m-1-a-3\mu/2$. It follows that $w_{a+m}\in H^{\mu/2-m+1/2}(\partial\Omega)$.

Theorem 2. *The integral equation* (3)

$$R_\Omega h=f\in H^a(\Omega)\,,\quad h\in H_0^{-a}(\Omega)$$

is equivalent to the following boundary-value problem:

$$\begin{cases} Qu=0 & \text{in } \Omega_-\,,\\ D_{\mathbf{n}}^j u=D_{\mathbf{n}}^j f & \text{on } \partial\Omega,\quad 0\le j\le a-1\,,\\ \sum\limits_{\ell=a+1}^{\nu}W_{m\ell}\left(\gamma_{\ell-1}u\right)=w_{a+m} & \text{on } \partial\Omega,\quad 1\le m\le\mu/2\,, \end{cases} \tag{19}$$

where the functions u, f *and* h *are related by the formulas*

$$h=P^{-1}QF,\quad F\in H^a(\mathbb{R}^n)\,,\quad F:=\begin{cases} f\in H^a(\Omega) & \text{in } \Omega,\\ u\in H^a(\Omega_-) & \text{in } \Omega_-\,.\end{cases}$$

Proof. Our starting point is Theorem 1. Consider the equation $Ph=QF$, $h\in H_0^{-a}(\Omega)$. Since $F\in H^a(\mathbb{R}^n)$ and $Qu=0$ in Ω_- by (4), then $QF\in H_0^{a-\nu}(\Omega)=H_0^{-a-\mu}(\Omega)$. By Lemma 1, a solution h to the equation $Ph=QF\in H_0^{-a-\mu}(\Omega)$ belongs to the space $H_0^{-a}(\Omega)$ if and only if QF satisfies the following $\mu/2$ boundary conditions:

$$r_{\partial\Omega}D_{\mathbf{n}}^{m-1}\Lambda_+^{-a-\mu/2}P^{-1}QF=0,\quad m=1,...,\mu/2\,. \tag{20}$$

Since $F=f^0+u^0$, one has $QF=Q\left(f^0\right)+Q\left(u^0\right)$. Substituting the last expression into (20) we have

$$\gamma_{m-1}\Lambda_+^{-a-\mu/2}P^{-1}Q\left(u^0\right)=-\gamma_{m-1}\Lambda_+^{-a-\mu/2}P^{-1}Q\left(f^0\right)\,,\quad m=1,...,\mu/2\,.$$

From (15) and (16), one gets:

$$\begin{aligned} i\gamma_{m-1}\Lambda_+^{-a-\mu/2}P^{-1}\sum_{j=1}^{\nu}Q_j\sum_{k=0}^{j-1}D_{\mathbf{n}}^k\left(\phi_{j-k}\delta_{\partial\Omega}\right) &\\ =\gamma_{m-1}\Lambda_+^{-a-\mu/2}P^{-1}\left(Q\left(f,\underline{\psi}\right)\right)^0 &\\ +\,i\gamma_{m-1}\Lambda_+^{-a-\mu/2}P^{-1}\sum_{j=1}^{\nu}Q_j\sum_{k=0}^{j-1}D_{\mathbf{n}}^k\left(\psi_{j-k}\delta_{\partial\Omega}\right). & \end{aligned} \tag{21}$$

Since

$$F := \begin{cases} f \in H^a(\Omega) & \text{in } \Omega, \\ u \in H^a(\Omega_-) & \text{in } \Omega_-, \end{cases} \quad \text{and } F \in H^a(\mathbb{R}^n),$$

it follows that $\gamma_{j-1}u = \gamma_{j-1}f$, $j = 1, ..., a$. Therefore, $\phi_j = \gamma_{j-1}u = \gamma_{j-1}f = \psi_j$, $j = 1, ..., a$.

