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A MIXED HILBERT-TYPE INEQUALITY WITH A BEST CONSTANT FACTOR

Bicheng Yang

Department of Mathematics
Guangdong Institute of Education
Guangzhou, Guangdong, 510303, P.R. CHINA
e-mail: bcyang@pub.guangzhou.gd.cn

Abstract: This paper deals with a mixed Hilbert type inequality by introducing a parameter and the beta function. As applications, the equivalent inequalities and some particular results are considered. All the theorems provide some new estimates on this type of inequalities.

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1. Introduction

If $a_n, b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then the famous Hilbert's inequality is given by

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi \left\{\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\right\}^{1/2}, \tag{1}$$

where the constant factor π is the best possible (see [3]). The equivalent form is

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^2 < \pi^2 \sum_{n=1}^{\infty} a_n^2, \tag{2}$$

where the constant factor π^2 is the best possible . Inequalities (1) and (2) are important in analysis and its applications (see [5]). In 1998, Yang [1] first introduced a parameter λ and the beta function for giving an extension of the integral analogue of (1). In 2001, by the same way and using Euler-Maclaurin's

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formula, Yang [2] gave an extension of (1) and (2) as:

If $0<\lambda\le 4, 0<\sum_{n=1}^{\infty} n^{1-\lambda}a_n^2<\infty$ and $0<\sum_{n=1}^{\infty} n^{1-\lambda}b_n^2<\infty$, then one has two equivalent inequalities as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}}<B(\frac{\lambda}{2},\frac{\lambda}{2})\left\{\sum_{n=1}^{\infty} n^{1-\lambda}a_n^2\sum_{n=1}^{\infty} n^{1-\lambda}b_n^2\right\}^{\frac{1}{2}}; \tag{3}$$

$$\sum_{n=1}^{\infty} n^{\lambda-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n)^{\lambda}}\right]^2<\left[B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^2\sum_{n=1}^{\infty} n^{1-\lambda}a_n^2, \tag{4}$$

where the constant factors $B(\frac{\lambda}{2},\frac{\lambda}{2})$ and $[B(\frac{\lambda}{2},\frac{\lambda}{2})]^2$ are the best possible. When $\lambda=1$, inequality (3) reduces to (1), and (4) reduces to (2). In 2003, Yang et al [6] provided and extensive account of the above results.

The main objective of this paper is to build a new mixed Hilbert's type inequality by introducing a parameter λ and the beta function, which is related to the double mixed form as

$$\sum_{n=n_0}^{\infty}\int_{n_0-1}^{\infty}\frac{a_n f(x)}{(1+u(n)u(x))^{\lambda}}dx \quad (\lambda>0)\,.$$

As applications, two equivalent inequalities and some particular results are considered. All the theorems provide some new estimates on this type of inequalities.

2. Main Results

First, we need the formula of the beta function as (cf. Wang et al [7]):

$$B(u,v)=\int_0^{\infty}\frac{1}{(1+t)^{u+v}}t^{u-1}dt=B(v,u) \quad (u,v>0). \tag{5}$$

Lemma 1. *If $u(t)$ is a differentiable strict increasing function in $(n_0-1,\infty)(n_0\in N)$, such that $u((n_0-1)+)=0$ and $u(\infty)=\infty$, and $\lambda>0, 0<\varepsilon<\lambda, (u(t))^{\frac{\lambda-2}{2}}u'(t)$ $(t\in(n_0-1,\infty))$ is decreasing; $c\in(n_0-1,\infty)$ and $n_1\in N$ satisfy $u(c)=1$ and $n_1\ge c+1$, setting I as*

$$I:=\int_{n_0-1}^{c}(u(x))^{\frac{\lambda+\varepsilon}{2}-1}u'(x)\sum_{n=n_1}^{\infty}\frac{(u(n))^{\frac{\lambda-\varepsilon}{2}-1}u'(n)}{(1+u(n)u(x))^{\lambda}}dx, \tag{6}$$

then one has

$$I > \frac{1}{\varepsilon(u(n_1))^{\varepsilon}} B\left(\frac{\lambda+\varepsilon}{2}, \frac{\lambda-\varepsilon}{2}\right) - \left(\frac{2}{\lambda+\varepsilon}\right)^2. \tag{7}$$

