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ON LOWER AND UPPER IRRESOLUTE MULTIFUNCTIONS

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Abstract: In the literature, several irresolute functions are introduced in general topology. In this paper, a new type of irresolute multifunctions called δ-semi-irresolute multifunctions is introduced.

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1. Introduction

Multifunctions and continuity are two of main topics of set valued analysis and general topology. Many author studied weaker and stronger forms of open sets and multifunction. Furthermore, a good many of papers in general topology have been extended to the setting of multifunctions. The aim of this paper is to investigate a new class of multifunctions called δ-semi-irresolute.

In this paper, spaces X and Y mean topological spaces. For a subset A of X, $\mathrm{cl}\,(A)$ and $\mathrm{int}\,(A)$ represent the closure of A and the interior of A, respectively. A subset A of a space X is said to be regular open if $A = \mathrm{int}\,(\mathrm{cl}\,(A))$ [5]. The δ-interior [6] of a subset A of (X, τ) is the union of all regular open sets of X contained in A is denoted by $\delta - \mathrm{int}\,(A)$. A subset A is called δ-open [6] if $A = \delta - \mathrm{int}\,(A)$. The complement of δ-open set is called δ-closed. The δ-closure of a subset A is defined by $\{x \in X : A \cap \mathrm{int}\,(\mathrm{cl}\,(U)) \neq \emptyset,\ U \in \tau \text{ and } x \in U\}$ and is denoted by $\delta - \mathrm{cl}\,(A)$. A subset A of a space X is said to be semi-open [2] (resp. δ-semi-open [4]) if $A \subset \mathrm{cl}\,(\mathrm{int}\,(A))$ (resp. $A \subset \mathrm{cl}\,(\delta - \mathrm{int}\,(A))$). The family of all δ-semi-open sets of X (resp. containing a point $x \in X$) is denoted

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by $\delta SO(X)$ (resp. $\delta SO(X,x)$). The family of all δ-open sets of X is denoted by $\delta O(X)$. The complement of a δ-semi-open set is said to be δ-semi-closed. The intersection of all δ-semi-closed sets of X containing A is called the δ-semi-closure of A and is denoted by $\delta - \operatorname{scl}(A)$. The union of all δ-semi-open sets of X contained A is called δ-semi-interior of A and is denoted by $\delta - \operatorname{sint}(A)$. A subset U of X is called a δ-semi-neighborhood of a point $x \in X$ if there exists a δ-semi-open set V such that $x \in V \subset U$.

Throughout this paper, $F : X \to Y$ presents a multifunction. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$, see [1]. For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$ and the subset $\{\{x\} \times F(x) : x \in X\} \subset X \times Y$ is called the multigraph of F and is denoted by $G(F)$.

Lemma 1. *Let A and Y be subsets of a space (X, τ). If $A \in \delta SO(X)$ and $Y \in \delta O(X)$, then $A \cap Y \in \delta SO(Y)$.*

Lemma 2. *Let $A \subset Y \subset X$. If $Y \in \delta O(X)$ and $A \in \delta SO(Y)$, then $A \in \delta SO(X)$.*

Lemma 3. *Let A and B be subsets of spaces (X, τ) and (Y, σ), respectively. If $A \in \delta SO(X)$ and $B \in \delta SO(Y)$, then $A \times B \in \delta SO(X \times Y)$.*

Lemma 4. *For a multifunction $F : X \to Y$, (1) $G_F^+(A \times B) = A \cap F^+(B)$; (2) $G_F^-(A \times B) = A \cap F^-(B)$ for any subsets $A \subset X$ and $B \subset Y$, see [3].*

2. Characterizations

In this section, characterizations and properties of lower and upper δ-semi-irresolute multifunctions are introduced.

Definition 5. A multifunction $F : X \to Y$ is said to be:

(a) lower δ-semi-irresolute at a point $x \in X$ if for each δ-semi-open set V of Y such that $x \in F^-(V)$, there exists $U \in \delta SO(X,x)$ such that $U \subset F^-(V)$,

(b) upper δ-semi-irresolute at a point $x \in X$ if for each δ-semi-open set V of Y such that $x \in F^+(V)$, there exists $U \in \delta SO(X,x)$ such that $U \subset F^+(V)$.