We identify the space $H^a(\Omega)$ with the subspace of $\mathfrak{H}^{a,(\nu)}(\Omega)$ of all $(f, \underline{\psi}) = (f, \psi_1, ..., \psi_\nu)$ such that $\psi_{a+1} = ... = \psi_\nu = 0$. Let $(f, \underline{\psi})$ belong to this subspace and $(u, \underline{\phi}) = (u, \phi_1, ..., \phi_\nu) \in \mathfrak{H}^{a,(\nu)}(\Omega_-)$. Then we can rewrite (21) as

$$\gamma_{m-1}\Lambda_+^{-a-\mu/2}P^{-1}\left(Q\left(f, \underline{\psi}\right)\right)^0 + i\gamma_{m-1}\Lambda_+^{-a-\mu/2}P^{-1}\sum_{j=1}^{\nu} Q_j \sum_{\ell=a+1}^{j} D_{\mathbf{n}}^{j-\ell}(\phi_\ell \delta_{\partial\Omega}) = 0.$$

Changing the order of the summation

$$\sum_{j=1}^{\nu}\sum_{\ell=a+1}^{j} = \sum_{\ell=a+1}^{\nu}\sum_{j=\ell}^{\nu},$$

we get

$$i\gamma_{m-1}\Lambda_+^{-a-\mu/2}P^{-1}\sum_{\ell=a+1}^{\nu}\sum_{j=\ell}^{\nu} Q_j D_{\mathbf{n}}^{j-\ell}(\phi_\ell \delta_{\partial\Omega}) = -\gamma_{m-1}\Lambda_+^{-a-\mu/2}P^{-1}\left(Q\left(f, \underline{\psi}\right)\right)^0, \quad (22)$$

where $m = 1, ..., \mu/2$. In view of (17) and (18), formula (22) can be rewritten as $\mu/2$ equations

$$\sum_{\ell=a+1}^{\nu} W_{m\ell}(\phi_\ell) = w_{a+m} \quad \text{on } \partial\Omega, \ \ m = 1, ..., \mu/2.$$

Since $\phi_j = \gamma_{j-1}u$ for $u \in S\left(\overline{\Omega}_-\right)$, we get

$$\sum_{\ell=a+1}^{\nu} W_{m\ell}(\gamma_{\ell-1}u) = w_{a+m} \quad \text{on } \partial\Omega, \ \ m = 1, ..., \mu/2.$$

These equations define $\mu/2$ extra boundary conditions for the boundary-value problem

$$\begin{cases} Qu = 0 & \text{in } \Omega_-, \\ D_{\mathbf{n}}^j u = D_{\mathbf{n}}^j f & \text{on } \partial\Omega, \ \ 0 \le j \le a-1. \end{cases}$$

Theorem 2 is proved. □

3. Isomorphism Property

We look for a solution $u \in H^{a}(\Omega_{-})$ to the boundary-value problem (19). Let us consider the following non-homogeneous boundary-value problem associated with (19):

$$\begin{cases} Qu = w & \text{in } \Omega_{-}\,, \\ \gamma_0 B_j u := \gamma_0 D_{\mathbf{n}}^{j-1} u = w_j & \text{on } \partial\Omega, \quad 1 \le j \le a\,, \\ \gamma_0 B_{a+m} u := \sum\limits_{\ell=a+1}^{\nu} W_{m\ell}\left(u_{\ell-1}\right) = w_{a+m} & \text{on } \partial\Omega, \quad 1 \le m \le \mu/2\,, \end{cases} \tag{23}$$

where $w,\ w_j\ \ (j = 1, ..., \nu/2)$ are arbitrary elements of the corresponding Sobolev spaces (see below Theorem 3 and Theorem 4).

For the formulation of the Shapiro-Lopatinskii condition we need some notation.

Let $\varepsilon > 0$ be a sufficiently small number. Denote by U (ε-conic neighborhood) the union of all balls $B(x, \varepsilon\langle x\rangle)$, centered at $x \in \partial\Omega$ with radius $\varepsilon\langle x\rangle$. Let $y = (y', y_n) = (y_1, ..., y_{n-1}, y_n)$ be normal coordinates in an ε-conic neighborhood U of $\partial\Omega$, that is, $\partial\Omega$ may be identified with $\{y_n = 0\}$, y_n is the normal coordinate, and the normal derivative $D_{\mathbf{n}}$ is D_{y_n} near $\partial\Omega$. Each differential operator on $\mathbb{R}^n$ with SG-symbol can be written in U as a differential operator with respect to $D_{y'}$ and D_{y_n} (see [1, p. 40]):

$$Q = \sum_{j=0}^{\nu} Q_j\left(y, D_{y'}\right) D_{y_n}^{j},$$

where $Q_j\left(y, D_{y'}\right)$ are differential operators with symbols belonging to $SG^{(\nu,0)}(\mathbb{R}^n)$. Let

$$q(y,\xi) = q\left(y, \xi', \xi_n\right) = \sum_{j=0}^{\nu} q_j\left(y, \xi'\right) \xi_n^{j}$$

be the symbol of Q, where ξ' and ξ_n are cotangent variables associated with y' and y_n.

