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MAXIMAL FUNCTIONS AND CARLESON MEASURES ON NONHOMOGENEOUS SPACES

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Abstract: Ruiz and Torrea showed the boundedness of some maximal operators of Fefferman and Stein type with respect to Carleson measure on R_+^{n+1}. In this paper we consider similar results on nonhomogeneous spaces.

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1. Introduction

Let M be the maximal operator such that

$$Mf(x,t) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)|dy \qquad (x \in R^n, t \geq 0),$$

where the supremum is taken over the balls Q centered at x and whose radii are greater than t. Carleson [2] proved the following:

$$\int_{R_+^{n+1}} Mf(x,t)^p d\mu(x,t) \leq C \int_{R^n} |f(x)|^p dx \qquad (1 < p < \infty), \tag{1}$$

where μ is a Carleson measure (for the precise definition, see Section 2).

Ruiz and Torrea [4], [5], [7], [6], [8] generalized this result to several directions. They studied fractional maximal operator M_α, generalized Carleson measure of order δ and weighted norm inequalities. They also considered (1) for doubling measure $d\sigma(x)$ in place of the Lebesgue measure dx.

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However, some recent results on Calderón-Zygmund operators and Hardy spaces (see for example [3] and [10]) show that it should be possible to dispense with the doubling condition for most of the classical theory. In this paper we shall show a version of the inequality (1) on nonhomogeneous space. As a corollary of our results (see Section 4), we obtain a generalization of Carleson's Theorem:

$$\int_{R_+^{n+1}} |P_t * f(x)|^p d\mu(x,t) \le C \int_{R^n} |f(x)|^p dx \qquad (1 < p < \infty),$$

where μ is a Carleson measure and $P_t * f(x)$ is a Poisson integral.

2. Definitions and Theorems

The following notation is used: We denote a ball radius r centered at x by $Q(x,r) = \{y \in R^n; |y-x| < r\}$ and rad (Q) is the radius of a ball Q. For a ball $Q = Q(x,r)$, we write $kQ = Q(x,kr)$. $R_+^{n+1} = R^n \times [0,\infty)$.

Throughout this paper we fix a positive Radon measure σ on R^n.

First we define the modified centered fractional maximal operator which is main object of this paper.

Definition 1. Let $k > 1$ and $0 < \alpha \le 1$. We define

$$M_\alpha^k f(x,t) = \sup_{r \ge t} \frac{1}{\sigma(Q(x,kr))^\alpha} \int_{Q(x,r)} |f(y)| d\sigma(y).$$

Nazarov, Treil and Volberg considered modified maximal operator in [3]. Next we define Carleson measure.

Definition 2. Given a ball Q in R^n, we define the cylindrical set

$$\widehat{Q} = \{(x,t) \in R_+^{n+1}; x \in Q, 0 \le t < 2\mathrm{rad}(Q)\}.$$

We may denote $\widehat{Q} = Q\hat{}$.

Definition 3. Let $\beta > 0$. We say a positive measure μ on R_+^{n+1} is a Carleson measure of order β (with respect to σ) if μ satisfies

$$\mu(\widehat{Q}) \le A_1 \sigma(Q)^\beta$$

for all balls Q in R^n, where A_1 is a constant independent of Q.

Furthermore we shall define generalized Carleson measure (see Ruiz and Torrea [6]).

Definition 4. Let μ be a positive measure on R_+^{n+1} and w be a positive σ-measurable function in R^n. Given $k > 1, 0 < \alpha \le 1, 1 < p < \infty$ and $1 \le q < \infty$. We say that the pair (μ, w) satisfies the condition $C_{q,p,\alpha,k}$ (or $(\mu, w) \in C_{q,p,\alpha,k}$) if for any ball Q in R^n,

$$\frac{\mu(\widehat{kQ})^{1/q}}{\sigma(kQ)^\alpha}\Big(\int_Q w(x)^{-p'/p} d\sigma(x)\Big)^{1/p'} \le A_2,$$

where $p' = p/(p-1)$.