Proof. Since $\lambda > 0$ and $(u(y))^{\frac{\lambda-\varepsilon}{2}-1}u'(y) = (u(y))^{\frac{\lambda-2}{2}}u'(y)\frac{1}{(u(y))^{\varepsilon/2}}$ ($y \in (n_0-1,\infty)$) is decreasing, one has

$$I > \int_{n_0-1}^{c} (u(x))^{\frac{\lambda+\varepsilon}{2}-1}u'(x) \int_{n_1}^{\infty} \frac{(u(y))^{\frac{\lambda-\varepsilon}{2}-1}u'(y)}{(1+u(y)u(x))^{\lambda}} dydx$$
$$= \int_{n_1}^{\infty} (u(y))^{\frac{\lambda-\varepsilon}{2}-1}u'(y) \left[\int_{n_0-1}^{c} \frac{(u(x))^{\frac{\lambda+\varepsilon}{2}-1}u'(x)}{(1+u(y)u(x))^{\lambda}} dx\right] dy.$$

Setting $t = \frac{1}{u(y)u(x)}$ in the above second integral, one has

$$I > \int_{n_1}^{\infty} (u(y))^{-1-\varepsilon}u'(y) \left[\int_{\frac{1}{u(y)}}^{\infty} \frac{t^{\frac{\lambda+\varepsilon}{2}-1}}{(1+t)^{\lambda}} dt\right] dy$$
$$= \int_{n_1}^{\infty} (u(y))^{-1-\varepsilon}u'(y) \left[\int_{0}^{\infty} \frac{t^{\frac{\lambda+\varepsilon}{2}-1}}{(1+t)^{\lambda}} dt\right] dy$$
$$- \int_{n_1}^{\infty} (u(y))^{-1-\varepsilon}u'(y) \left[\int_{0}^{\frac{1}{u(y)}} \frac{t^{\frac{\lambda+\varepsilon}{2}-1}}{(1+t)^{\lambda}} dt\right] dy$$
$$> \frac{1}{\varepsilon(u(n_1))^{\varepsilon}} B\left(\frac{\lambda+\varepsilon}{2}, \frac{\lambda-\varepsilon}{2}\right) - \int_{c}^{\infty} (u(y))^{-1}u'(y)[\int_{0}^{\frac{1}{u(y)}} t^{\frac{\lambda+\varepsilon}{2}-1}dt]dy$$
$$= \frac{1}{\varepsilon(u(n_1))^{\varepsilon}} B\left(\frac{\lambda+\varepsilon}{2}, \frac{\lambda-\varepsilon}{2}\right) - \left(\frac{2}{\lambda+\varepsilon}\right)^2.$$

The lemma is proved. □

Theorem 1. *If* $u(t)$ *is a differentiable strict increasing function in* (n_0-1,∞) $(n_0 \in N)$, *such that* $u((n_0-1)+) = 0$ *and* $u(\infty) = \infty$, *and* $\lambda > 0, (u(t))^{\frac{\lambda-2}{2}}u'(t)$ $(t \in (n_0-1,\infty))$ *is decreasing;* $f(x), a_n \geq 0$, *satisfy*

$$0 < \int_{n_0-1}^{\infty} \frac{(u(x))^{1-\lambda}}{u'(x)} f^2(x)dx < \infty \text{ and } 0 < \sum_{n=n_0}^{\infty} \frac{(u(n))^{1-\lambda}}{u'(n)} a_n^2 < \infty,$$

then, one has

$$\sum_{n=n_0}^{\infty} \int_{n_0-1}^{\infty} \frac{a_n f(x)}{(1+u(n)u(x))^{\lambda}} dx < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \sum_{n=n_0}^{\infty} \frac{(u(n))^{1-\lambda} a_n^2}{u'(n)} \int_{n_0-1}^{\infty} \frac{(u(x))^{1-\lambda} f^2(x)}{u'(x)}\, dx \right\}^{\frac{1}{2}}, \quad (8)$$

where the constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ *is the best possible. In particular,*