Assumption 1. We assume that the operator Q is md-properly elliptic (cf. [1, Assumption 1, p. 40]), that is, for all large $|y| + |\xi'|$ the polynomial $q(y, \xi', z)$ with respect to the complex variable z has exactly $\nu/2$ zeros with positive imaginary parts $\tau_1(y', \xi'), ..., \tau_{\nu/2}(y', \xi')$.

We conclude from Assumption 1 that the polynomial $q(y, \xi', z)$ has no real zeros and it has exactly $\nu/2$ zeros with negative imaginary part for all large $|y| + |\xi'|$.

In particular, the Laplacian Δ in the space $\mathbb{R}^n$ $(n \geq 2)$ is elliptic in the usual sense but not md-property elliptic, while the operator $I - \Delta$ is md-properly elliptic.

Let

$$\chi\left(y', \xi'\right) := \left(1 + \sum_{i,j=1}^{n-1} \xi_i \left(g\left(y\right)^{-1}\right)_{ij} \xi_j\right)^{1/2},$$

where $g = (g_{ij})$ is a Riemannian metric on $\partial\Omega$. We denote

$$q^+\left(y', \xi', z\right) := \prod_{j=1}^{\nu/2} \left(z - \chi\left(y', \xi'\right)^{-1} \tau_j\left(y', \xi'\right)\right).$$

Consider the operators B_m $(m = 1, ..., \nu/2)$ from (23). Each of them is of the form

$$B_m = \sum_{j=0}^{\nu-1} B_{mj}\left(y', D_{y'}\right) D_{y_n}^j$$

in the normal coordinates $y = (y', y_n) = (y_1, ..., y_{n-1}, y_n)$ in an ε-conic neighborhood of $\partial\Omega$. Here $B_{mj}\left(y', D_{y'}\right)$ is a pseudodifferential operator of order $\rho_m - j$ $(\rho_m \in \mathbb{N})$ acting on $\partial\Omega$. Let $b_{mj}(y', \xi')$ denote the principal symbol of $B_{mj}\left(y', D_{y'}\right)$. The operators B_m in the boundary-value problem (23) are operators of this type. We set

$$b_m\left(y', \xi', z\right) := \sum_{j=0}^{\nu-1} b_{mj}\left(y', \xi'\right) \chi\left(y', \xi'\right)^{-\rho_m + j} z^j.$$

Define the following polynomials with respect to z:

$$r_m\left(y', \xi', z\right) = \sum_{j=1}^{\nu/2} r_{mj}\left(y', \xi'\right) z^{j-1}$$

as the residues of $b_m(y', \xi', z)$ modulo $q^+(y', \xi', z)$, i.e. we get $r_{mj}(y', \xi')$ representing $b_m(y', \xi', z)$ in the form

$$b_m\left(y', \xi', z\right) = q_m\left(z\right) q^+\left(y', \xi'\right) + \sum_{j=1}^{\nu/2} r_{mj}\left(y', \xi'\right) z^{j-1},$$

where $q_m(z)$ is a polynomial in z.

Assumption 2. (Shapiro-Lopatinskii Condition) The determinant $\det(r_{mj}(y',\xi'))$ is bounded and bounded away from zero, that is, there exist two positive constants c and C such that

$$0 < c \leq \det\left(r_{mj}\left(y',\xi'\right)\right) \leq C.$$

Remark. The following Theorem 3 has been proved in [1, Theorem 3.1] in the more general case of the SG-manifold. The latter includes the exterior of bounded domains which is a particular case of the SG-manifolds. This particular case was chosen for simplicity of the exposition. Moreover, the results in [1, Theorem 3.1], [12] have been obtained for operators acting in weighted Sobolev spaces. The usual Sobolev spaces in Theorem 3 are particular cases of the weighted Sobolev spaces with zero order of the weight.

