International Journal of Pure and Applied Mathematics
Volume 21 No. 1 2005, 105-116

SENSITIVITY ANALYSIS OF SOLUTIONS FOR GENERALIZED PARAMETRIC NONLINEAR QUASIVARIATIONAL INEQUALITIES

Shin Min Kang[1 §], Haiyan Gao[2], Zeqing Liu[3], Sang Keun Lee[4]
[1,4]Department of Mathematics
Research Institute of Natural Sciences
Gyeongsang National University
Chinju, 660-701, KOREA
[1]e-mail: smkang@nongae.gsnu.ac.kr
[2,3]Department of Mathematics
Liaoning Normal University
P.O. Box 200, Dalian, Liaoning, 116029, P.R. CHINA
[2]e-mail: haiyangao@mail.china.com
[3]e-mail: zeqingliu@dl.cn

Abstract: In this paper, we introduce and study a new class of generalized parametric nonlinear quasivariational inequalities, and prove an existence theorem of solutions for this kind of variational inequalities involving strongly monotone mappings. We also discuss the sensitivity of solutions for the generalized parametric nonlinear quasivariational inequality. Our results extend and unify the corresponding results due to Verma, Huang and Cho-Kim-Huang-Kang and others.

AMS Subject Classification: 47J20, 49J40
Key Words: generalized parametric nonlinear quasivariational inequality, sensitivity analysis, contraction mapping

Received: April 26, 2005

[§]Correspondence author

1. Introduction

Generalized variational inequalities provided an important and effective technique for studying nonlinear problems arising from applied mathematics, physics, optimization and controls, economics, transportation equilibrium and engineering, science, etc. Cho-Kim-Huang-Kang [2], Huang [5] and Verma [28]-[30] have investigated the existence of solutions for different kinds of variational inequalities. Liu-Debnath-Kang-Ume [8]-[11], Liu-Kang-Ume [15]-[18] and Liu-Ume-Kang [20]-[25] introduced the general variational inclusions, completely generalized multivalued nonlinear quasivariational inclusions and general strongly nonlinear quasivariational inequalities and established a few existence theorems of solutions for these kinds of variational inclusions and inequalities, respectively. Dong-Lee-Huang [4] not only established some existence theorems of solutions for a class of generalized parametric implicit quasivariational inequalities but also discussed the sensitivity of solutions for the generalized parametric implicit quasivariational inequality.

Motivated and inspired by the results in [1]-[30], in this paper, we introduce and study a new class of generalized parametric nonlinear quasivariational inequalities, which include the variational inequalities of Cho-Kim-Huang-Kang [2], Huang [5] and Verma [28]-[30] as special cases. We establish an existence theorem of solutions for this kind of generalized parametric nonlinear quasivariational inequalities involving strongly monotone mappings, and discuss the sensitivity of solutions for the generalized parametric nonlinear quasivariational inequality.

2. Preliminaries

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $(\cdot,\cdot)$, respectively. The pairing between H^* and H is denoted by $\langle\cdot,\cdot\rangle$, where H^* denotes the dual of H. Let $\wedge$ be a canonical isomorphism from H^* onto H defined by $\langle u,x\rangle = (\wedge u, x)$ for all $x \in H$ and $u \in H^*$. Then $\| \wedge \|_{H^*} = \| \wedge^{-1} \|_H = 1$. Let K be a nonempty convex subset of H, P_K denote the projection of H onto K and P be an open subset of H in which the parameter λ takes values. Suppose that $b : H \times H \times P \to (-\infty,+\infty)$ is nondifferential and satisfies the following conditions:

(c1) b is linear in the first argument;

(c2) b is convex in the second argument;

(c3) b is bounded, that is, there exists a constant $l > 0$ satisfying

$$|b(u, v, \lambda)| \leq l\|u\|\|v\|, \quad \forall (u, v, \lambda) \in H \times H \times P;$$

(c4) $b(u, v, \lambda) - b(u, w, \lambda) \leq b(u, v - w, \lambda)$, $\forall (u, v, w, \lambda) \in H \times H \times H \times P$.

