International Journal of Pure and Applied Mathematics
Volume 21 No. 1 2005, 89-92

A NOTE ON MINIMAL SURFACES IN CP^n

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Abstract: In this note, we will use the method of harmonic map to discuss the minimal immersion in CP^n and obtain that compact minimal flat surfaces with some constant Kähler angle are never isotropic.

AMS Subject Classification: 53C40
Key Words: minimal surface, Kähler angle, isotropic

1. Introduction

Recently, Miyata in [2] proved that compact, totally real, minimal flat surfaces in $CP^n(4)$ are never isotropic. Now, in this paper, we obtain a generalization of Miyata's result as following

Theorem. *Let M be a compact minimal flat surfaces in $CP^n(4)$ with constant Kähler angle $\theta : \tan^2 \frac{\theta}{2} \neq \frac{k+1}{k}$ or $\frac{k}{k+1}$ $(k \neq 0, \pm 1)$, then M is not isotropic.*

Remark. When $\tan^2 \frac{\theta}{2} = 1$, we obtain the result of Miyata (see Proposition 4.2 of [2]).

2. Preliminaries

Let M be a smooth manifold and V be a complex vector subbundle of the trivial bundle $\underline{C}^{n+1} = M \times C^{n+1}$ over M. Then V has a connection ∇, induced from the trivial connection on $\underline{C}^{n+1}$, given by $\nabla s = \pi_V ds$, where s is a section of V and $\pi_V : \underline{C}^{n+1} \to V$ denotes orthogonal projection onto V.

Received: April 6, 2005

Let L be the universal line bundle over CP^n defined by $L = \{(p,v) \in CP^n \times C^{n+1} | v \in p$, then both L and its orthogonal complement $L^\perp$ have induced connections and Hermitian metrics. Let $T^{(1,0)}CP^n$ (resp. $T^{(0,1)}CP^n$) denote (1,0)-part (resp. (0,1)-part) of the complexification TCP^{nC} of TCP^n. Thus we have a hermitian metric and a connection of $\mathrm{Hom}\,(L, L^\perp)$ and there is a canonical isomorphism $h\colon T^{(1,0)}CP^n \to \mathrm{Hom}\,(L, L^\perp)$ given by $h(X)s = \pi_{L^\perp} ds(X)$, where $X \in T^{(1,0)}CP^n$ and s is a local section of L.

For a smooth manifold M, there is a bijective correspondence between (smooth) complex line subbundles of $\underline{C}^{n+1}$ and smooth maps $\varphi\colon M \to CP^n$, given by $\varphi \leftrightarrow \varphi^* L$. If M is a Riemannian surface, we have (see [2]):

$$\partial\colon T^{(1,0)}M \otimes \varphi^* L \to \varphi^* L^\perp, \quad \partial(\frac{\partial}{\partial z} \otimes s) = (h \circ d^{(1,0)}\varphi(\frac{\partial}{\partial z}))(s) = \pi_{L^\perp} ds(\frac{\partial}{\partial z}),$$

and

$$\overline{\partial}\colon T^{(0,1)}M \otimes \varphi^* L \to \varphi^* L^\perp, \quad \overline{\partial}(\frac{\partial}{\partial \overline{z}} \otimes s) = (h \circ d^{(0,1)}\varphi(\frac{\partial}{\partial \overline{z}}))(s) = \pi_{L^\perp} ds(\frac{\partial}{\partial \overline{z}}).$$

Wolfson in [3] shows that φ is harmonic if and only if ∂ (resp. $\overline{\partial}$) is a holomorphic (resp. an antiholomorphic) bundle map and construct inductively an associated sequence:

$$\cdots\cdots L_{-2}, L_{-1}, L_0, L_1, L_2, \cdots\cdots$$

and bundle maps

$$\partial_p\colon T^{(1,0)}M \otimes L_p \to L_{p+1}\,, \quad \overline{\partial}_p\colon T^{(0,1)}M \otimes L_p \to L_{p-1}.$$

Here $L_p = \varphi_p^* L$ for a suitable harmonic map $\varphi_p\colon M \to CP^n$ and ∂_p (resp. $\overline{\partial}_p$) is essentially the map ∂ (resp. $\overline{\partial}$) defined above for the map φ_p. Then ∂_p (resp. $\overline{\partial}_p$) is a holomorphic (resp. antiholomorphic) bundle map. We call the sequence $\{\varphi_p\}$ the *harmonic sequence* determined by φ with $\varphi = \varphi_p$ for some p, and the sequence $\{L_p\}$ the *associated bundle sequence*. φ_p is conformal if and only if $L_{(p+1)} \perp L_{(p-1)}$. If $\{\varphi_p\}$ terminates at one end, then it terminates at both ends and all the element of $\{L_p\}$ are mutually orthogonal, i.e. $L_{(p)} \perp L_{(q)}$ for $p \neq q$. If $\{\varphi_p\}$ satisfies this condition, φ is called *isotropic*, so that each $\{\varphi_p\}$ is conformal.