(c) lower (upper) δ-semi-irresolute if F has this property at each point of X.

Theorem 6. *Let $F : X \to Y$ be a multifunction. Then the following statements are equivalent:*

(1) F is upper δ-semi-irresolute;

(2) for each $x \in X$ and for each δ-semi-open set V such that $F(x) \subset V$, there exists $U \in \delta SO(X,x)$ such that if $y \in U$, then $F(y) \subset V$;

(3) for each $x \in X$ and for each δ-semi-closed set K such that $x \in F^+(Y\backslash K)$, there exists a δ-semi-closed set H such that $x \in X\backslash H$ and $F^-(K) \subset H$;

(4) $F^+(V) \in \delta SO(X)$ for any δ-semi-open set $V \subset Y$;

(5) $F^-(K) \in \delta SC(X)$ for any δ-semi-closed set $K \subset Y$;

(6) for each point x of X and each δ-semi-neighbourhood V of $F(x)$, $F^+(V)$ is a δ-semi-neighbourhood of x;

(7) for each point x of X and each δ-semi-neighbourhood V of $F(x)$, there exists a δ-semi-neighbourhood U of x such that $F(U) \subset V$;

(8) $\delta\text{-scl}\,(F^-(B)) \subset F^-(\delta\text{-scl}\,(B))$ for any subset B of Y;

(9) $F^+(\delta\text{-sint}\,(N)) \subset \delta\text{-sint}\,(F^+(N))$ for any subset N of Y.

Proof. (1)⇔(2). Obvious.

(2)⇔(3). Let $x \in X$ and K be a δ-semi-closed set of Y such that $x \in F^+(Y\backslash K)$. By (2), there exists $U \in \delta SO(X,x)$ such that $F(U) \subset Y\backslash K$. We have $U \subset F^+(Y\backslash K) = X\backslash F^-(K)$. We obtain $F^-(K) \subset X\backslash U$. Take $H = X\backslash U$. Then, $x \in X\backslash H$ and H is a δ-semi-closed set.

Converse is similar.

(1)⇒(4). Let V be any δ-semi-open set of Y and $x \in F^+(V)$. By (1), there exists $U_x \in \delta SO(X,x)$ such that $U_x \subset F^+(V)$. Therefore, we obtain $F^+(V) \in \delta SO(X)$.

(4)⇒(1). Let V be any δ-semi-open set of Y and $x \in F^+(V)$. By (4), $F^+(V) \in \delta SO(X)$. Take $U = F^+(V)$. Then, $F(U) \subset V$. Hence, F is upper δ-semi-irresolute.

(4)⇔(5). Let K be any δ-semi-closed set of Y. Then, $Y\backslash K$ is a δ-semi-open set of Y. By (5), $F^+(Y\backslash K) \in \delta SO(X)$. It follows that $F^+(Y\backslash K) = X\backslash F^-(K)$. We obtain that $F^-(K)$ is δ-semi-closed in X.

Converse is similar.

(4)⇒(6). Let $x \in X$ and V be a δ-semi-neighbourhood of $F(x)$. Then there exists a δ-semi-open set G of Y such that $F(x) \subset G \subset V$. Therefore, we obtain $x \in F^+(G) \subset F^+(V)$. Since $F^+(G) \in \delta SO(X)$, $F^+(V)$ is a δ-semi-neighbourhood of x.

(6)⇒(7). Let $x \in X$ and V be a δ-semi-neighbourhood of $F(x)$. By (6), $F^+(V)$ is a δ-semi-neighbourhood of x. Take $U = F^+(V)$. Then $F(U) \subset V$.

(7)⇒(1). Let $x \in X$ and V be any δ-semi-open set of Y such that $F(x) \subset V$. Then V is a δ-semi-neighbourhood of $F(x)$. By (7), there exists a δ-semi-neighbourhood U of x such that $F(U) \subset V$. Therefore, there exists $G \in$

$\delta SO(X)$ such that $x \in G \subset U$ and hence $F(G) \subset F(U) \subset V$. We obtain that F is upper δ-semi-irresolute.

