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# DEVELOPABLE SURFACES WHICH ARE SURFACES OF REVOLUTION

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**Abstract:** Let $c$ be a curve in the three dimensional Euclidean space $\Re^3$. A surface of revolution generated by revolving this curve around a fixed axis. The given curve is a profile curve while the axis is the axis of revolution. The aim of the present paper is to determine the form of the revolution surface which is also developable surface. We give some applications of this theory in the industry.

**AMS Subject Classification:** 53A04, 53A05, 53A99
**Key Words:** surfaces of revolution, developable surfaces, ruled surface, fundamental magnitudes

## 1. Introduction

A developable surface is a surface, which can be unfolded (developed) into a plane without stretching or tearing. It is well known in elementary differential geometry that under the assumption of sufficient differentiability, a developable surface is a plane, conical surface, cylindrical surface or tangent surface of a curve or a composition of these types. Thus a developable surface is a ruled surface, where all points of the same generator line share a common tangent plane. The rulings are principal curvature lines with vanishing normal curvature and the Gaussian curvature vanishes at all surface points. Therefore developable surfaces are also called single-curved surfaces, as opposed to double-curved surfaces.

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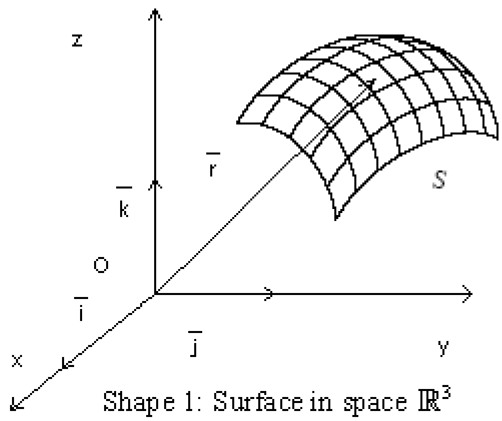



Shape 1: Surface in space $\mathbb{R}^3$

We can ask which the form of this surface is in order to be also surface of revolution.

The purpose of this paper is to determine the form of the tangent developable surface of a curve so as to be also revolution surface. The paper is organized as follows.

The Section 1 is the introduction.

Some basic elements of the theory of surfaces, and developable surfaces are contained in the Section 2.

Determination of developable Surfaces which are also surfaces of revolution are given in the Section 3.

The last section contains some applications in the industry.

## 2. Basic Elements of the Theory of Surfaces and Developable Surfaces

Let $S$ be a surface in the three dimensional Euclidean space $\Re^3$ which is referred to the orthogonal coordinate system $Oxyz$. The surface $S$ can be represented with respect to $Oxyz$ as follows (Shape 1):

$$\overrightarrow{r} = \overrightarrow{r}(u,v) = x(u,v)\overrightarrow{i} + y(u,v)\overrightarrow{j} + z(u,v)\overrightarrow{k}. \tag{1}$$

The fundamental magnitudes of the first order of the surface $S$ are given by

$$E = \left(\frac{\partial \overrightarrow{r}}{\partial u}\right)^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, \tag{2}$$

$$F = \frac{\partial \overrightarrow{r}}{\partial u} \cdot \frac{\partial \overrightarrow{r}}{\partial v} = \frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \cdot \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \cdot \frac{\partial z}{\partial v}, \tag{3}$$

$$G = \left(\frac{\partial \overrightarrow{r}}{\partial v}\right)^2 = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2. \tag{4}$$

The unit vector field $\overrightarrow{\ell_o}$ to the surface $S$ has the form

$$\overrightarrow{\ell_o} = \frac{\overrightarrow{\ell}}{\left|\overrightarrow{\ell}\right|} = \frac{\frac{\partial \overrightarrow{r}}{\partial u} \times \frac{\partial \overrightarrow{r}}{\partial v}}{\left|\frac{\partial \overrightarrow{r}}{\partial u} \times \frac{\partial \overrightarrow{r}}{\partial v}\right|} = \frac{1}{\sqrt{E\,G - F^2}} \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}. \tag{5}$$

The fundamental magnitudes of the second order of the surface $S$ are given by

$$L = \frac{d^2 \overrightarrow{r}}{d\,u^2} \cdot \overrightarrow{\ell_o} = \frac{1}{\sqrt{E\,G - F^2}} \begin{vmatrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 y}{\partial u^2} & \frac{\partial^2 z}{\partial u^2} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}, \tag{6}$$

$$M = \frac{d^2 \overrightarrow{r}}{du\,dv} \cdot \overrightarrow{\ell_o} = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix} \frac{\partial^2 x}{\partial u \partial v} & \frac{\partial^2 y}{\partial u \partial v} & \frac{\partial^2 z}{\partial u \partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}, \tag{7}$$

