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MODIFIED DECOMPOSITION METHOD FOR SOLVING A SYSTEM OF THIRD-ORDER OBSTACLE PROBLEMS

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Abstract: In this paper, we present an efficient numerical algorithm for approximate solutions of a system of third-order boundary value problems associated with obstacle, unilateral and contact problems. The Adomian decomposition method and a modified form of this method are applied to construct the numerical solution. A numerical example is given to illustrate the applicability and efficiency of the technique.

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1. Introduction

In this paper, we use Adomian decomposition method (in short ADM) [3, 4] for obtaining approximate solutions of a system of third-order boundary value problem of the type

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$$u''' = \begin{cases} f(x), & a \leq x \leq c, \\ p(x)u(x) + f(x) + r, & c \leq x \leq d, \\ f(x), & d \leq x \leq b, \end{cases} \tag{1.1}$$

with the boundary conditions

$$u(a) = \alpha, \quad u'(a) = \beta_1, \text{ and } \quad u'(b) = \beta_2, \tag{1.2}$$

and the continuity conditions of u, u' and u'' at c and d. Here, f and p are continuous functions on $[a, b]$ and $[c, d]$, respectively. The parameters r, α, β_1 and β_2 are real finite constants. Such type of systems arises in the study of obstacle, unilateral, moving and free boundary value problems and has important applications in other branches of pure and applied sciences (see, for example [3-8, 10-12] and the references therein). In general it is not possible to obtain the analytical solution of (1.1) for arbitrary choices of $f(x)$ and $p(x)$. We usually resort to some numerical methods for obtaining an approximate solution of (1.1). Noor and Khalifa [20] have used collocation method with quintic B-splines as basis functions to solve a special form of (1.1), namely,

$$u''' = \begin{cases} 0, & 0 \leq x \leq \frac{1}{4}, \\ u(x) - 1, & \frac{1}{4} \leq x \leq \frac{3}{4}, \\ 0, & \frac{3}{4} \leq x \leq 1, \end{cases} \tag{1.3}$$

with the boundary conditions

$$u(0) = 0, \quad u'(0) = 0, \text{ and } \quad u'(1) = 0, \tag{1.4}$$

and the continuity conditions of u, u' and u'' at $1/4$ and $3/4$. After this, Al-Said and co-workers [4-7] have developed first- and second-order methods for solving problems (1.1) and (1.3). More recently Siraj-ul-Islam et al [14] have applied non polynomial spline functions for solving such a type of third-order system of differential equations associated with obstacle and unilateral problems.

The Adomian decomposition method [3, 4] will be effectively used to approach problem (1.1). The Adomian algorithm assumes a series solution for the unknown function $u(x)$. Unlike the method of separation of variables that require initial and boundary conditions, the decomposition method may provide an analytic solution by using the initial conditions only. The boundary conditions can be used only to justify the obtained result. The Adomian decomposition method has many advantages over the classical techniques mainly, it avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculations.

The convergence of the decomposition series have been investigated by several authors, see [14-18]. They obtained some results about the speed of convergence of this method. Abbaoui and Cherruault [1] have proposed a new approach of convergence of the decomposition series. The authors have given a new condition for obtaining convergence of the decomposition series to the classical presentation of the ADM in [1].

The paper is organized as follows. In Section 2 we extend application of the decomposition method to construct our numerical solutions for of the third-order boundary value problem. In Section 3, we present a numerical experiments to illustrate the efficiency and simplicity of the method.

2. Decomposition Method

In this section, we outline the steps to obtain approximate solution of (1.1) using the ADM. To begin, it is convenient to rewrite the third-order boundary value problem in the standard operator from

$$L_x u = \begin{cases} f(x), & a \leq x \leq c, \\ p(x)u(x) + f(x) + r, & c \leq x \leq d, \\ f(x), & d \leq x \leq b, \end{cases} \tag{2.1}$$

where the differential operator L_x is given by

$$L_x = \frac{d^3}{dx^3}. \tag{2.2}$$

Equation (2.1) can be expressed in terms of unit step function as

$$L_x u = f(x)(y(x-a) - y(x-c)) + (p(x)u(x) + f(x) + r)(y(x-c) \tag{2.3}$$

$$-y(x-d)) + f(x)(y(x-d) - y(x-b)),$$

where $y(x)$ is the Heaviside function.

The inverse operator L_x^{-1} is therefore considered a three-fold integral operator defined by

$$L_x^{-1}(.) = \int_a^x \int_a^x \int_a^x (.)dxdxdx\,. \tag{2.4}$$

Operating with L_x^{-1} on (2.3) and using the boundary conditions at $x = b$ yields

$$u(x) = \alpha + \beta_1(x-a) + \frac{1}{2!}A(x-a)^2 + L_x^{-1}\Big[f(x)(y(x-a) - y(x-c)) + (p(x)u(x) + f(x) + r)(y(x-c) - y(x-d)) + f(x)(y(x-d) - y(x-b))\Big], \quad (2.5)$$

where the constant

$$A = u'(b), \quad (2.6)$$

will be determined later by using the boundary condition at $x = b$.