Given singlevalued mappings $g, A, B : H \times P \to H$, $f \in H^*$, $N : H \times H \times P \to H^*$ and a multivalued mapping $K : H \times P \to 2^H$ with nonempty closed convex values. We consider the following problem: For each given $\lambda \in P$, find $u \in H$ such that $g(u, \lambda) \in K(u, \lambda)$ and

$$\begin{aligned} &\langle N(A(u, \lambda), B(u, \lambda), \lambda) - f, v - g(u, \lambda)\rangle \\ \geq \quad & b(u, g(u, \lambda), \lambda) - b(u, v, \lambda), \quad \forall v \in K(u, \lambda), \end{aligned} \tag{2.1}$$

which is called *generalized parametric nonlinear quasivariational inequality.*

Special Cases

(A) If $K(x, \lambda) = K(x)$, $b = f = 0$, $g(x, \lambda) = gx$ and $N(A(x, \lambda), B(x, \lambda), \lambda) = A(gx) - Ax + tBx$, $\forall (x, \lambda) \in H \times P$, where $t > 0$ is a constant, then the generalized parametric nonlinear quasivariational inequality (2.1) is equivalent to finding $u \in H$ such that $gu \in K(u)$ and

$$\langle A(gu) - Au + tBu, v - gu\rangle \geq 0, \quad \forall v \in K(u),$$

which was introduced by Verma, see [30].

(B) If $K(x, \lambda) = K(x)$, $A = B = I$, $b = f = 0$, $N(A(x, \lambda), B(x, \lambda), \lambda) = x - N(x, x)$ and $g(x, \lambda) = gx$, $\forall (x, \lambda) \in H \times P$, where I denotes the identity mapping on H, then the generalized parametric nonlinear quasivariational inequality (2.1) is equivalent to finding $u \in H$ such that $gu \in K(u)$ and

$$\langle u - N(u, u), v - gu\rangle \geq 0, \quad \forall v \in K(u),$$

which was considered by Cho-Kim-Huang-Kang, see [2].

(C) If $K(x, \lambda) = K(x)$, $b = f = 0$, $N(A(x, \lambda), B(x, \lambda), \lambda) = Ax + Bx$ and $g(x, \lambda) = gx$, $\forall (x, \lambda) \in H \times P$, then the generalized parametric nonlinear quasivariational inequality (2.1) is equivalent to finding $u \in H$ such that $gu \in K(u)$ and

$$\langle Au + Bu, v - gu\rangle \geq 0, \quad \forall v \in K(u),$$

which was introduced by Huang [5].

Now let us recall the following definitions and lemmas.

Definition 2.1. Let $\lambda \in P$ and $g, A : H \times P \to H$, $N : H \times H \times P \to H^*$ be mappings.

(1) g is said to be r-*Lipschitz continuous* in the first argument if there exists a constant $r > 0$ such that

$$\|g(x,\lambda) - g(y,\lambda)\| \leq r\|x - y\|, \quad \forall (x,y,\lambda) \in H \times H \times P;$$

(2) g is said to be s-*strongly monotone* in the first argument if there exists a constant $s > 0$ such that

$$\langle g(x,\lambda) - g(y,\lambda), x - y\rangle \geq s\|x - y\|^2, \quad \forall (x,y,\lambda) \in H \times H \times P;$$

(3) g is called *continuous* (resp., *uniformly continuous* and *Lipschitz continuous*) in the second argument if, for each given $x \in H$, $g(x,\cdot)$ is continuous (resp., uniformly continuous and Lipschitz continuous).

(4) N is said to be r-Lipschitz continuous in the first argument if there exists a constant $r > 0$ such that

$$\|N(x,u,\lambda) - N(y,u,\lambda)\| \leq r\|x - y\|, \quad \forall (u,x,y,\lambda) \in H \times H \times H \times P;$$

(5) N is said to be s-*strongly monotone* with respect to A in the first argument if there exists a constant $s > 0$ such that

$$\langle N(Ax,u,\lambda) - N(Ay,u,\lambda), x - y\rangle \geq s\|x - y\|^2, \quad \forall (u,x,y,\lambda) \in H \times H \times H \times P.$$

In a similar way, we can define that N is Lipschitz continuous with respect to the second argument.

Lemma 2.1. (see [27]) *If $K \subset H$ is a closed convex subset and $z \in H$ is a given point, then there exists $x \in K$ such that*

$$\langle x - z, y - x\rangle \geq 0, \quad \forall y \in K,$$

if and only if $x = P_K z$. Furthermore, the mapping P_K is nonexpansive, that is,

$$\|P_K x - P_K y\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Lemma 2.2. *Let $\lambda \in P$ and ρ be positive parameters. Then, the following statements are equivalent:*

(a) *the generalized parametric nonlinear quasivariational inequality* (2.1) *has a solution $u \in H$ with $g(u,\lambda) \in K(u,\lambda)$;*