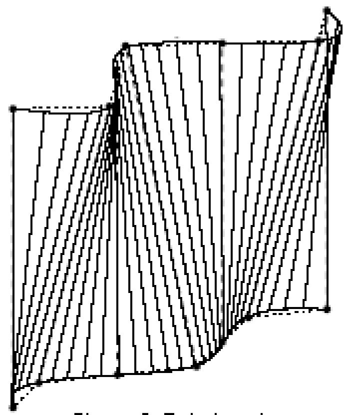
$$N = \frac{d^2 \overrightarrow{r}}{d\,v^2} \cdot \overrightarrow{\ell_o} = \frac{1}{\sqrt{E\,G - F^2}} \begin{vmatrix} \frac{\partial^2 x}{\partial v^2} & \frac{\partial^2 y}{\partial v^2} & \frac{\partial^2 z}{\partial v^2} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}. \tag{8}$$

A developable surface is a ruled surface with no curvature. A ruled surface (Shape 2) is a tangent developable of a curve. A tangent developable is a flat surface. So, the plane is an excellent example. Therefore, the surface $S$ is developable, which means is isometric onto a plane, if and only if the Gaussian curvature is zero (Shape 3). The Gaussian curvature is given by

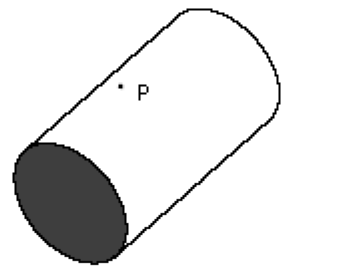
$$K = \frac{L\,N - M^2}{E\,G - F^2}, \tag{9}$$

where $L$, $M$, $N$ are the fundamental magnitudes of the second order and $E$, $F$, $G$ are the fundamental magnitudes of the first order of the surface $S$, that means

$$K = 0 \ \text{ or } \ LN - M^2 = 0. \tag{10}$$



Shape 2: Ruled surface




Shape 3: Surface with zero Gauss curvature

### 3. Determination of Developable Surfaces which are Also Surfaces of Revolution

A surface of revolution is a surface generated by rotating a planar curve about an axis of revolution, lying in the plane and rigidly connected with the curve, through a complete revolution. We shall choose the axis of revolution to serve as the $z$-axis of the coordinate system. The resulting surface therefore always has azimuthally symmetry. Examples of surfaces of revolution include the cylinder (Shape 3) excluding the ends, and pseudosphere (Shape 4), which is the surface of revolution of a tractrix (Shape 5).

The standard parameterization of a surface of revolution $M$ can be represented by the equations (Gray 1993)

$$(x\,(u,v)\,,\ y\,(u,v)\,,\ z\,(u,v)) = (g\,(u)\,,\ h\,(u)\cos v,\ h\,(u)\sin v)\,, \tag{11}$$

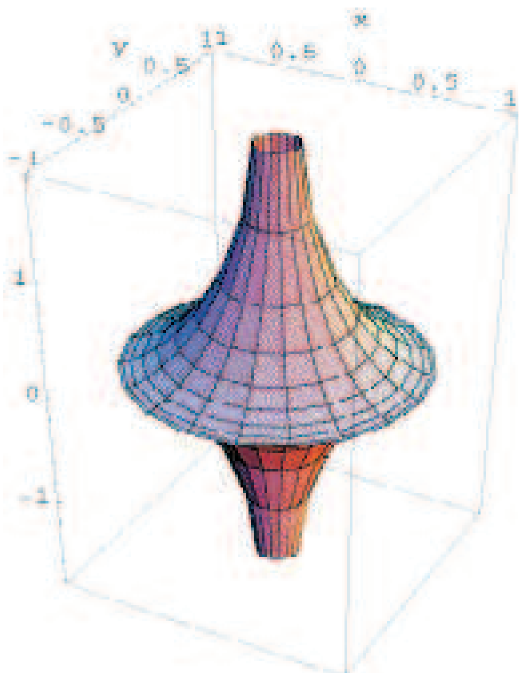
where $g\,(u)\,,\ h\,(u)$ are arbitrary functions of $u$. These functions have special meaning, that is $h\,(u)$ is the radius of the parallel of $M$ at distance along the axis of revolution.

The fundamental magnitudes of first order of this surface are given by

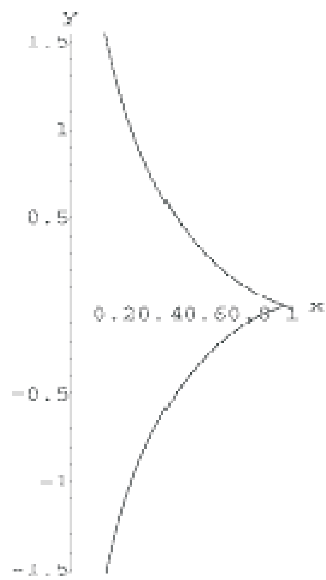
$$E = g'^{\,2} + h'^{\,2}, \quad F = 0, \quad G = h^2\,. \tag{12}$$

Wherever $h$ and $g'^{\,2} + h'^{\,2}$ are nonzero, then the surface is regular and the fundamental magnitudes of second order of this surface are given by

$$L = \frac{-g'\,h'' + g''\,h'}{\sqrt{g'^{\,2} + h'^{\,2}}}, \quad M = 0, \quad N = \frac{g'\,h}{\sqrt{g'^{\,2} + h'^{\,2}}}\,. \tag{13}$$