(5)⇒(8). For any subset B of Y, δ-scl$\,(B)$ is δ-semi-closed in Y and then $F^-(\delta\text{-scl}\,(B))$ is δ-semi-closed in X. Hence $\delta\text{-scl}\,(F^-(B)) \subset F^-(\delta\text{-scl}\,(B))$.

(8)⇒(5). Let K be any δ-semi-closed set in Y. Then $\delta\text{-scl}\,(F^-(K)) \subset F^-(\delta\text{-scl}\,(K)) = F^-(K)$ and hence $F^-(K)$ is a δ-semi-closed set in X.

(4)⇒(9). For any subset B of Y, δ-sint$\,(B)$ is δ-semi-open in Y and then $F^+(\delta\text{-sint}\,(B))$ is δ-semi-open in X. Hence $F^+(\delta\text{-sint}\,(B)) \subset \delta\text{-sint}\,(F^+(B))$.

(9)⇒(4). Let V be any δ-semi-open set of Y. Then $F^+(V) = F^+(\delta\text{-sint}\,(V)) \subset \delta\text{-sint}\,(F^+(V))$ and hence $F^+(V) \in \delta SO(X)$. □

Theorem 7. *Let $F : X \to Y$ be a multifunction. Then the following statements are equivalent:*

(1) F is lower δ-semi-irresolute;

(2) for each $x \in X$ and for each δ-semi-open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in \delta SO(X, x)$ such that if $y \in U$, then $F(y) \cap V \neq \emptyset$;

(3) for each $x \in X$ and for δ-semi-closed set K such that $x \in F^-(Y \backslash K)$, there exists a δ-semi-closed set H such that $x \in X \backslash H$ and $F^+(K) \subset H$;

(4) $F^-(G) \in \delta SO(X)$ for any δ-semi-open set G of Y;

(5) $F^+(K) \in \delta SC(X)$ for any δ-semi-closed set K of Y;

(6) $\delta\text{-scl}\,(F^+(B)) \subset F^+(\delta\text{-scl}\,(B))$ for any subset B of Y;

(7) $F^-(\delta\text{-sint}\,(N)) \subset \delta\text{-sint}\,(F^-(N))$ for any subset N of Y.

Theorem 8. *Let $F : X \to Y$ be a multifunction and let A be a δ-open set in X. If F is a lower (upper) δ-semi-irresolute, then the restriction multifunction $F\,|_A : A \to Y$ is a lower (resp. upper) δ-semi-irresolute.*

Proof. Suppose that V is an δ-semi-open set in Y. Let $x \in A$ and let $x \in (F\,|_A)^-(V)$. Since F is a lower δ-semi-irresolute multifunction, it follows that there exists a δ-semi-open set G such that $x \in G \subset F^-(V)$. By Lemma 1, we obtain that $x \in G \cap A \in \delta SO(A)$ and $G \cap A \subset (F\,|_A)^-(V)$. Thus, we show that the restriction multifunction $F\,|_A$ is a lower δ-semi-irresolute.

The proof of upper δ-semi-irresolute multifunctions is similar. □

Theorem 9. *Let $\{U_i\}_{i \in I}$ be a δ-open cover of a space X. Then a multifunction $F : X \to Y$ is upper δ-semi-irresolute (resp. lower δ-semi-irresolute) if and only if the restriction $F\,|_{U_i} : U_i \to Y$ is upper δ-semi-irresolute (resp. lower δ-semi-irresolute) for each $i \in I$.*

Proof. Let $i \in I$ and V be any δ-semi-open set of Y. Since F is upper δ-semi-irresolute, $F^+(V)$ is δ-semi-open in X. By Lemma 1, $(F\,|_{U_i})^+(V) = F^+(V) \cap U_i$ is δ-semi-open in U_i and hence $F\,|_{U_i}$ is upper δ-semi-irresolute.