(i) setting $u(t) = t^{\alpha}$ $(\alpha > 0; t \in (0, \infty))$, *then for* $0 < \lambda \le \frac{2}{\alpha}$, *one has*

$$\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{a_n f(x)}{[1+(nx)^{\alpha}]^{\lambda}} dx < \frac{1}{\alpha} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \sum_{n=1}^{\infty} n^{1-\alpha\lambda} a_n^2 \int_{0}^{\infty} x^{1-\alpha\lambda} f^2(x) dx \right\}^{\frac{1}{2}}; \quad (9)$$

(ii) setting $u(t) = \ln t$ $(t \in (1, \infty))$, *then for* $0 < \lambda \le 2$, *one has*

$$\sum_{n=2}^{\infty} \int_{1}^{\infty} \frac{a_n f(x)}{(1+\ln n \ln x)^{\lambda}} dx < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \sum_{n=2}^{\infty} \frac{n a_n^2}{(\ln n)^{\lambda-1}} \int_{1}^{\infty} \frac{x f^2(x)}{(\ln x)^{\lambda-1}}\, dx \right\}^{\frac{1}{2}}. \quad (10)$$

Proof. Setting $F(y) = a_n, U(y) = u(n), y \in [n, n+1)$ $(n = n_0, n_0+1, \cdots)$, by Cauchy's inequality with weight (see [4]), one has

$$\begin{aligned} \sum_{n=n_0}^{\infty} \int_{n_0-1}^{\infty} \frac{a_n f(x)}{(1+u(n)u(x))^{\lambda}} dx &= \int_{n_0}^{\infty} \int_{n_0-1}^{\infty} \frac{f(x)F(y)}{(1+U(y)u(x))^{\lambda}} dxdy \\ &= \int_{n_0}^{\infty} \int_{n_0-1}^{\infty} \frac{1}{(1+U(y)u(x))^{\lambda}} \left[\left(\frac{U(y)}{u(x)}\right)^{\frac{2-\lambda}{4}} \left(\frac{u'(x)}{U'(y)}\right)^{\frac{1}{2}} F(y) \right] \\ &\quad \times \left[\left(\frac{u(x)}{U(y)}\right)^{\frac{2-\lambda}{4}} \left(\frac{U'(y)}{u'(x)}\right)^{\frac{1}{2}} f(x) \right] dxdy \\ &\le \left\{ \int_{n_0}^{\infty} \int_{n_0-1}^{\infty} \frac{1}{(1+U(y)u(x))^{\lambda}} \left(\frac{U(y)}{u(x)}\right)^{\frac{2-\lambda}{2}} \frac{u'(x)}{U'(y)} F^2(y) dxdy \right\}^{\frac{1}{2}} \end{aligned}$$

$$\times\left\{\int_{n_0}^{\infty}\int_{n_0-1}^{\infty}\frac{1}{(1+U(y)u(x))^{\lambda}}\left(\frac{u(x)}{U(y)}\right)^{\frac{2-\lambda}{2}}\frac{U'(y)}{u'(x)}f^2(x)dxdy\right\}^{\frac{1}{2}}$$

$$=\left\{\sum_{n=n_0}^{\infty}\left[\int_{n_0-1}^{\infty}\frac{1}{(1+u(n)u(x))^{\lambda}}\left(\frac{u(n)}{u(x)}\right)^{\frac{2-\lambda}{2}}\frac{u'(x)}{u'(n)}dx\right]a_n^2\right\}^{\frac{1}{2}}$$