Theorem 3. (cf. [1, Theorem 3.1], [12]) *If the differential operator Q of even order ν satisfies Assumptions 1 and 2, that is Q is md-properly elliptic and the Shapiro-Lopatinskii condition holds for the operator $\left(Q,\gamma_0 B_1,...,\gamma_0 B_{\nu/2}\right)$, then the mapping*

$$\left(Q,\gamma_0 B_1,...,\gamma_0 B_{\nu/2}\right) : H^s\left(\Omega_-\right) \rightarrow H^{s-\nu}\left(\Omega_-\right) \times \prod_{j=1}^{\nu/2} H^{s-\rho_j-1/2}\left(\partial\Omega\right),\ s \geq \nu,$$

is a Fredholm operator.

Assumption 3. The Fredholm operator $\left(Q,\gamma_0 B_1,...,\gamma_0 B_{\nu/2}\right)$ has the trivial kernel and cokernel.

For example, if the kernel $R(x,y)$ has the property $(Rh,h) \geq c||h||^2_{H_0^{-a}}$ for all $h \in H_0^{-a}$, where c =const> 0 does not depend on h, then the operator in Assumption 3 is invertible (see [7]).

Corollary. *Under the assumptions of Theorem 3 and in addition under Assumption 3, for any $s \in \mathbb{R}$, there exists a bounded (Poisson) operator*

$$K : \prod_{j=0}^{m-1} H^{s+2m-j-1/2}\left(\partial\Omega\right) \rightarrow H^{s+2m}\left(\Omega_-\right), \tag{24}$$

which gives a unique solution $u = K\underline{\chi}$ to the boundary-value problem

$$Qu = 0 \text{ in } \Omega_-, \quad \gamma_0 B_1 u = \chi_1, ..., \gamma_0 B_{\nu/2} u = \chi_{\nu/2} \tag{25}$$

with

$$\underline{\chi} = \left(\chi_1,...,\chi_{\nu/2}\right) \in \prod_{j=0}^{\nu/2-1} H^{s+\nu-j-1/2}\left(\partial\Omega\right).$$

More precisely, the operator $u = K\underline{\chi}$ solves the problem with $s < 0$ in the sense that $u = K\underline{\chi}$ is the limit in the space $H^{s+\nu}(\Omega_-)$ of a sequence u_n in $H^{\nu}(\Omega_-)$ with

$$Qu_n = 0, \ \ \gamma_0 B_j u_n = \chi_{j,n} \ \ (j = 1, ..., m)\,,$$
$$\lim_{n\to\infty} \underline{\chi}_n \to \underline{\chi} \ \ in \prod_{j=0}^{\nu/2-1} H^{s+\nu-j-1/2}(\partial\Omega)\,.$$

Proof. The statement of Corollary is an immediate consequence of Theorem 3 due to the fact that the solution operator to the boundary-value problem (25) with homogeneous equation $Qu = 0$ in Ω_- is a Poisson operator. The latter acts in the full scale of Sobolev spaces [12], that is, (24) holds for all $s \in \mathbb{R}$. □

Theorem 4. *Under the assumptions of Theorem 3 and in addition under Assumption 3, the mapping R_Ω, defined in Introduction, is an isomorphism: $H_0^{-a}(\Omega) \to H^a(\Omega)$.*

Proof. Let us consider the operator $\left(Q, \gamma_0 B_1, ..., \gamma_0 B_{\nu/2}\right)$ generated by the boundary value problem (23). Taking into account that

$$\begin{array}{ll} \rho_j = \text{order}\, B_j = j - 1 & \text{for } j = 1, ..., a, \\ \rho_j = \text{order}\, B_j = j - \mu + \nu/2 - 2 & \text{for } j = a+1, ..., \nu/2, \end{array}$$

one concludes, by Theorem 3, that the mapping

$$\left(u, \underline{\phi}\right) \mapsto \left(Q\left(u, \underline{\phi}\right), \gamma_0 B_1\left(u, \underline{\phi}\right), ..., \gamma_0 B_{\nu/2}\left(u, \underline{\phi}\right)\right) = \left(w, w_1, ..., w_{\nu/2}\right)$$

is a Fredholm operator. It maps the space $H^s(\Omega_-)$ to the space

$$H^{s-\nu}(\Omega_-) \times \prod_{j=1}^{a} H^{s-j+1/2}(\partial\Omega) \times \prod_{j=a+1}^{\nu/2} H^{s-j+\mu-\nu/2+3/2}(\partial\Omega) \quad (s \geq \nu)\,.$$

