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A NOTE ON INTEGRATION WITH RESPECT TO VECTOR MEASURES

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Abstract: Let X be a separable barrelled locally convex $C(K)$-module and X' be the continuous dual of X. In this paper, we concern with integration with respect to the dual space X'-valued vector measure.

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1. Introduction

Let X be a locally convex Hausdorff space and $L(X)$ be the continuous linear operators on X. The identity operator in X is denoted by I and X' will denote the continuous dual of X. The adjoint of an operator $T \in L(X)$ is denoted by T'. Let $\sum$ be a σ-algebra of subsets of a non-empty set Ω. A function $m : \sum \to X$ is called a vector measure if it is σ-additive set function. Given $x' \in X'$, let $< x', m >$ denote the complex measure $E \to < x', m(E) >$ for $E \in \sum$. Its total variation measure is denoted by $| < x', m > |$. A $\sum$-measurable function $f : \Omega \to C$ is called m-integrable if it is $< x', m >$-integrable for every $x' \in X'$ and if, for each $E \in \sum$, there exists an element $\int_E f dm$ in X such that

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$< x', \int_E f dm >= \int_E f d < x', m >$, for $x' \in X'$, where C is complex numbers. The linear space of all m-integrable functions is denoted by $L^1(m)$. A set $E \in \sum$ is called m-null if $m(F) = 0$ for every $F \in \sum$ such that $F \subseteq E$. A C-valued, $\sum$-measurable function on Ω is said to be m-essentially bounded if it is bounded off an m-null set. The space of all m-essentially bounded functions is denoted by $L^\infty(m)$. A function $P : \sum \to L(X)$ is called a spectral measure if P is σ-additive and multiplicative, that is, $P(E \cap F) = P(E)P(F)$ for all $E, F \in \sum$, and $P(\Omega) = I$. The multiplicativity of P implies that $E \in \sum$ is P-null if and only if $P(E) = 0$. Integrability with respect to general vector measure is not simpler than for spectral measure P. Given $f \in L^1(P)$, the continuous operator $\int_\Omega f dP$ is also denoted by $P(f)$. Since a spectral measure $P : \sum \to L(X)$ is a vector measure it has an associated space $L^1(P)$ of P-integrable functions. For each $x \in X$, there is an induced X-valued vector measure $Px : \sum \to X$ defined by $Px : E \to P(E)x$, for $E \in \sum$, and its associated space $L^1(Px)$ of Px-integrable functions.

For undefined notions and terminology we refer to Ricker [4], [5], and Schaefer [6].

Let X be a barrelled space. Then X' is $\sigma(X', X)$-quasicomplete, Schaefer [6, p. 148]. Let K be a compact Hausdorff space and let $C(K)$ be the set of all continuous real or complex valued functions. We say that X is a barrelled locally convex $C(K)$-module, if the following conditions are satisfied:

(i) $h : C(K) \times X \to X, (a, x) \to a.x$, is a bilinear mapping.

(ii) Bilinear mapping h is separately continuous.

(iii) $(ab).x = a.(b.x)$ for all $a, b \in C(K), x \in X$.

(iv) $1.x = x$, for $1 \in C(K), x \in X$.

We accomplish the other bilinear mappings:

$$X \times X' \to C(K)', (x, x') \to (x.x')(a) = x'(a.x); \tag{1}$$

$$C(K)'' \times X' \to X', (a, x') \to (a.x')(x) = a(x.x'). \tag{2}$$

By the bilinear mapping h we define a mapping $n : C(K) \to L(X)$, $n(a)x = a.x$, which is (norm to strong operator topology) continuous such that n is unital. The bilinear mapping (2) defines a mapping $n^\star : C(K)'' \to L(X'), n^\star(a)x' = a.x'$. It is known that $C(K)$ is $|\sigma|(C(K)'', C(K)')$ dense in $C(K)''$ (C.D. Aliprantis et al [1, Theorem 11.16]), and $C(K)''$ is isomorphic to $C(S)$ with S hyperstonian (C.D. Aliprantis et al [1, Theorem 15.7]). We give some properties of $n^\star$:

(i) $n^\star$ is $(\sigma(C(K)'', C(K)')$-weak* operator topology) continuous linear mapping.

(ii) $n^\star(1) = I'$, where $I' \in L(X')$ (identity operator).