We say that the pair (μ, w) satisfies the condition $C_{q,1,\alpha,k}$ (or $(\mu, w) \in C_{q,1,\alpha,k}$) if

$$\sup_{Q \ni x} \frac{\mu(\widehat{kQ})^{1/q}}{\sigma(kQ)^\alpha} \le A_3 w(x) \qquad \text{a.e. } x.$$

We obtain the following theorem.

Theorem 1. *Let $k > 1, 0 < \alpha \le 1$ and $1 \le p \le q < \infty$. If $(\mu, w) \in C_{q,p,\alpha,k}$, then*

$$\mu(\{(x,t) \in R_+^{n+1}; M_\alpha^k f(x,t) > \lambda\}) \le \frac{A_{n,p,q,k}}{\lambda^q}\Big(\int_{R^n} |f(x)|^p w(x) d\sigma(x)\Big)^{q/p},$$

where $A_{n,p,q,k}$ is a constant depending only on n, p, q, k, A_2 and A_3.

Following corollaries are easily deduced from our theorem.

Corollary 1. *Let $0 < \alpha \le \beta$. If μ is a Carleson measure of order β, then*

$$\mu(\{(x,t) \in R_+^{n+1}; M_\alpha^k f(x,t) > \lambda\}) \le \frac{A_{n,k}}{\lambda^{\beta/\alpha}}\Big(\int_{R^n} |f(x)| d\sigma(x)\Big)^{\beta/\alpha}.$$

Corollary 2. *If μ is a Carleson measure of order 1 then*

$$\mu(\{(x,t) \in R_+^{n+1}; M_1^k f(x,t) > \lambda\}) \le \frac{A_{n,k}}{\lambda}\int_{R^n} |f(x)| d\sigma(x).$$

By using interpolation theorem we obtain the next corollary.

Corollary 3. *If μ is a Carleson measure of order 1 then*

$$\int_{R_+^{n+1}} \big(M_1^k f(x,t)\big)^p d\mu(x,t) \le A_{n,p,k}\int_{R^n} |f(x)|^p d\sigma(x) \qquad (1 < p < \infty).$$

3. Proof

To prove our theorem, the following Covering Lemma due to Sawano [9] is essential.

Lemma 1. (Sawano) *For any $k > 1$, there exists an integer $S_{n,k}$, depending only on n and k which satisfies the following:*

Let $\{B(x_\lambda, r_\lambda)\}_{\lambda \in L}$ be a family of balls. Suppose $\sup_{\lambda\in L} r_\lambda < \infty$. Then we can take disjoint subfamilies

$$\{b(x_\rho, r_\rho)\}_{\rho\in L_1}, \{B(x_\rho, r_\rho)\}_{\rho\in L_2}, \dots, \{B(x_\rho, r_\rho)\}_{\rho\in L_{S_{n,k}}},$$

such that

$$\bigcup_{\lambda\in L} B(x_\lambda, r_\lambda) \subset \bigcup_{i=1}^{S_{n,k}} \bigcup_{\rho\in L_i} B(x_\rho, kr_\rho),$$

where L_i's are countable subsets of L, and balls in $\{B(x_\rho, r_\rho)\}_{\rho\in L_i}$ are disjoint for each i.

Now we begin to prove the theorem. We follow the argument in Ruiz and Torrea [6], Section 5. First we shall define restricted maximal operator as follows.

Definition 5. Let $R > 0$, we define

$$M_\alpha^{k,R} f(x,t) = \sup_{t\le r<R} \frac{1}{\sigma(Q(x,kr))^\alpha} \int_{Q(x,r)} |f(y)|d\sigma(y).$$

Note that if $R < t$, then $M_\alpha^{k,R} f(x,t) = 0$.

Next we define two modified <u>uncentered</u> maximal operators.