Shape 4: pseudosphere




Shape 5: tractrix

As we mentioned before, in order this surface to be developable the Gaussian curvature has to be zero, so, we must have

$$L\,N - M^2 = 0 \;\text{ or }\; L\,N = 0\,, \tag{14}$$

which implies

$$N = 0 \quad \text{or} \quad L = 0\,. \tag{15}$$

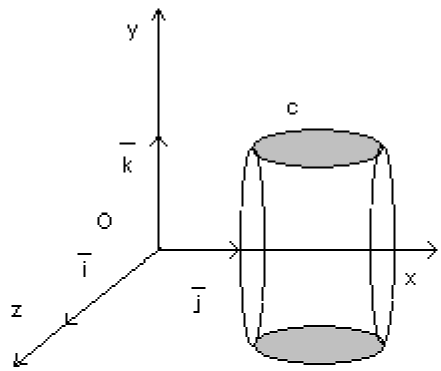
The first of (15) implies

$$g'\,h = 0 \Longrightarrow g' = 0 \Longrightarrow g = c\,. \tag{16}$$

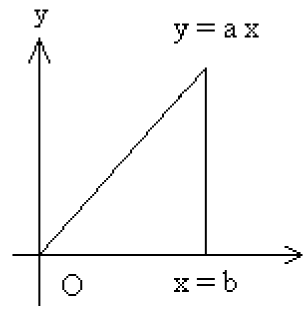
In this case $h\,(u)$ is an arbitrary function of $u$. Therefore

$$L = M = N = 0\,. \tag{17}$$

Under these conditions the surface is plane. The plane is of course an excellent example of a developable surface. But the problem does not have meaning.




Shape 6: Revolution of a curve




Shape 7: Revolution of a line

On the other way (15) we have

$$-g'\,h''+g''\,h'=0 \quad \text{or} \quad g'\,h''=g''\,h'\,, \tag{18}$$

which implies

$$\frac{h'}{h''}=\frac{g'}{g''}\Longrightarrow \ln h'=\ln c\,g' \quad \text{or} \quad h'=c\,g'\,, \tag{19}$$

and finally

$$h=c\,g+c_1\,. \tag{20}$$

Therefore we have the following proposition.

**Proposition 1.** *The surfaces of revolution which are developable have the form*

$$\left(x=g\left(u\right),\ y=\left[c\,g\left(u\right)+c_1\right]\cos v,\ \left[c\,g\left(u\right)+c_1\right]\sin v\right)\,. \tag{21}$$

Now, let us consider the following problem. Which is the developable surface of the form (21) that comes through the curves

$$z=0 \quad \text{and} \quad f\left(x,y\right)=0\,. \tag{22}$$

Let $c$ be a curve on the plane $Oxy$ defined by the equation

$$c\longrightarrow f\left(x,y\right)=0\,. \tag{23}$$

Each circle (Shape 6) is defined by

$$x=a, \quad x^2+y^2+z^2=R^2\,. \tag{24}$$

The curve in the $Oxyz$ is defined by

$$f(x,y)=0, \quad z=0\,. \tag{25}$$

From (24) and (25) we obtain

$$y=\sqrt{R^2-a^2}\,. \tag{26}$$

The first of (25) by means of (24) amd (26) becomes

$$f\left(a,\sqrt{R^2-a^2}\right)=0\,. \tag{27}$$

Therefore, the required equation becomes

$$f\left(x,\sqrt{y^2+z^2}\right)=0\,. \tag{28}$$

If we put

$$y=h(u)\sin v, \quad z=h(u)\cos v\,, \tag{29}$$

then (28) becomes

$$f(x,h(u))=0\,. \tag{30}$$

Now, if we put

$$x=g(u)\;. \tag{31}$$

**Remark 2.** The parametric equations of the requested surface have the form

$$(x=g(u)\,,\ y=h(u)\sin v,\ z=h(u)\ ,\cos v)\ , \tag{32}$$

where $g(u)$ is determined by (31). Therefore, the de1velopable surface is also a surface of revolution.

As an example we consider (Shape 7)

$$f(x,y)=y-a\,x=0\,,$$

then by means of (28) the equation becomes

$$\sqrt{y^2+z^2}-a\,x=0\,,$$

which by means of (29) becomes

$$h(u)-a\,x=0 \Longleftrightarrow x=\frac{1}{a}h(u)=g(u)\;. \tag{33}$$

**Remark 3.** By means of (33) the parametric equations of the surface have the form

$$(x=g(u)\,,\ y=h(u)\sin v,\ z=h(u)\cos v)\;. \tag{34}$$

## 4. Applications in the Technology

It is known that in many branches of constructions we need pieces of surfaces, which pass through a closed curve and the same time these surfaces are coming from a revolution of a curve.

From these requirements we conclude that we need surfaces that are revolution surfaces and the same time developable.

Therefore, the above proposition solves this problem, which means we have determined surfaces that are developable and surfaces of revolution.

Developable surfaces possess a wide range of applications, for example in sheet-metal and plate-metal based industries [9], in the design of boat hulls and shoes and clothing, in windshield designs and as binder (blank holder) surfaces for sheet metal deforming process and duct work.

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