(b) *there exists $u \in H$ such that $g(u,\lambda) \in K(u,\lambda)$ and*

$$(u - \wedge\varphi(u,\lambda), v - g(u,\lambda)) \geq 0, \quad \forall v \in K(u,\lambda),$$

where $\varphi : H \times P \to H^*$ *satisfies the following relations:*

$$\langle \varphi(u,\lambda), v\rangle = (u,v) - b(u,v,\lambda) - \langle N(A(u,\lambda), B(u,\lambda),\lambda) - f, v\rangle \tag{2.2}$$

for all $(u,v,\lambda) \in H \times H \times P$;

(c) *there exists* $u \in H$ *such that* $g(u,\lambda) \in K(u,\lambda)$ *and*

$$g(u,\lambda) = P_{K(u,\lambda)}(g(u,\lambda) - \rho u + \rho \wedge \varphi(u,\lambda));$$

(d) *the mapping* $F : H \times P \to H$, *defined by*

$$F(x,\lambda) = x - g(x,\lambda) + P_{K(x,\lambda)}(g(x,\lambda) - \rho x + \rho \wedge \varphi(x,\lambda)), \quad \forall x \in H, \tag{2.3}$$

has a fixed point $u \in H$.

Proof. It is easy to see that (c) and (d) are equivalent. Notice that

$$\begin{aligned} & (u - \wedge\varphi(u,\lambda), v - g(u,\lambda)) \\ = \; & (u,v) - (u, g(u,\lambda)) - (u,v) + \rho b(u,v,\lambda) \\ & + \rho\langle N(A(u,\lambda), B(u,\lambda),\lambda) - f, v\rangle + (u, g(u,\lambda)) \\ & - \rho b(u, g(u,\lambda),\lambda) - \rho\langle N(A(u,\lambda), B(u,\lambda),\lambda) - f, g(u,\lambda)\rangle \\ = \; & \rho b(u,v,\lambda) - \rho b(u, g(u,\lambda),\lambda) \\ & + \rho\langle N(A(u,\lambda), B(u,\lambda),\lambda) - f, v - g(u,\lambda)\rangle \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} & \rho(u - \wedge\varphi(u,\lambda), v - g(u,\lambda)) \\ = \; & (g(u,\lambda) - (g(u,\lambda) - \rho u + \rho \wedge \varphi(u,\lambda)), v - g(u,\lambda)) \end{aligned} \tag{2.5}$$

for all $(u,v,\lambda) \in H \times H \times P$. It follows from (2.4), (2.5) and Lemma 2.1 that statements (a), (b), and (c) are equivalent. □

3. Existence Theorem and Sensitivity Analysis

In this section, we give an existence theorem of solutions for the generalized parametric nonlinear quasivariational inequality (2.1) and discuss the sensitivity of solutions for the generalized parametric nonlinear quasivariational inequality (2.1).

Theorem 3.1. *Suppose that* $b : H \times H \times P \to (-\infty, +\infty)$ *satisfies* (c1)-(c4), $g, A, B : H \times P \to H$, $N : H \times H \times P \to H^*$ *are mappings, and* $f \in H^*$. *Assume that* $g : H \times P \to H$ *is* α-*strongly monotone and* β-*Lipschitz* *continuous in the first argument, respectively. Let N be ξ-Lipschitz continuous in the first argument, η-Lipschitz continuous in the second argument, and be p-strongly monotone with respect to A in the first argument. Suppose that A is r-Lipschitz continuous in the first argument, B is s-Lipschitz continuous in the first argument. Assume that there exists some $\mu > 0$ such that*

$$\|P_{K(x,\lambda)}z - P_{K(y,\lambda)}z\| \leq \mu\|x - y\| \tag{3.1}$$

for all $(x, y, z, \lambda) \in H \times H \times H \times P$.

Let $j = l + \sqrt{1 - 2p + \xi^2 r^2} + \eta s$ and $k = \sqrt{1 - 2\alpha + \beta^2} + \mu$. Assume that either condition (I) *or condition* (II) *below is satisfied:*

(I) *There exists a constant $\rho > 0$ satisfying:*

$$\rho j < 1 - k \leq \beta \tag{3.2}$$

and one of the following conditions:

$$\begin{gathered} \left|\rho - \frac{\alpha - (1-k)j}{1 - j^2}\right| < \frac{\sqrt{[\alpha - (1-k)j]^2 - [\beta^2 - (1-k)^2](1 - j^2)}}{1 - j^2}, \\ |\alpha - (1-k)j| > \sqrt{[\beta^2 - (1-k)^2](1 - j^2)}, \quad j < 1; \end{gathered} \tag{3.3}$$

$$\begin{gathered} \left|\rho - \frac{(1-k)j - \alpha}{j^2 - 1}\right| > \frac{\sqrt{[\alpha - (1-k)j]^2 + [\beta^2 - (1-k)^2](j^2 - 1)}}{j^2 - 1}, \\ j > 1, \end{gathered} \tag{3.4}$$