$$\times\left\{\int_{n_0-1}^{\infty}\left[\sum_{n=n_0}^{\infty}\frac{1}{(1+u(n)u(x))^{\lambda}}\left(\frac{u(x)}{u(n)}\right)^{\frac{2-\lambda}{2}}\frac{u'(n)}{u'(x)}\right]f^2(x)dx\right\}^{\frac{1}{2}}$$

$$=\left\{\sum_{n=n_0}^{\infty}\omega_{\lambda}(n)a_n^2\int_{n_0-1}^{\infty}\widetilde{\omega}_{\lambda}(x)f^2(x)dx\right\}^{\frac{1}{2}},\quad(11)$$

where the weight functions $\omega_{\lambda}(x)$ and $\widetilde{\omega}_{\lambda}(x)$ $(x>n_0-1)$ are defined respectively by

$$\omega_{\lambda}(x):=\int_{n_0-1}^{\infty}\frac{1}{(1+u(x)u(y))^{\lambda}}\left(\frac{u(x)}{u(y)}\right)^{\frac{2-\lambda}{2}}\frac{u'(y)}{u'(x)}dy,\qquad(12)$$

$$\widetilde{\omega}_{\lambda}(x):=\sum_{n=n_0}^{\infty}\frac{1}{(1+u(n)u(x))^{\lambda}}\left(\frac{u(x)}{u(n)}\right)^{\frac{2-\lambda}{2}}\frac{u'(n)}{u'(x)}.\qquad(13)$$

Setting $t=\frac{1}{u(x)u(y)}$ in (12), one has

$$\omega_{\lambda}(x)$$
$$=\left[\int_0^{\infty}\frac{1}{(1+t)^{\lambda}}t^{\frac{\lambda}{2}-1}\,dt\right]\frac{(u(x))^{1-\lambda}}{u'(x)}=B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\frac{(u(x))^{1-\lambda}}{u'(x)}.\quad(14)$$

Since $(u(t))^{\frac{\lambda-2}{2}}u'(t)$ $(t\in(n_0-1,\infty))$ is decreasing, by the same way, one has

$$\widetilde{\omega}_{\lambda}(x)<\omega_{\lambda}(x)=B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\frac{(u(x))^{1-\lambda}}{u'(x)}\quad(x>n_0-1).\qquad(15)$$

By (11), (15) and (14) (for $x=n$), one has (8).

For $0<\varepsilon<\lambda$, setting $\widetilde{a}_n$ and $\widetilde{f}(x)$ as

$$\begin{aligned}\widetilde{a}_n &= 0,\ n=n_0,\cdots,n_1-1;\ \widetilde{a}_n=(u(n))^{\frac{\lambda-\varepsilon}{2}-1}u'(n),\ n\geq n_1(n\in N),\\ \widetilde{f}(x) &= (u(x))^{\frac{\lambda+\varepsilon}{2}-1}u'(x),\ x\in(n_0-1,c];\ \widetilde{f}(x)=0,\ x\in(c,\infty).\end{aligned}$$

Since
$$\frac{u'(x)}{(u(x))^{1+\varepsilon}} = \frac{1}{(u(x))^{\varepsilon+\lambda/2}}(u(x))^{\frac{\lambda-2}{2}}u'(x)$$
and $(u(x))^{\frac{\lambda-2-\varepsilon}{2}}u'(x) = \frac{1}{(u(x))^{\varepsilon/2}}(u(x))^{\frac{\lambda-2}{2}}u'(x)$ are decreasing, one finds

$$\left\{\sum_{n=n_0}^{\infty} \frac{(u(n))^{1-\lambda}\,\widetilde{a}_n^2}{u'(n)} \int_{n_0-1}^{\infty} \frac{(u(x))^{1-\lambda}\,(\widetilde{f}\,(x))^2}{u'(x)}\,dx\right\}^{\frac{1}{2}}$$
$$= \left\{\sum_{n=n_1}^{\infty} \frac{u'(n)}{(u(n))^{1+\varepsilon}} \int_{n_0-1}^{c} \frac{u'(x)}{(u(x))^{1-\varepsilon}}\,dx\right\}^{\frac{1}{2}} < \left\{\int_{c}^{\infty} \frac{u'(x)}{(u(x))^{1+\varepsilon}}\,dx \cdot \frac{1}{\varepsilon}\right\}^{\frac{1}{2}}$$
$$= \frac{1}{\varepsilon}. \quad (16)$$