Definition 6.

$$\widetilde{M}_\alpha^{k,R} f(x,t) = \sup_{Q\ni x, t\le \mathrm{rad}(Q)<R} \frac{1}{\sigma(kQ)^\alpha} \int_Q |f(y)|d\sigma(y),$$

$$\widetilde{m}_\alpha^{k,R} f(x) = \sup_{Q\ni x, \mathrm{rad}(Q)<R} \frac{1}{\sigma(kQ)^\alpha} \int_Q |f(y)|d\sigma(y).$$

Let

$$E_\lambda^R = \{(x,t) \in R_+^{n+1};\ M_\alpha^{k,R} f(x,t) > \lambda\},$$
$$\widetilde{E}_\lambda^R = \{(x,t) \in R_+^{n+1};\ \widetilde{M}_\alpha^{k,R} f(x,t) > \lambda\},$$
$$\widetilde{A}_\lambda^R = \{x \in R^n;\ \widetilde{m}_\alpha^{k,R} f(x) > \lambda\}.$$

For $x \in \widetilde{A}_\lambda^R$, we set $t(x, \lambda, R) = \sup\{t;\ (x,t) \in \widetilde{E}_\lambda^R\}$. Throughout our proof we fix λ and R, so we denote $t(x, \lambda, R) = t(x)$ for the simplicity of notation. Then $t(x) \leq R$.

For any $x \in \widetilde{A}_\lambda^R$, there exists a ball $Q^*(x) = Q^*(x, \lambda, R)$ such that $x \in Q^*(x)$,

$$\mathrm{rad}(Q^*(x)) > t(x)/2, \tag{2}$$

and

$$\frac{1}{\sigma(kQ^*(x))^\alpha} \int_{Q^*(x)} |f(y)| d\sigma(y) > \lambda. \tag{3}$$

Note that the center of $Q^*(x)$ may not be x.

By Sawano's Covering Lemma, we can choose a sequence $\{x_j\} \subset \widetilde{A}_\lambda^R$ such that

$$\widetilde{A}_\lambda^R \subset \bigcup_j kQ^*(x_j), \tag{4}$$

and

the family $\{Q^*(x_j)\}$ can be distributed in $S_{n,k}$ disjoint subfamilies. (5)

To prove our theorem we need the following lemma.

Lemma 2.

$$E_\lambda^R \subset \bigcup_j (kQ^*(x_j))\widehat{\ }.$$

Assuming this lemma, we shall continue the proof. It is suffices to show that the next inequality

$$\mu(E_\lambda^R) \leq \frac{A_{n,p,q,k}}{\lambda^q} \Big(\int_{R^n} |f(x)|^p w(x) d\sigma(x) \Big)^{q/p} \tag{6}$$

is satisfied. By Lemma 2 and (3), we have

$$\begin{aligned} \mu(E_\lambda^R) &\leq \sum_j \mu((kQ^*(x_j))\widehat{\ }) \\ &\leq \frac{1}{\lambda^q} \sum_j \frac{\mu((kQ^*(x_j))\widehat{\ })}{\sigma(kQ^*(x_j))^{\alpha q}} \Big(\int_{Q^*(x_j)} |f(y)| d\sigma(y) \Big)^q. \end{aligned}$$

When $p = 1$, by the condition $C_{q,1,\alpha,k}$ we have

$$\frac{\mu((kQ^*(x_j))\widehat{\ })^{1/q}}{\sigma(kQ^*(x_j))^\alpha} \int_{Q^*(x_j)} |f(y)| d\sigma(y) \leq A_3 \int_{Q^*(x_j)} |f(y)| w(y) d\sigma(y).$$

By using (5) we obtain

$$\mu(E_\lambda^R) \le \frac{A_3^q}{\lambda^q} \sum_j \Big(\int_{Q^*(x_j)} |f(y)| w(y) d\sigma(y) \Big)^q$$
$$\le \frac{A_{n,q,k}}{\lambda^q} \Big(\int_{R^n} |f(y)| w(y) d\sigma(y) \Big)^q.$$

When $p > 1$, by Hölder's inequality we have

$$\int_{Q^*(x_j)} |f(y)| d\sigma(y)$$
$$\le \Big(\int_{Q^*(x_j)} |f(y)|^p w(y) d\sigma(y) \Big)^{1/p} \Big(\int_{Q^*(x_j)} w(y)^{-p'/p} d\sigma(y) \Big)^{1/p'},$$

and by the condition $C_{q,p,\alpha,k}$ we obtain the desired result. □

Finally we shall prove Lemma.