(II) *There exists a constant $\rho > 0$ satisfying:*

$$\max\{\beta, \rho j\} < 1 - k \tag{3.5}$$

and one of the following conditions:

$$\begin{gathered} \left|\rho - \frac{\alpha - (1-k)j}{1 - j^2}\right| < \frac{\sqrt{[\alpha - (1-k)j]^2 + [(1-k)^2 - \beta^2](1 - j^2)}}{1 - j^2}, \\ j < 1; \end{gathered} \tag{3.6}$$

$$\begin{aligned}&\left|\rho-\frac{(1-k)j-\alpha}{j^2-1}\right|\\&\quad>\frac{\sqrt{[\alpha-(1-k)j]^2-[(1-k)^2-\beta^2](j^2-1)}}{j^2-1},\\&|\alpha-(1-k)j|>\sqrt{[(1-k)^2-\beta^2](j^2-1)},\quad j>1,\end{aligned}\tag{3.7}$$

then for each $\lambda \in P$*, the generalized parametric nonlinear quasivariational inequality* (2.1) *has a unique solution* $u \in H$*.*

Proof. For each given $\lambda \in P$, we prove that $F(\cdot,\lambda) : H \to H$ defined by (2.3) is a contraction mapping. In view of Lemma 2.1, (2.3) and (3.1), we infer that for any $x, y \in H$

$$\begin{aligned}&\|F(x,\lambda)-F(y,\lambda)\|\\=\;&\|x-g(x,\lambda)+P_{K(x,\lambda)}(g(x,\lambda)-\rho x+\rho\wedge\varphi(x,\lambda))\\&-y+g(y,\lambda)-P_{K(y,\lambda)}(g(y,\lambda)-\rho y+\rho\wedge\varphi(y,\lambda))\|\\\le\;&\|x-y-(g(x,\lambda)-g(y,\lambda))\|\\&+\|P_{K(x,\lambda)}(g(x,\lambda)-\rho x+\rho\wedge\varphi(x,\lambda))\\&-P_{K(y,\lambda)}(g(x,\lambda)-\rho x+\rho\wedge\varphi(x,\lambda))\|\\&+\|P_{K(y,\lambda)}(g(x,\lambda)-\rho x+\rho\wedge\varphi(x,\lambda))\\&-P_{K(y,\lambda)}(g(y,\lambda)-\rho y+\rho\wedge\varphi(y,\lambda))\|\\\le\;&\|x-y-(g(x,\lambda)-g(y,\lambda))\|+\mu\|x-y\|\\&+\|g(x,\lambda)-g(y,\lambda)-\rho(x-y)\\&+\rho(\wedge\varphi(x,\lambda)-\wedge\varphi(y,\lambda))\|.\end{aligned}\tag{3.8}$$

Since N is p-strongly monotone with respect to A in the first argument, A is r-Lipschitz continuous in the first argument, B is s-Lipschitz continuous in the first argument, N is ξ-Lipschitz continuous in the first argument and η-Lipschitz continuous in the second argument, we obtain that

$$\begin{aligned}&|\langle\varphi(x,\lambda)-\varphi(y,\lambda),z\rangle|\\=\;&|(x-y,z)-b(x-y,z,\lambda)-\langle N(A(x,\lambda),B(x,\lambda),\lambda)\\&-N(A(y,\lambda),B(y,\lambda),\lambda),z\rangle|\\\le\;&l\|x-y\|\|z\|+|\langle x-y-(N(A(x,\lambda),B(x,\lambda),\lambda)\\&-N(A(y,\lambda),B(x,\lambda),\lambda)),z\rangle|\\&+|\langle N(A(y,\lambda),B(x,\lambda),\lambda)-N(A(y,\lambda),B(y,\lambda),\lambda),z\rangle|\\\le\;&l\|x-y\|\|z\|+\|x-y-(N(A(x,\lambda),B(x,\lambda),\lambda)\\&-N(A(y,\lambda),B(x,\lambda),\lambda))\|\|z\|\end{aligned}$$

$$\begin{aligned} &+\|N(A(y,\lambda),B(x,\lambda),\lambda)-N(A(y,\lambda),B(y,\lambda),\lambda)\|\|z\| \\ \leq\ & j\|x-y\|\|z\|, \end{aligned} \tag{3.9}$$

which implies that

$$\|\varphi(x,\lambda)-\varphi(y,\lambda)\| = \sup_{\|z\|\leq 1} |\langle\varphi(x,\lambda)-\varphi(y,\lambda),z\rangle| \leq j\|x-y\|. \tag{3.10}$$