Now, we decompose the unknown function $u(x)$ a sum of components defined by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \quad (2.7)$$

The 0-th component is usually taken to be all terms arise from the initial conditions and the integration of the source term in (2.1), i.e.,

$$u_0 = \alpha + \beta_1(x-a) + \frac{1}{2!}A(x-a)^2 + L_x^{-1}\Big[f(x)(y(x-a) - y(x-c)) + (f(x) + r)(y(x-c) - y(x-d)) + f(x)(y(x-d) - y(x-b))\Big]. \quad (2.8)$$

The remaining components $u_n(x)$, $n \geq 1$, can be completely determined such that each term is computed by using the previous term. Since u_0 is known,

$$u_n = L_x^{-1}\Big[p(x)u_{n-1}(y(x-c) - y(x-d))\Big], \quad n \geq 1. \quad (2.9)$$

A slight modification to the ADM was proposed by Wazwaz [23] that gives some flexibility in the choice of the 0-th component u_0 to be any simple term and modify the term u_1 accordingly. Since the computation in (2.9) depends heavily on u_0 the whole computations to find the solution will be simplified considerably. For example an alternative to (2.9) might be

$$\begin{aligned} u_0 &= \alpha + \beta_1(x-a) + \tfrac{1}{2!}A(x-a)^2, \\ u_1 &= L_x^{-1}\Big[f(x)(y(x-a) - y(x-c)) + (f(x) + r)(y(x-c) - y(x-d)) \\ &\quad + f(x)(y(x-d) - y(x-b))\Big] + L_x^{-1}\Big[p(x)u_0(y(x-c) - y(x-d))\Big], \\ u_n &= L_x^{-1}\Big[p(x)u_{n-1}(y(x-c) - y(x-d))\Big], \quad n \geq 2. \end{aligned} \quad (2.10)$$

Finally an N-term approximate solution is given by

$$\Phi_N(x) = \sum_{n=0}^{N-1} u_n(x), \quad N \geq 1, \tag{2.11}$$

and the exact solution is $u(x) = \lim_{N\to\infty} \Phi_N$.

To show the effectiveness of the proposed decomposition method and to give a clear overview of the methodology, one example of the third-order obstacle boundary value problems (1.1) will be discussed in the following section.

3. Applications and Numerical Results

To illustrate the application of the numerical method developed in the previous sections we consider the third-order obstacle boundary value problem of finding u such that

$$\left.\begin{array}{ll} -u''' \geq f(x)\,, & \text{on } \Omega = [0,1]\,, \\ u \geq \psi(x)\,, & \text{on } \Omega = [0,1]\,, \\ [-u''' - f(x)][u - \psi(x)] = 0\,, & \text{on } \Omega = [0,1]\,, \\ u(0) = 0\,, \quad u'(0) = 0, \quad u'(1) = 0\,, & \end{array}\right\} \tag{3.1}$$

where $f(x)$ is a continuous function and $\psi(x)$ is the obstacle function. We study the problem (3.1) in the framework of variational inequality approach. To do so, we first define the set K as

$$K = \{v : v \in H_0^2(\Omega) : v \geq \psi \quad \text{on } \Omega\},$$

which is a closed convex set in $H_0^2(\Omega)$, where $H_0^2(\Omega)$ is a Sobolev space, which is in fact a Hilbert space (for more details, see [9, 12, 13]). One can easily show that the energy functional associated with the problem (3.1) is

$$\begin{aligned} I[v] &= -\int_0^1 \left(\frac{d^3v}{dx^3}\right)\left(\frac{dv}{dx}\right)dx - 2\int_0^1 f(x)\left(\frac{dv}{dx}\right)dx, \quad \text{for all } \tfrac{dv}{dx} \in K \\ &= \int_0^1 \left(\frac{d^2v}{dx^2}\right)^2 dx - 2\int_0^1 f(x)\left(\frac{dv}{dx}\right)dx \\ &= \langle Tv, g(v)\rangle - 2\langle f, g(v)\rangle\,, \end{aligned} \tag{3.2}$$

where

$$\langle Tu, g(v)\rangle = \int_0^1 \left(\frac{d^2u}{dx^2}\right)\left(\frac{d^2v}{dx^2}\right)dx, \tag{3.3}$$

and $g = \dfrac{d}{dx}$ is the linear operator.