Conversely, let V be any δ-semi-open set of Y. Since $F\ |_{U_i}$ is upper δ-semi-irresolute for each $i \in I$, $(F\ |_{U_i})^+(V) = F^+(V) \cap U_i$ is δ-semi-open in U_i. By Lemma 2, $(F\ |_{U_i})^+(V)$ is δ-semi-open in X for each $i \in I$. We obtain that $F^+(V) = \underset{i\in\Lambda}{\cup}(F\ |_{U_i})^+(V)$ is δ-semi-open in X. Hence, F is upper δ-semi-irresolute. □

Theorem 10. *Let $F : X \to Y$ and $G : Y \to Z$ be multifunctions. If is upper (lower) δ-semi-irresolute and G is upper (lower) δ-semi-irresolute, then $G \circ F : X \to Z$ is an upper (lower) δ-semi-irresolute multifunction.*

Proof. Let $V \subset Z$ be any δ-semi-open set. From the definition of $G \circ F$, we have $(G\circ F)^+(V) = F^+(G^+(V))$ (resp. $(G\circ F)^-(V) = F^-(G^-(V))$). Since G is an upper (lower) δ-semi-irresolute multifunction, it follows that $G^+(V)$ (resp. $G^-(V)$) is a δ-semi-open set. Since F is an upper (lower) δ-semi-irresolute multifunction, it follows that $F^+(G^+(V))$ (resp. $F^-(G^-(V))$) is a δ-semi-open set. It shows that $G\circ F$ is an upper (resp. lower) δ-semi-irresolute multifunction. □

Definition 11. A multifunction $F : X \to Y$ is said to be: (a) lower irresolute at a point $x \in X$ if for each semi-open set V of Y such that $x \in F^-(V)$, there exists a semi-open set U containing x such that $U \subset F^-(V)$; (b) upper irresolute at a point $x \in X$ if for each semi-open set V of Y such that $x \in F^+(V)$, there exists a semi-open set U containing x such that $U \subset F^+(V)$.

Definition 12. Let (X, τ) be a topological space. The collection of all regular open sets forms a base for a topology τ^*. It is called the semi-regularization. In case when $\tau = \tau^*$, the space (X, τ) is called semi-regular, see [5].

Theorem 13. *Let $F : X \to Y$ be a multifunction and X and Y be semi-regular. Then F is a lower (resp. upper) δ-semi-irresolute multifunction if and only if F is lower (resp. upper) irresolute.*

Theorem 14. *Suppose that for each $i \in I$, (X_i, τ_i), (Y_i, υ_i) are topological spaces. Let $F_i : X_i \to Y_i$ be a multifunction for each $i \in I$ and let $F : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ be defined by $F((x_i)) = \prod_{i\in I} F_i(x_i)$ from the product space $\prod_{i\in I} X_i$ to the product space $\prod_{i\in I} Y_i$. If F is an upper (lower) δ-semi-irresolute multifunction, then each F_i is an upper (resp. lower) δ-semi-irresolute multifunction for each $i \in I$.*

Proof. Let $V_i \subset Y_i$ be a δ-semi-open set. Then $V_i \times \prod_{i\neq j} Y_j$ is a δ-semi-open set. Since F is an upper (lower) δ-semi-irresolute multifunction, it follows that

$$F^+(V_i \times \prod_{i\neq j} Y_j) = F_i^+(V_i) \times \prod_{i\neq j} X_j$$

(resp. $F^-(V_i \times \prod_{i\neq j} Y_j) = F_i^-(V_i) \times \prod_{i\neq j} X_j$) is a δ-semi-open set. Consequently, we obtain that $F_i^+(V_i)$ (resp. $F_i^-(V_i)$) is a δ-semi-open set. Thus, we show that F_i is an upper (resp. lower) δ-semi-irresolute multifunction. □

Theorem 15. *Suppose that (X,τ) and (X_i,τ_i) are topological spaces where $i \in I$. Let $F : X \to \prod_{i\in I} X_i$ be a multifunction from X to the product space $\prod_{i\in I} X_i$ and let $P_i : \prod_{i\in I} X_i \to X_i$ be the projection for each $i \in I$. If F is an upper (lower) δ-semi-irresolute multifunction, then $P_i \circ F$ is an upper (resp. lower) δ-semi-irresolute multifunction for each $i \in I$.*

Proof. Let V_{i_0} be a δ-semi-open set in (X_{i_0}, τ_{i_0}). Then $(P_{i_0} \circ F)^+(V_{i_0}) = F^+(P_{i_0}^+(V_{i_0})) = F^+(V_{i_0} \times \prod_{i\neq i_0} X_i)$. Since F is an upper δ-semi-irresolute multifunction and since $V_{i_0} \times \prod_{i\neq i_0} X_i$ is a δ-semi-open set, it follows that $F^+(V_{i_0} \times \prod_{i\neq i_0} X_i)$ is δ-semi-open in (X,τ). It shows that $P_{i_0} \circ F$ is an upper δ-semi-irresolute multifunction. Hence, we obtain that $P_i \circ F$ is an upper δ-semi-irresolute multifunction for each $i \in I$.