Proof of Lemma. Suppose $(x,t) \in E_\lambda^R$ then $x \in \widetilde{A}_\lambda^R$. Therefore there exists x_j such that $x \in kQ^*(x_j)$.

If $(x,t) \notin (kQ^*(x_j))\widehat{\ }$. By the definition of cylindrical set and (2), we have

$$t \ge 2\mathrm{rad}(kQ^*(x_j)) > \frac{2k}{2} t(x_j) > t(x_j). \tag{7}$$

On the contrary, since $(x,t) \in E_\lambda^R$, there exists a ball $Q(x,r)$ centered at x such that $t \le r < R$ and

$$\frac{1}{\sigma(k(Q(x,r)))^\alpha} \int_{Q(x,r)} |f(y)| d\sigma(y) > \lambda.$$

Since $|x - x_j| \le (1+k)\mathrm{rad}(Q^*(x_j)) < 2k\mathrm{rad}(Q^*(x_j)) \le t \le r$, we have $x_j \in Q(x,r)$. Therefore $\widetilde{M}_\alpha^{k,R} f(x_j, r) > \lambda$. But $r \ge t > t(x_j)$ by (7). This is a contradiction to the definition of $t(x_j)$. □

4. Applications

In the previous sections, we do not assume any conditions on measure σ. In this section we assume some growth condition. Then we can define Poisson-like operator (see [1] and [6]).

Definition 7. Let $d > 0$. We say a measure σ satisfies the growth condition of order d if σ satisfies

$$\sigma(Q(x,r)) \leq A_4 r^d.$$

This condition is natural when we consider non-doubling measure (see [3] and [10]). Next we define Poisson-like operator.

Definition 8. Let $d > 0, 0 < \alpha \leq 1$ and $\varepsilon > 0$. A function $K(x,y,t)$ defined on $R^n \times R^n \times [0,\infty)$ is called a (d,α,ε)-Poisson-like kernel if K satisfies

$$|K(x,y,t)| \leq \frac{A_5 t^\varepsilon}{(|x-y|+t)^{d\alpha+\varepsilon}}.$$

Definition 9. For a (d,α,ε)-Poisson-like kernel, we shall define (d,α,ε)-Poisson-like operator by

$$Tf(x,t) = \int_{R^{n+1}_+} K(x,y,t) f(y) d\sigma(y).$$

As corollaries of our theorem we obtain the following results.

Corollary 4. *Let $d > 0, 0 < \alpha \leq \beta$ and $\varepsilon > 0$. We assume that σ satisfies the growth condition of order d and let T be a (d,α,ε)-Poisson-like operator. If μ is a Carleson measure of order β then*

$$\mu(\{(x,t) \in R^{n+1}_+; |Tf(x,t)| > \lambda\}) \leq \frac{A_{n,k,\varepsilon}}{\lambda^{\beta/\alpha}} \Big(\int_{R^n} |f(x)| d\sigma(x) \Big)^{\beta/\alpha}.$$

Proof. By the growth condition we have

$$|Tf(x,t)| \leq A_{n,k,\varepsilon} M^k_\alpha f(x,t).$$ □

Corollary 5. *We assume that σ satisfies the growth condition of order d, and let T be a $(d,1,\varepsilon)$-Poisson-like operator. If μ is a Carleson measure of order 1 then*

$$\int_{R^{n+1}_+} |Tf(x,t)|^p d\mu(x,t) \leq A_{n,p,k,\varepsilon} \int_{R^n} |f(x)|^p d\sigma(x) \qquad (1 < p < \infty).$$

Proof. Note that $\int_{R^n} t^\varepsilon/(|x|+t)^{(d+\varepsilon)} d\sigma(x) \leq A_{n,k,\varepsilon}$ and use Interpolation Theorem. □

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