It is clear that the operator T defined by (3.3) is linear, g-symmetric and g-positive. Using the technique of Noor [16, 17], one can easily show that the minimum of the functional $I[v]$ defined by (3.2) associated with the problem (3.1) on the closed convex set K can be characterized by the variational inequality

$$\langle Tu, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle, \quad \text{for all } g(v) \in K. \tag{3.4}$$

Now using the idea of Lewy and Stampacchia [15], the problem (3.1) may be written as

$$\left.\begin{array}{l} -u''' + \nu\{u-\psi\}(u-\psi) = f, \quad 0 < x < 1\,, \\ u(0) = u'(0) = u'(1) = 0\,, \end{array}\right\} \tag{3.5}$$

where

$$\nu(t) = \begin{cases} 1 & \text{for } t \geq 0\,, \\ 0 & \text{for } t < 0\,, \end{cases} \tag{3.6}$$

is a discontinuous function and is known as the penalty function, and $\psi(x)$ is the given obstacle function defined by

$$\psi(x) = \begin{cases} -1, & \text{for } 0 \leq x \leq \frac{1}{4} \text{ and } \frac{3}{4} \leq x \leq 1\,, \\ 1, & \text{for } \frac{1}{4} \leq x \leq \frac{3}{4}. \end{cases} \tag{3.7}$$

From equations (3.5)-(3.7), we obtain the following system of differential equations

$$u''' = \begin{cases} f & \text{for } 0 \leq x \leq \frac{1}{4} \text{ and } \frac{3}{4} \leq x \leq 1, \\ u + f - 1 & \text{for } \frac{1}{4} \leq x \leq \frac{3}{4}, \end{cases} \tag{3.8}$$

with the boundary conditions

$$u(0) = u'(0) = u'(1) = 0 \tag{3.9}$$

and the condition of continuity of u, u' and u'' at $x = \frac{1}{4}$ and $\frac{3}{4}$. Note that the problem (3.8) is a special form of the system (1.1) with $p(x) = 1$ and $r = -1$.

Example. When $f = 0$, the system of differential equations (3.8) reduces to

$$u''' = \begin{cases} 0, & \text{for } 0 \leq x \leq \frac{1}{4} \text{ and } \frac{3}{4} \leq x \leq 1\,, \\ u - 1, & \text{for } \frac{1}{4} \leq x \leq \frac{3}{4}\,, \end{cases} \tag{3.10}$$

with the boundary conditions (3.9). The analytical solution for this problem is

$$u(x) = \begin{cases} \frac{1}{2}a_1 x^2, & 0 \le x \le \frac{1}{4}\,, \\ 1 + a_2 e^x + e^{-\frac{x}{2}}[a_3 \cos\frac{\sqrt{3}}{2}x + a_4 \sin\frac{\sqrt{3}}{2}x], & \frac{1}{4} \le x \le \frac{3}{4}\,, \\ \frac{1}{2}a_5 x(x-2) + a_6, & \frac{3}{4} \le x \le 1. \end{cases} \tag{3.11}$$

To find the constants $a_i,\ i = 1, 2, \ldots, 6$, we apply the continuity conditions of u, u' and u'' at $x = \frac{1}{4}$ and $\frac{3}{4}$, which leads to the following system of linear equations

$$\begin{bmatrix} \frac{1}{32} & -S_1 & -S_2 CS_1 & -S_2 SC_1 & 0 & 0\,, \\ \frac{1}{4} & -S_1 & \frac{1}{2}S_2(\sqrt{3}SC_1 + CS_1) & -\frac{1}{2}S_2(\sqrt{3}CS_1 - SC_1) & 0 & 0\,, \\ 1 & -S_1 & -\frac{1}{2}S_2(\sqrt{3}SC_1 - CS_1) & \frac{1}{2}S_2(\sqrt{3}CS_1 + SC_1) & 0 & 0 \\ 0 & S_3 & S_4 CS_2 & S_4 SC_2 & \frac{15}{32} & -1 \\ 0 & S_3 & -\frac{1}{2}S_4(\sqrt{3}SC_2 + CS_2) & \frac{1}{2}S_4(\sqrt{3}CS_2 - SC_2) & \frac{1}{4} & 0 \\ 0 & S_3 & \frac{1}{2}S_4(\sqrt{3}SC_2 - CS_2) & \frac{1}{2}S_4(-\sqrt{3}CS_2 - SC_2) & -1 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

where

$$S_1 = \exp(\frac{1}{4}), \quad S_2 = \exp(-\frac{1}{8}), \quad S_3 = \exp(\frac{3}{4}), \quad S_4 = \exp(-\frac{3}{8}),$$

$$CS_1 = \cos\frac{\sqrt{3}}{8}, \quad SC_1 = \sin\frac{\sqrt{3}}{8}, \quad CS_2 = \cos\frac{3\sqrt{3}}{8}, \quad \text{and} \quad SC_2 = \sin\frac{3\sqrt{3}}{8}.$$

One can find the exact solution of this system of linear equation by using Gaussian elimination.