Since g is α-strongly monotone and β-Lipscitz continuous in the first argument, respectively, by (3.8) and (3.10) we infer that

$$\|F(x,\lambda)-F(y,\lambda)\| \leq \theta\|x-y\|, \tag{3.11}$$

where

$$\theta = k+\sqrt{\rho^2-2\rho\alpha+\beta^2}+\rho j. \tag{3.12}$$

It is easy to verify that $\theta < 1$ if and only if (3.2) and one of (3.3) and (3.4) hold, or (3.5) and one of (3.6) and (3.7) hold. Hence for each given $\lambda \in P$, $F(\cdot,\lambda) : H \times P \to H$ is a contraction mapping and $F(\cdot,\lambda)$ has a unique fixed point $u \in H$, which shows that the generalized parametric nonlinear quasivariational inequality (2.1) has a unique solution $u \in H$. □

Next we analyze the sensitivity of solutions for the generalized parametric nonlinear quasivariational inequality (2.1).

Theorem 3.2. *Let f, g, A, B, K and N be the same as in Theorem 3.1 and* (3.1) *hold. Suppose that there exists some $\nu > 0$ such that*

$$\|P_{K(x,\lambda)}y - P_{K(x,\bar{\lambda})}y\| \leq \nu\|\lambda-\bar{\lambda}\|, \tag{3.13}$$

for all $(x,y,\lambda,\bar{\lambda}) \in H \times H \times P \times P$. Assume that there exists $\rho > 0$ satisfying (3.2) *and one of* (3.3) *and* (3.4), *or satisfying* (3.5) *and one of* (3.6) *and* (3.7). *If g and φ defined by* (2.2) *are continuous (resp., uniformly continuous or Lipschitz continuous) in the second argument, respectively, then the solutions of the generalized parametric nonlinear quasivariational inequality* (2.1) *are continuous (resp., uniformly continuous or Lipschitz continuous).*

Proof. Let F be defined by (2.3). It follows from Theorem 3.1 that for each given $\lambda \in P$, the generalized parametric nonlinear quasivariational inequality (2.1) has a unique solution $u = u(\lambda) \in H$ such that $u(\lambda) = F(u(\lambda),\lambda)$. Hence for each $\lambda, \bar{\lambda} \in P$, we have

$$u(\lambda) = F(u(\lambda),\lambda), \quad u(\bar{\lambda}) = F(u(\bar{\lambda}),\bar{\lambda})$$

and

$$\begin{aligned}\|u(\lambda)-u(\bar{\lambda})\| &\leq \|F(u(\lambda),\lambda)-F(u(\lambda),\bar{\lambda})\| \\ &\quad +\|F(u(\lambda),\bar{\lambda})-F(u(\bar{\lambda}),\bar{\lambda})\|. \end{aligned} \tag{3.14}$$

By virtue of Lemma 2.1 and (3.13), we get that

$$\begin{aligned} &\|F(u(\lambda),\lambda)-F(u(\lambda),\bar{\lambda})\| \\ = \; &\|u(\lambda)-g(u(\lambda),\lambda)+P_{K(u(\lambda),\lambda)}(g(u(\lambda),\lambda)-\rho u(\lambda) \\ &+\rho\wedge\varphi(u(\lambda),\lambda))-u(\lambda)+g(u(\lambda),\bar{\lambda}) \\ &-P_{K(u(\lambda),\bar{\lambda})}(g(u(\lambda),\bar{\lambda})-\rho u(\lambda)+\rho\wedge\varphi(u(\lambda),\bar{\lambda}))\| \\ \leq \; &\|g(u(\lambda),\lambda)-g(u(\lambda),\bar{\lambda})\| \\ &+\|P_{K(u(\lambda),\lambda)}(g(u(\lambda),\lambda)-\rho u(\lambda)+\rho\wedge\varphi(u(\lambda),\lambda)) \\ &-P_{K(u(\lambda),\bar{\lambda})}(g(u(\lambda),\bar{\lambda})-\rho u(\lambda)+\rho\wedge\varphi(u(\lambda),\bar{\lambda}))\| \\ \leq \; &\|g(u(\lambda),\lambda)-g(u(\lambda),\bar{\lambda})\|+\nu\|\lambda-\bar{\lambda}\| \\ &+\|g(u(\lambda),\lambda)-g(u(\lambda),\bar{\lambda})+\rho(\wedge\varphi(u(\lambda),\lambda)-\wedge\varphi(u(\lambda),\bar{\lambda}))\| \\ \leq \; &2\|g(u(\lambda),\lambda)-g(u(\lambda),\bar{\lambda})\|+\nu\|\lambda-\bar{\lambda}\| \\ &+\rho\|\varphi(u(\lambda),\lambda)-\varphi(u(\lambda),\bar{\lambda})\|. \end{aligned} \tag{3.15}$$