The proof for lower δ-semi-irresolute multifunction is similar. □

Theorem 16. *Suppose that (X,τ), (Y,υ), (Z,ω) are topological spaces and $F_1 : X \to Y$, $F_2 : X \to Z$ are multifunctions. Let $F_1 \times F_2 : X \to Y \times Z$ be a multifunction which is defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is an upper (lower) δ-semi-irresolute multifunction, then F_1 and F_2 are upper (resp. lower) δ-semi-irresolute multifunctions.*

Proof. Let $x \in X$ and let $K \subset Y$, $H \subset Z$ be δ-semi-open sets such that $x \in F_1^+(K)$ and $x \in F_2^+(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and so $F_1(x) \times F_2(x) = (F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is an upper δ-semi-irresolute multifunction, it follows that there exists a δ-semi-open set U containing x such that $U \subset (F_1 \times F_2)^+(K \times H)$. We obtain that $U \subset F_1^+(K)$ and $U \subset F_2^+(H)$. Thus, we obtain that F_1 and F_2 are upper δ-semi-irresolute multifunctions.

The other proof is similar. □

Theorem 17. *Let $F : X \to Y$ be a multifunction. If the graph multifunction of F is upper δ-semi-irresolute, F is upper δ-semi-irresolute.*

Proof. Suppose that $G_F : X \to X \times Y$ is upper δ-semi-irresolute. Let $x \in X$ and V be any δ-semi-open set of Y containing $F(x)$. Since $X \times V$ is δ-semi-open in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in \delta SO(X, x)$ such that $G_F(U) \subset X \times V$. By Lemma 4, we have $U \subset G_F^+(X \times V) = F^+(V)$ and $F(U) \subset V$. This shows that F is upper δ-semi-irresolute. □

Theorem 18. *A multifunction $F : X \to Y$ is lower δ-semi-irresolute if $G_F : X \to X \times Y$ is lower δ-semi-irresolute.*

Definition 19. The δ-semi-frontier of a subset A of a space X, denoted by $\delta - \mathrm{sFr}\,(A)$, is defined by $\delta - \mathrm{sFr}\,(A) = \delta - \mathrm{scl}\,(A) \cap \delta - \mathrm{scl}\,(X \backslash A) = \delta - \mathrm{scl}\,(A) \backslash \delta - \mathrm{sint}\,(A)$.

Theorem 20. *The set all points of X at which a multifunction $F : X \to Y$ is not upper δ-semi-irresolute (lower δ-semi-irresolute) is identical with the union of the δ-semi-frontier of the upper (lower) inverse images of δ-semi-open sets containing (meeting) $F(x)$.*

Proof. Let $x \in X$ at which F is not upper δ-semi-irresolute. Then there exists a δ-semi-open set V of Y containing $F(x)$ such that $U \cap (X \backslash F^+(V)) \neq \emptyset$ for every $U \in \delta SO(X, x)$. Therefore, we have $x \in \delta - \mathrm{scl}\,(X \backslash F^+(V)) = X \backslash \delta - \mathrm{sint}\,(F^+(V))$ and x$\in F^+(V)$. Thus, we obtain $x \in \delta - \mathrm{sFr}\,(F^+(V))$.

Conversely, suppose that V is a δ-semi-open set of Y containing $F(x)$ such that $x \in \delta - \mathrm{sFr}\,(F^+(V))$. If F is upper δ-semi-irresolute at x, then there exists $U \in \delta SO(X, x)$ such that $U \subset F^+(V)$; hence $x \in \delta - \mathrm{sint}\,(F^+(V))$. This is a contradiction and hence F is not upper δ-semi-irresolute at x.

The case for lower δ-semi-irresolute is similar. □

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