Assumption 3 implies that this mapping is an isomorphism. By Corollary, the operator K, solving the boundary-value problem

$$Qu = 0, \ \ \gamma_0 B_j u = \chi_j \quad (j = 1, ..., m)\,,$$

is a Poisson operator

$$K : \prod_{j=0}^{m-1} H^{s+2m-j-1/2}(\partial\Omega) \to H^{s+2m}(\Omega_-) \quad (s \in \mathbb{R})\,.$$

Choosing $s = a$ and using Theorem 2, we conclude, that for any $f \in H^a(\Omega)$ the function u is a unique solution to the boundary-value problem (19). Therefore, again by Theorem 2, the operator R_Ω is an isomorphism of the space $H_0^{-a}(\Omega)$ onto $H^a(\Omega)$. Theorem 4 is proved. □

Example. Let $P = I$ be the identity operator (its order $\mu = 0$) and $Q = I - \Delta$ $(\nu = \operatorname{ord} Q = 2)$. Then, by Theorem 4, the corresponding operator R_Ω is an isomorphism: $H_0^{-1}(\Omega) \to H^1(\Omega)$.

Under the assumptions of Theorem 4 there exists a unique solution to the integral equation (3). Let us find this solution.

Examples of analytical formulas for the solution to the integral equation (3) can be found in [7]. The analytical formulas for the solution in the cases when the corresponding boundary-value problems are solvable analytically, can be obtained only for domains Ω of special shape, for example, when Ω is a ball, and for special operators Q and P, for example, for operators with constant coefficients.

We give such a formula for the solution of equation (3) assuming $P = I$ and $Q = -\Delta + a^2 I$. Consider the equation

$$R_\Omega h(x) = \int_\Omega \frac{\exp(-a|x-y|)}{4\pi|x-y|} h(y)dy = f(x), \quad x \in \overline{\Omega} \subset \mathbb{R}^3, \quad a > 0, \tag{26}$$

with the kernel $R(x,y) := \exp(-a|x-y|)/(4\pi|x-y|)$, $P = I$, and $Q = -\Delta + a^2 I$. By Theorem 1, one obtains a unique solution to the equation (26) in $H_0^{-1}(\Omega)$:

$$h(x) = (-\Delta + a^2)f + \left(\frac{\partial f}{\partial \mathbf{n}} - \frac{\partial u}{\partial \mathbf{n}}\right)\delta_{\partial\Omega}, \tag{27}$$

where u is a unique solution to the exterior Dirichlet boundary-value problem

$$(-\Delta + a^2)u = 0 \quad \text{in} \quad \Omega_-, \quad u|_{\partial\Omega} = f|_{\partial\Omega}. \tag{28}$$

For any $\varphi \in C_0^\infty(\mathbb{R}^n)$ one has:

$$\begin{aligned}
&\big((-\Delta + a^2)Rh, \varphi\big) = \big(Rh, (-\Delta + a^2)\varphi\big) \\
&= \int_\Omega f(-\Delta + a^2)\overline{\varphi} dx + \int_{\Omega_-} u(-\Delta + a^2)\overline{\varphi} dx \\
&= \int_\Omega (-\Delta + a^2) f\overline{\varphi} dx + \int_{\Omega_-} (-\Delta + a^2) u\overline{\varphi} dx \\
&- \int_{\partial\Omega} (f\partial_{\mathbf{n}}\overline{\varphi} - \partial_{\mathbf{n}} f\overline{\varphi})\, ds + \int_{\partial\Omega} (u\partial_{\mathbf{n}}\overline{\varphi} - \partial_{\mathbf{n}} u\overline{\varphi})\, ds \\
&= \int_\Omega (-\Delta + a^2) f\overline{\varphi} dx + \int_{\partial\Omega} (\partial_{\mathbf{n}} f - \partial_{\mathbf{n}} u)\, ds,
\end{aligned}$$

where the condition $u = f$ on $\partial\Omega$ was used. Thus, we have checked that formula (27) gives the unique in $H_0^{-1}(\Omega)$ solution to equation (26). This solution has minimal order of singularity.