(iii) The mapping

$$C(K)'' \to X', a \to a.x', \text{ is } (\sigma(C(K)'', C(K)') - \sigma(X', X)) \text{ continuous.}$$

(iv) For all $a \in C(K), (n(a))^\star = n^\star(a)$, where $(n(a))^\star$ is the adjoint of $n(a)$ in $L(X)$.

We denote by B the idempotents in $C(K)''$. We note that $n^\star(B)$ is strongly equicontinuous in $L(X')$,Dodds et al [2].

Definition. (see [3]) A locally convex Hausdorff space X is said to have the closed graph property if every closed, linear map of X into X is continuous. The vector space of all closed, linear maps of all of X into X is denoted by $G(X)$. If Y is a locally convex Hausdorff space and $m : \sum \to Y$ is a vector measure, then the sequential closure of the linear span of $m(\sum)$ is denoted by $[Y]_m$. A spectral measure P in X has the bounded-pointwise intersection property if

$$L^\infty(P) \cap (\cap_{x \in X} L^1(Px)) \subseteq L^1(P).$$

By $Co(S)$ we denote the collection of all subsets of S which are simultaneously open and closed (clopen). $\sigma(Co(S))$ denotes the σ-algebra generated by $Co(S)$. Put $\sum = \sigma(Co(S))$. Above similar definitions can be done for locally convex $C(S)$-module X'. Let $P : \sum \to L(X')$ be a spectral measure. Since spectral measure P is a vector measure, it has an associated space $L^1(P)$ of P-integrable functions. For each $x' \in X'$, there is an induced X'-valued vector measure $Px' : \sum \to X'$ defined by $Px' : E \to P(E)x'$, for $E \in \sum$, and its associated space $L^1(Px')$ of Px'-integrable functions. Let f be a complex valued $\sum$-measurable function so that $f \in L^1(Px')$ for each $x' \in X'$. Then we define the everywhere defined mapping $P_{[f]} : X' \to X'$ by $P_{[f]}x' = \int_S f d(Px')$ for $x' \in X'$.

Now we can state and prove the main result of this paper.

Theorem 1. (see [3]) *Let X be a separable barrelled locally convex $C(K)$-module and $P : \sum \to L(X')$ be a spectral measure, where $\sum = \sigma(Co(S))$. Suppose that any one of the following conditions is satisfied:*

(i) X' has the closed graph property and $\{P_{[f]} : f \in \cap_{x' \in X'} L^1(Px')\} \subseteq G(X')$.

(ii) $[L(X')]_P$ is sequentially complete.

Then the linear operator $P_{[f]}$ is weak*-weak*-continuous, for every $f \in \cap_{x' \in X'} L^1(Px')$. In particular,

$$L^1(P) = \cap_{x' \in X'} L^1(Px').$$

Proof. It is easy to see that every complex(C) valued function $f \in L^1(P)$ necessarily belongs to $L^1(Px')$, for each $x' \in X'$, and that the continuous linear operator $P(f) = \int_S f dP$ in X' satisfies $P(f)x' = \int_S f d(Px')$, for each $x' \in X'$.

Conversely, suppose that f is a C-valued, $\sum$-measurable function which belongs to $L^1(Px')$, for each $x' \in X'$. Then the everywhere defined map $P_{[f]} : X' \to X'$ defined by $P_{[f]}(x') = \int_S f d(Px')$, $\quad x' \in X'$ is linear. Suppose that (i) is satisfied. If $f \in \cap_{x' \in X'} L^1(Px')$, then (i) implies that $P_{[f]} \in G(X')$ and by the closed graph property of X' we have $P_{[f]} \in L(X')$. By Ricker [3, Lemma 2], we have $f \in L^1(P)$. Assume that (ii) is given. The sequential completeness of $[L(X')]_P$ implies that $L^\infty(P) \subseteq L^1(P)$. This implies that P has the bounded-pointwise intersection property. Let $f \in \cap_{x' \in X'} L^1(Px')$. By Ricker [3, Lemma 6], $\{P(f^{[n]})\}_{n=1}^\infty$ is a Cauchy sequence in $[L(X')]_P$, where $f^{[n]} = f\chi_{E(n)}, E(n) = \{y \in S : |f(y)| \leq n\}$ for each $n \in N$. So, there exists $T \in L(X')$ such that $P(f^{[n]}) \to T$ in $L(X')$. Since $P(f^{[n]}) \to P_{[f]}$ pointwise on X', it follows by Ricker [3, Lemma 6] that $T = P_{[f]}$ and $P_{[f]} \in L(X')$. By Ricker [3, Lemma 2], $f \in L^1(P)$. □

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