If the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ in (8) is not the best possible, then there exists a positive constant K, with $K < B(\frac{\lambda}{2},\frac{\lambda}{2})$, such that (8) is still valid if one replaces $B(\frac{\lambda}{2},\frac{\lambda}{2})$ by K. In particular, one has

$$I = \sum_{n=n_0}^{\infty} \int_{n_0-1}^{\infty} \frac{\widetilde{a}_n\widetilde{f}\,(x)}{(1+u(n)u(x))^{\lambda}}dx$$
$$< K\left\{\sum_{n=n_0}^{\infty} \frac{(u(n))^{1-\lambda}\,\widetilde{a}_n^2}{u'(n)} \int_{n_0-1}^{\infty} \frac{(u(x))^{1-\lambda}\,(\widetilde{f}\,(x))^2}{u'(x)}\,dx\right\}^{\frac{1}{2}}.$$

In view of (7) and (16), one has
$$\frac{1}{(u(n_1))^{\varepsilon}}B\left(\frac{\lambda+\varepsilon}{2},\frac{\lambda-\varepsilon}{2}\right) - \varepsilon\left(\frac{2}{\lambda+\varepsilon}\right)^2 < K,$$
and then $B(\frac{\lambda}{2},\frac{\lambda}{2}) \le K$ ($\varepsilon \to 0^+$). This contradicts $K < B(\frac{\lambda}{2},\frac{\lambda}{2})$. Hence the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ in (8) is the best possible. The theorem is proved. □

3. Two Equivalent Inequalities

Theorem 2. *If $u(t)$ is a differentiable strict increasing function in $(n_0 - 1,\infty)$ ($n_0 \in N$), such that $u((n_0-1)+) = 0$ and $u(\infty) = \infty$, and $\lambda >$*

$0, (u(t))^{\frac{\lambda-2}{2}}u'(t)$ $(t \in (n_0-1, \infty))$ *is decreasing;* $a_n \geq 0$, *satisfying* $0 < \sum_{n=n_0}^{\infty} \frac{(u(n))^{1-\lambda}}{u'(n)} a_n^2 < \infty$, *then one has the equivalent form of (8) as*

$$\int_{n_0-1}^{\infty} \frac{u'(x)}{(u(x))^{1-\lambda}} \left[\sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(n)u(x))^{\lambda}}\right]^2 dx < \left[B(\frac{\lambda}{2}, \frac{\lambda}{2})\right]^2 \sum_{n=n_0}^{\infty} \frac{(u(n))^{1-\lambda}}{u'(n)} a_n^2, \quad (17)$$

where the constant factor $\left[B(\frac{\lambda}{2}, \frac{\lambda}{2})\right]^2$ *is the best possible. In particular:*

(i) setting $u(t) = t^{\alpha}$ $(\alpha > 0; t \in (0, \infty))$, *then for* $0 < \lambda \leq \frac{2}{\alpha}$, *one has*

$$\int_0^{\infty} x^{\alpha\lambda-1} \left\{\sum_{n=1}^{\infty} \frac{a_n}{[1+(nx)^{\alpha}]^{\lambda}}\right\}^2 dx < \left[\frac{1}{\alpha} B(\frac{\lambda}{2}, \frac{\lambda}{2})\right]^2 \sum_{n=1}^{\infty} n^{1-\alpha\lambda} a_n^2; \quad (18)$$