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Appendix

We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{C}$ the set of complex numbers. Let $\mathbb{Z} := \{0, \pm 1, \pm 2, ...\}$, $\mathbb{N} := \{0, 1, ...\}$, $\mathbb{N}_+ := \{1, 2, ...\}$, $\mathbb{R}^n := \{x = (x_1, ..., x_n) : x_i \in \mathbb{R},\ i = 1, ..., n\}$.

Let α be a multi-index, $\alpha := (\alpha_1, \ldots, \alpha_n),\ \alpha_j \in \mathbb{N},\ |\alpha| := \alpha_1 + \ldots + \alpha_n$, $i := \sqrt{-1}$; $D_j := i^{-1}\partial/\partial x_j$; $D^\alpha := D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$.

Let $C^\infty\left(\overline{\Omega}\right)$ be the space of infinitely differentiable up to the boundary functions in $\overline{\Omega}$. Near $\partial\Omega$ there is defined a normal vector field $\mathbf{n}(x) = (n_1(x), ..., n_n(x))$, is defined in a neighborhood of the boundary $\partial\Omega$ as follows: for $x_0 \in \partial\Omega$, $\mathbf{n}(x_0)$ is the unit normal to $\partial\Omega$, pointing into the exterior of Ω. We set

$$\mathbf{n}(x) := \mathbf{n}(x_0) \text{ for } x \text{ of the form } x = x_0 + s\mathbf{n}(x_0) =: \zeta(x_0, s),$$

where $x_0 \in \partial\Omega$, $s \in (-\delta, \delta)$. Here $\delta > 0$ is taken so small that the representation of x in terms of $x_0 \in \partial\Omega$ and $s \in (-\delta, \delta)$ is unique and smooth, that is, ζ is bijective and C^{∞} with C^{∞} inverse, from $\partial\Omega \times (-\delta, \delta)$ to the set $\zeta\left(\partial\Omega \times (-\delta, \delta)\right) \subset \mathbb{R}^n$.

We call differential operators *tangential* when, for $x \in \zeta\left(\partial\Omega \times (-\delta, \delta)\right)$, they are either of the form

$$Af = \sum_{j=1}^{n} a_j(x) \frac{\partial f}{\partial x_j}(x) + a_0(x) f \quad \text{with} \quad \sum_{j=1}^{n} a_j(x) n_j(x) = 0,$$

or they are products of such operators. The derivative along $\mathbf{n}$ is denoted $\partial_{\mathbf{n}}$:

$$\partial_{\mathbf{n}} f := \sum_{j=1}^{n} n_j(x) \frac{\partial f}{\partial x_j}(x)$$

for $x \in \zeta\left(\partial\Omega \times (-\delta, \delta)\right)$. Let $D_{\mathbf{n}} := i^{-1}\partial_{\mathbf{n}}$.

Let $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ denote the exterior of the domain Ω, $r_{\partial\Omega}$, r_{Ω} be respectively the restriction operators to $\partial\Omega$, $\Omega : r_{\partial\Omega} f := f|_{\partial\Omega}$, $r_{\Omega} f := f|_{\Omega}$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the space of rapidly decreasing functions, that is the space of all $u \in C^{\infty}(\mathbb{R}^n)$ such that

$$\sup_{|\alpha| \le k} \sup_{x \in \mathbb{R}^n} \left| \left(1 + |x|^2\right)^m D^{\alpha} u(x) \right| < \infty \quad \text{for all } k, m \in \mathbb{N}.$$

Let $\mathcal{S}\left(\overline{\Omega}_-\right)$ be the space of restrictions of the elements $u \in \mathcal{S}(\mathbb{R}^n)$ to $\overline{\Omega}_-$ (this space is equipped with the factor topology).