To calculate the components of the decomposition series (2.7) for $u(x)$ by using the decomposition method outlined in the previous section, we consider the following three cases:

Case I. For $0 \le x \le \frac{1}{4}$. In this case we implement the Adomian decomposition method and obtain the recursive relation

$$\begin{aligned} u_0 &= \alpha + \beta_1 x + \frac{1}{2!}u''(0)x^2 \\ u_n &= L_x^{-1}(u_{n-1}), \quad n \ge 1. \end{aligned} \tag{3.12}$$

The initial condition $u''(0)$ is directly taken from the analytical solution (3.11). Nonetheless, such approach is needed to evaluate the accuracy of the numerical scheme.

Case II. For $\frac{1}{4} \leq x \leq \frac{3}{4}$. To determine the components u_n, $n \geq 0$, the modified decomposition method will be implemented in this case. Although a slight change is made in this recently developed modification [23], it introduces a qualitative tool that facilitates the computational work. In this approach, we split the terms (2.8) into two parts, one is assigned to the 0-th component $u_0(x)$and the remaining part is assigned to $u_1(x)$ among other terms. On these identifications, we obtain the recursive relation

$$\begin{aligned} u_0 &= u(\frac{1}{4}), \\ u_1 &= u'(\frac{1}{4})(x-\frac{1}{4}) + \frac{1}{2}u''(\frac{1}{4})(x-\frac{1}{4})^2 + L_x^{-1}(-1) + L_x^{-1}(u_0), \\ u_n &= L_x^{-1}(u_{n-1}), \quad n \geq 2. \end{aligned} \tag{3.13}$$

The initial conditions $u(\frac{1}{4})$, $u'(\frac{1}{4})$ and $u''(\frac{1}{4})$ are taken directly from the approximate solution obtained in Case I.

Case III. For $\frac{3}{4} \leq x \leq 1$. As in the previous case, we obtain the recursive relation

$$\begin{aligned} u_0 &= u(\frac{3}{4}), \\ u_1 &= u'(\frac{3}{4})(x-\frac{3}{4}) + \frac{1}{2}u''(\frac{3}{4})(x-\frac{3}{4})^2 + L_x^{-1}(u_0), \\ u_n &= L_x^{-1}(u_{n-1}), \quad n \geq 2. \end{aligned} \tag{3.14}$$

The initial conditions $u(\frac{3}{4})$, $u'(\frac{3}{4})$ are taken directly from the approximate solution obtained in Case II and $u''(\frac{3}{4})$ was determined by using the boundary condition at $x = 1$.

Table 1 shows the analytical solution, the numerical solution and the absolute errors obtained using the decomposition method. It is to be noted that three terms only were used in evaluating the numerical solutions. We achieved a very good approximation with the actual solution of the equation by using only 3-terms of the decomposition series solution derived above. It is evident that the overall errors can be made smaller by adding new terms of the decomposition series.

Numerical solutions of Example above show a high degree of accuracy and in most cases Φ_N, the N-term approximations accurate for quite low values of N. The solutions are very rapidly convergent by utilizing the ADM. The numerical

x	Analytical solution	Numerical solution	Absolute error
0.0	0.00000000000	0.00000000000	0.0000E-00
0.1	0.00121955481	0.00121957513	2.0325E-08
0.2	0.00487821924	0.00487886968	6.5044E-07
0.3	0.01095533525	0.01095533525	2.3765E-16
0.4	0.01895608939	0.01895608939	1.7694E-16
0.5	0.02791612717	0.02791612717	1.3183E-16
0.6	0.03685884727	0.03685884727	2.7755E-17
0.7	0.04481674960	0.04481674960	1.4294E-15
0.8	0.05085037524	0.05085139301	1.0177E-06
0.9	0.05448245887	0.05451065438	2.8195E-05
1.0	0.05569315342	0.05582638144	1.3322E-04

Table 1: Numerical results for equation (3.10)

results we obtained justify the advantage of this methodology, even in the few terms approximation is accurate. Furthermore, as the decomposition method does not require discretization of the variable, i.e., time and space, it is not effected by computation round off error and one is not faced with necessity of large computer memory and time. Also, the illustrations show the dependence of the rapid convergence depend on the character and behavior of the solutions just as in a closed form solutions.

4. Conclusions

In this paper we presented an efficient algorithm for the solution of third order boundary value problems based on Adomian decomposition method and a modified form of this method. The present method enables us to approximate the solution at every interval of the range of integration. The approximate solutions are calculated in the form of a convergent series with easily computable components. The results obtained are very encouraging and our method provides highly accurate numerical solutions without spatial discretization of the problem.

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