(ii) setting $u(t) = \ln t$ $(t \in (1, \infty))$, *then for* $0 < \lambda \leq 2$, *one has*

$$\int_1^{\infty} \frac{(\ln x)^{\lambda-1}}{x} \left[\sum_{n=2}^{\infty} \frac{a_n}{(1+\ln n \ln x)^{\lambda}}\right]^2 dx < \left[B(\frac{\lambda}{2}, \frac{\lambda}{2})\right]^2 \sum_{n=2}^{\infty} \frac{n a_n^2}{(\ln n)^{\lambda-1}} . \quad (19)$$

Proof. Set $f(x)$ as

$$f(x) := \frac{u'(x)}{(u(x))^{1-\lambda}} \left[\sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(n)u(x))^{\lambda}}\right],$$

and use (8) to obtain

$$\begin{aligned} 0 < \int_{n_0-1}^{\infty} \frac{(u(x))^{1-\lambda} f^2(x)}{u'(x)} dx &= \int_{n_0-1}^{\infty} \frac{u'(x)}{(u(x))^{1-\lambda}} \left[\sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(n)u(x))^{\lambda}}\right]^2 dx \\ &= \sum_{n=n_0}^{\infty} \int_{n_0-1}^{\infty} \frac{a_n f(x)}{(1+u(n)u(x))^{\lambda}} dx \\ &\leq B(\frac{\lambda}{2}, \frac{\lambda}{2}) \left\{\sum_{n=n_0}^{\infty} \frac{(u(n))^{1-\lambda} a_n^2}{u'(n)} \int_{n_0-1}^{\infty} \frac{(u(x))^{1-\lambda} f^2(x)}{u'(x)} dx\right\}^{\frac{1}{2}}; \quad (20) \end{aligned}$$

$$0 < \left\{ \int_{n_0-1}^{\infty} \frac{(u(x))^{1-\lambda} f^2(x)}{u'(x)}\, dx \right\}^{\frac{1}{2}}$$
$$= \left\{ \int_{n_0-1}^{\infty} \frac{u'(x)}{(u(x))^{1-\lambda}} \left[\sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(n)u(x))^{\lambda}} \right]^2 dx \right\}^{\frac{1}{2}}$$
$$\le B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \sum_{n=n_0}^{\infty} \frac{(u(n))^{1-\lambda} a_n^2}{u'(n)} \right\}^{\frac{1}{2}} < \infty. \quad (21)$$

Hence (20) takes strict inequality by using (8); so does (21). It follows that (17) holds.

On the other hand, if (17) holds, by Cauchy's inequality, one has

$$\int_{n_0-1}^{\infty} \sum_{n=n_0}^{\infty} \frac{a_n f(x)}{(1+u(n)u(x))^{\lambda}} dx$$
$$= \int_{n_0-1}^{\infty} \left\{ \left[\frac{u'(x)}{(u(x))^{1-\lambda}} \right]^{\frac{1}{2}} \sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(n)u(x))^{\lambda}} \right\}$$
$$\times \left\{ \left[\frac{(u(x))^{1-\lambda}}{u'(x)} \right]^{\frac{1}{2}} f(x) \right\} dx \le \left\{ \int_{n_0-1}^{\infty} \frac{u'(x)}{(u(x))^{1-\lambda}} \right.$$
$$\left. \times \left[\sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(n)u(x))^{\lambda}} \right]^2 dx \int_{n_0-1}^{\infty} \frac{(u(x))^{1-\lambda} f^2(x)}{u'(x)}\, dx \right\}^{\frac{1}{2}}. \quad (22)$$

By (17), one has (8). Hence (17) and (8) are equivalent.

If the constant factor in (17) is not the best possible, one can get a contradiction that the constant factor in (8) is not the best possible by using (22). The theorem is proved. □

By the symmetry and the same way of Theorem 2, one has the following result.