Now we give a local description of the harmonic sequence of an isotropic harmonic map φ. Let z be a local complex coordinate on M. Then, for example, we can choose a meromorphic local section f_p of L_p such that

$$f_{p+1} = \partial_p(\frac{\partial}{\partial z} \otimes f_p).$$

Defining function γ_p by

$$\gamma_p = \begin{cases} \frac{|f_{p+1}|^2}{|f_p|^2}, & f_p \not\equiv 0\,, \\ 0, & f_p \equiv 0\,, \end{cases}$$

then we have

$$\frac{\partial}{\partial z} f_p = f_{p+1} + \frac{\partial}{\partial z} \lg |f_p|^2 f_p \;, \tag{1}$$

$$\frac{\partial}{\partial \bar{z}} f_p = -\gamma_{p-1} f_{p-1}. \tag{2}$$

Since $\frac{\partial^2}{\partial z \partial \bar{z}} f_p = \frac{\partial^2}{\partial \bar{z} \partial z} f_p$, we have

$$\frac{\partial^2}{\partial z \partial \bar{z}} \lg |f_p|^2 = \gamma_p - \gamma_{p-1} \tag{3}$$

and the unintegrated Plücker formula

$$\frac{\partial^2}{\partial z \partial \bar{z}} \lg \gamma_p = \gamma_{p+1} - 2\gamma_p + \gamma_{p-1}. \tag{4}$$

Let g_p and θ_p denote the induced metric of M by φ_p and the Kähler angle of φ_p, respectively. Let $\triangle_p$ and K_P denote the Laplacian and the Gaussian curvature of (M, g_p), respectively. Then we have (see [1]):

$$g_p = \sigma_p dz d\bar{z}, \quad \sigma_p = \gamma_p + \gamma_{p-1}\,, \tag{5}$$

$$\tan^2 \frac{\theta_p}{2} = \frac{\gamma_{p-1}}{\gamma_p}\,, \tag{6}$$

$$\triangle_p = -\frac{4}{\sigma_p} \frac{\partial^2}{\partial z \partial \bar{z}}\,, \tag{7}$$

$$K_p = -\frac{2}{\sigma_p} \frac{\partial^2}{\partial z \partial \bar{z}} \lg \sigma_p\,. \tag{8}$$

3. Proof of Theorem

Let $\tan^2 \frac{\theta}{2} = a$ =constant, then $a \neq 2$ or $\frac{1}{2}$. From (6), we have $\gamma_{-1} = a\gamma_0$. Using $K_0 = 0$ and (8), we get $\gamma_1 = (2-a)\gamma_0 \neq 0$. From (4) and $\gamma_{-1} = a\gamma_0$, we have $\gamma_{-2} = (2a-1)\gamma_0 \neq 0$. Then, from these, we have

$$
\begin{aligned}
K_1 &= -\frac{2}{\gamma_1+\gamma_0}\frac{\partial^2}{\partial z \partial \bar{z}} \lg(\gamma_1+\gamma_0) = 0, \\
\tan^2\frac{\theta_1}{2} &= \frac{\gamma_0}{\gamma_1} = \text{constant}(\neq 2, \frac{1}{2}), \\
K_{-1} &= -\frac{2}{\gamma_{-1}+\gamma_{-2}}\frac{\partial^2}{\partial z \partial \bar{z}} \lg(\gamma_{-1}+\gamma_{-2}) = 0, \\
\tan^2\frac{\theta_{-1}}{2} &= \frac{\gamma_{-2}}{\gamma_{-1}} = \text{constant}(\neq 2, \frac{1}{2}),
\end{aligned}
$$

i.e. each of φ_1, φ_{-1} has constant Kähler angle and the induced metric is flat. Then we can iterate this procedure and obtain that each φ_p has constant Kähler angle and the induced metric is flat. So the sequence $\{\varphi_p\}$ has no end and M is not isotropic.

Acknowledgements

Research is supported by National Natural Science Foundation of P.R. China (No. 10301008) and Support Program for outstanding young teachers of Southeast University.

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