Let $u \in C^{\infty}\left(\overline{\Omega}\right)$ and $v \in \mathcal{S}\left(\overline{\Omega}_-\right)$, then we set $\gamma_k u := r_{\partial\Omega} D_{\mathbf{n}}^k u = \left(D_{\mathbf{n}}^k u\right)\big|_{\partial\Omega}$, $\gamma_k v := r_{\partial\Omega} D_{\mathbf{n}}^k v = \left(D_{\mathbf{n}}^k v\right)\big|_{\partial\Omega}$.

The Sobolev Spaces

Let $H^s(\mathbb{R}^n)$ $(s \in \mathbb{R})$ be the usual Sobolev space:

$$H^s(\mathbb{R}^n) := \left\{ f \in \mathcal{S}' \mid \mathcal{F}^{-1}\left(1 + |\xi|^2\right)^{s/2} \mathcal{F} f \in L_2(\mathbb{R}^n) \right\},$$
$$\|f\|_{H^s(\mathbb{R}^n)} := \left\| \mathcal{F}^{-1}\left(1 + |\xi|^2\right)^{s/2} \mathcal{F} f \right\|_{L_2(\mathbb{R}^n)},$$

where $\mathcal{F}$ denotes the Fourier transform $f \mapsto \mathcal{F}_{x\to\xi} f(x) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x) dx$, $\mathcal{F}^{-1}$ its inverse and $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ denote the space of tempered distributions which is dual to the space $\mathcal{S}(\mathbb{R}^n)$.

Let $H^s(\Omega)$ and $H^s(\Omega_-)$ $(0 \le s \in \mathbb{R})$ be respectively the spaces of restrictions of elements of $H^s(\mathbb{R}^n)$ to Ω and Ω_-. The norms in the spaces $H^s(\Omega)$ and $H^s(\Omega_-)$ are defined by the relations

$$\|f\|_{H^s(\Omega)} := \inf \|g\|_{H^s(\mathbb{R}^n)} \quad (s \ge 0),$$

$$\|f\|_{H^s(\Omega_-)} := \inf \|g\|_{H^s(\mathbb{R}^n)} \quad (s \ge 0),$$

where infimum is taken over all elements $g \in H^s(\mathbb{R}^n)$ which are equal to f in Ω respectively in Ω_-.

By $H_0^s(\Omega)$ $(s \in \mathbb{R})$ and $H_0^s(\Omega_-)$, we denote the closed subspaces of the space $H^s(\mathbb{R}^n)$ which consist of the elements with supports respectively in $\overline{\Omega}$ or in $\overline{\Omega}_-$, that is,

$$H_0^s(\Omega) := \left\{f \in H^s(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\Omega}\right\} \subset H^s(\mathbb{R}^n), \ s \in \mathbb{R},$$
$$H_0^s(\Omega_-) := \left\{f \in H^s(\mathbb{R}^n) : \operatorname{supp} f \subseteq \overline{\Omega}_-\right\} \subset H^s(\mathbb{R}^n), \ s \in \mathbb{R}.$$

We define the spaces

$$\mathcal{H}^s(\Omega) := \begin{cases} H^s(\Omega), & s > 0, \\ H_0^s(\Omega), & s \le 0, \end{cases}$$

$$\mathcal{H}^s(\Omega_-) := \begin{cases} H^s(\Omega_-), & s > 0, \\ H_0^s(\Omega_-), & s \le 0. \end{cases}$$

For $s \ne k + 1/2$ $(k = 0, 1, ..., \ell - 1)$, we define the spaces $\mathfrak{H}^{s,\ell}(\Omega)$ and $\mathfrak{H}^{s,\ell}(\Omega_-)$, respectively, as the sets of all

$$\left(u, \underline{\phi}\right) = (u, \phi_1, ..., \phi_\ell) \text{ and } \left(v, \underline{\psi}\right) = (v, \psi_1, ..., \psi_\ell),$$

where $u \in \mathcal{H}^s(\Omega)$, $v \in \mathcal{H}^s(\Omega_-)$, $\underline{\phi} = (\phi_1, ..., \phi_\ell)$ and $\underline{\psi} = (\psi_1, ..., \psi_\ell)$ are vectors in $\prod_{j=1}^{\ell} H^{s-j+1/2}(\partial\Omega)$ satisfying the condition

$$\phi_j = D_{\mathbf{n}}^{j-1} u\big|_{\partial\Omega}, \quad \psi_j = D_{\mathbf{n}}^{j-1} v\big|_{\partial\Omega} \qquad \text{for } j < \min(s, \ell).$$