Theorem 3. *If* $u(t)$ *is a differentiable strict increasing function in* $(n_0 - 1, \infty)$ $(n_0 \in N)$, *such that* $u((n_0-1)+) = 0$ *and* $u(\infty) = \infty$, *and* $\lambda > 0, (u(t))^{\frac{\lambda-2}{2}} u'(t)$ $(t \in (n_0-1, \infty))$ *is decreasing*; $f(x) \ge 0$, *satisfy* $0 < \int_{n_0-1}^{\infty} \frac{(u(x))^{1-\lambda} f^2(x)}{u'(x)}\, dx < \infty$, *then, one has the equivalent form of (8) as*

$$\sum_{n=n_0}^{\infty} \frac{u'(n)}{(u(n))^{1-\lambda}} \left[\int_{n_0-1}^{\infty} \frac{f(x)}{(1+u(n)u(x))^{\lambda}} dx \right]^2$$

$$< \left[B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^2 \int_{n_0-1}^{\infty} \frac{(u(x))^{1-\lambda} f^2(x)}{u'(x)}\,dx, \quad (23)$$

where the constant factor $\left[B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^2$ *is the best possible. In particular:*

(i) setting $u(t) = t^{\alpha}$ $(\alpha > 0; t \in (0,\infty))$, *then for* $0 < \lambda \le \frac{2}{\alpha}$, *one has*

$$\sum_{n=1}^{\infty} n^{\alpha\lambda-1} \left\{ \int_0^{\infty} \frac{f(x)}{[1+(nx)^{\alpha}]^{\lambda}} dx \right\}^2 < \left[\frac{1}{\alpha} B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^2 \int_0^{\infty} x^{1-\alpha\lambda} f^2(x)dx; \quad (24)$$

(ii) setting $u(t) = \ln t$ $(t \in (1,\infty))$, *then for* $0 < \lambda \le 2$, *one has*

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\lambda-1}}{n} \left[\int_1^{\infty} \frac{f(x)}{(1+\ln n \ln x)^{\lambda}} dx \right]^2 < \left[B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^2 \int_1^{\infty} \frac{x f^2(x)}{(\ln x)^{\lambda-1}} dx\ . \quad (25)$$

Remarks. (i) By the same way of Theorem 1, one still can get the following inequality:

$$\sum_{n=n_0}^{\infty} \sum_{m=n_0}^{\infty} \frac{a_m b_n}{(1+u(m)u(n))^{\lambda}} < B(\frac{\lambda}{2},\frac{\lambda}{2}) \left\{ \sum_{n=n_0}^{\infty} \frac{(u(n))^{1-\lambda} a_n^2}{u'(n)} \sum_{n=n_0}^{\infty} \frac{(u(n))^{1-\lambda} b_n^2}{u'(n)} \right\}^{\frac{1}{2}}. \quad (26)$$

But one cannot show that the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ in (26) is the best possible by using the method of Theorem 1.

(ii) By the assumption of Theorem 1, one can conclude that (8), (17) and (23) are equivalent. In particular, for $\lambda = \alpha = 1$ in (9), (18) and (24), one can give three equivalent inequalities as follows:

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} \frac{f(x)}{1+nx} dx < \pi \left\{ \sum_{n=1}^{\infty} a_n^2 \int_0^{\infty} f^2(x)dx \right\}^{\frac{1}{2}}, \quad (27)$$

$$\int_0^{\infty} \left(\sum_{n=1}^{\infty} \frac{a_n}{1+nx} \right)^2 dx < \pi^2 \sum_{n=1}^{\infty} a_n^2\,, \quad (28)$$

$$\sum_{n=1}^{\infty}\left(\int_0^{\infty}\frac{f(x)}{1+nx}dx\right)^2 < \pi^2\int_0^{\infty}f^2(x)dx\,, \tag{29}$$

where the constant factors in the above inequalities are the best possible.

(iii) Since all the obtaining inequalities with the best constant factors, one may gives some new results.

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