The norms in $\mathfrak{H}^{s,\ell}(\Omega)$ and $\mathfrak{H}^{s,\ell}(\Omega_-)$ can be defined as

$$\left\|\left(u, \underline{\phi}\right)\right\|^2_{\mathfrak{H}^{s,\ell}(\Omega)} = \|u\|^2_{\mathcal{H}^s(\Omega)} + \sum_{j=1}^{\ell} \|\phi_j\|^2_{H^{s-j+1/2}(\partial\Omega)},$$

$$\left\| \left(v, \underline{\psi}\right) \right\|^2_{\mathfrak{H}^{s,\ell}(\Omega)} = \|v\|^2_{\mathcal{H}^s(\Omega)} + \sum_{j=0}^{\ell} \|\psi_j\|^2_{H^{s-j+1/2}(\partial\Omega)}.$$

Since only the components ϕ_j and ψ_j with index $j < s$ can be chosen independently of u, we can identify $\mathfrak{H}^{s,\ell}(\Omega)$ and $\mathfrak{H}^{s,\ell}(\Omega_-)$ with the following spaces.

For $s \neq k + 1/2 \ (k = 0, 1, ..., \ell - 1)$,

$$\mathfrak{H}^{s,\ell}(\Omega) = \begin{cases} \mathcal{H}^s(\Omega), \ \ell = 0, \\ \mathcal{H}^s(\Omega), \ 1 \leq \ell < s + 1/2, \\ \mathcal{H}^s(\Omega) \times \prod\limits_{j=[s+1/2]+1}^{\ell} H^{s-j+1/2}(\partial\Omega), \ 0 < \left[s + \frac{1}{2}\right] < \ell, \\ \mathcal{H}^s(\Omega) \times \prod\limits_{j=1}^{\ell} H^{s-j+1/2}(\partial\Omega), \ s < \frac{1}{2}, \end{cases}$$

$$\mathfrak{H}^{s,\ell}(\Omega_-) = \begin{cases} \mathcal{H}^s(\Omega_-), \ \ell = 0, \\ \mathcal{H}^s(\Omega_-), \ 1 \leq \ell < s + 1/2, \\ \mathcal{H}^s(\Omega_-) \times \prod\limits_{j=[s+1/2]+1}^{\ell} H^{s-j+1/2}(\partial\Omega), \ 0 < \left[s + \frac{1}{2}\right] < \ell, \\ \mathcal{H}^s(\Omega_-) \times \prod\limits_{j=1}^{\ell} H^{s-j+1/2}(\partial\Omega), \ s < \frac{1}{2}. \end{cases}$$

Finally, for $s = k + 1/2 \ (k = 0, 1, ..., \ell - 1)$, we define the spaces $\mathfrak{H}^{s,\ell}(\Omega)$, $\mathfrak{H}^{s,\ell}(\Omega_-)$ by the method of complex interpolation.

Let us note that for $s \neq k + 1/2 \ (k = 0, 1, ..., \ell - 1)$, the spaces $\mathfrak{H}^{s,\ell}(\Omega)$, $\mathfrak{H}^{s,\ell}(\Omega_-)$ are completion of $C^\infty\left(\overline{\Omega}\right)$, $\mathcal{S}\left(\overline{\Omega}_-\right)$, respectively, in the norms

$$\|(u, \gamma_0 u, ..., \gamma_{\ell-1} u)\|^2_{\mathfrak{H}^{s,\ell}(\Omega)} = \|u\|^2_{\mathcal{H}^s(\Omega)} + \sum_{j=0}^{\ell-1} \|\gamma_j u\|^2_{H^{s-j-1/2}(\partial\Omega)},$$

$$\|v, \gamma_0 v, ..., \gamma_{\ell-1} v\|^2_{\mathfrak{H}^{s,\ell}(\Omega_-)} = \|v\|^2_{\mathcal{H}^s(\Omega_-)} + \sum_{j=0}^{\ell-1} \|\gamma_j v\|^2_{H^{s-j-1/2}(\partial\Omega)}.$$