Note that (3.8) ensures that

$$\|F(u(\lambda),\bar{\lambda})-F(u(\bar{\lambda}),\bar{\lambda})\| \leq \theta\|u(\lambda)-u(\bar{\lambda})\|, \tag{3.16}$$

where θ is defined by (3.12). Combining (3.14), (3.15) and (3.16), we deduce that

$$\begin{aligned}\|u(\lambda)-u(\bar{\lambda})\| &\leq \frac{1}{1-\theta}[2\|g(u(\lambda),\lambda)-g(u(\lambda),\bar{\lambda})\|+\nu\|\lambda-\bar{\lambda}\| \\ &\quad +\rho\|\varphi(u(\lambda),\lambda)-\varphi(u(\lambda),\bar{\lambda})\|], \end{aligned}$$

and the desired conclusion follows from the continuities of g and φ (resp., uniform continuities or Lipschitz continuities) in the second argument, respectively. This completes the proof. □

Acknowledgements

This work was supported by the Science Research Foundation of Educational Department of Liaoning Province (2004C063).

References

[1] S. Adly, Perturbed algorithm and sensitivity analysis for a general class of varitional inclusions, *J. Math. Anal. Appl.*, **201** (1996), 609-630.

[2] Y.J. Cho, J.H. Kim, N.J. Huang, S.M. Kang, Ishikawa and Mann iterative processes with errors for generalized nonlinear implicit quasi-variational inequalities, *Publ. Math. Debrecen*, **58** (2001), 639-649.

[3] S. Dafermos, Sensitivity analysis in variational inequalities, *Math. Oper. Res.*, **13** (1988), 421-434.

[4] H. Dong, B.S. Lee, N.J. Huang, Sensitivity analysis for generalized parametric implicit quasivariational inequalities, *Nonlinear Anal. Forum*, **6**, No. 2 (2001), 313-320.

[5] N.J. Huang, On the generalized implicit quasivariational inequalities, *J. Math. Anal. Appl.*, **216** (1997), 197-210.

[6] N.J. Huang, M.R. Bai, Y.J. Cho, S.M. Kang, Generalized nonlinear mixed quasivariational inequalities, *Computers Math. Appl.*, **160** (1978), 139-161.

[7] Z. Liu, R.P. Agarwal, S.M. Kang, On solvability of functional equations and system of functional equations arising in dynamic programming, *J. Math. Anal. Appl.*, **297** (2004), 111-130.

[8] Z. Liu, L. Debnath, S.M. Kang, J.S. Ume, Completely generalized multivalued nonlinear quasi-variational inclusions, *IJMMS*, **30**, No. 10 (2002), 593-604.

[9] Z. Liu, L. Debnath, S.M. Kang, J.S. Ume, On the generalized nonlinear quasi-variational inclusions, *Acta. Math. Informatica Universitatis Ostraviensis*, **11** (2003), 81-90.

[10] Z. Liu, L. Debnath, S.M. Kang, J.S. Ume, Sensitivity analysis for parametric completely generalized nonlinear implicit quasivariational inclusions, *J. Math. Anal. Appl.*, **277** (2003), 142-154.

[11] Z. Liu, L. Debnath, S.M. Kang, J.S. Ume, Generalized mixed quasivariational inclusions and generalized mixed resolvent equations for fuzzy mappings, *Appl. Math. Comput.*, **149** (2004), 879-891.

[12] Z. Liu, S.M. Kang, Comments on the papers involving variational and quasivariational inequalities for fuzzy mappings, *Math. Sci. Res. J.*, **7**, No. 10 (2003), 394-339.

[13] Z. Liu, S.M. Kang, Generalized multivalued nonlinear quasi-variational inclusions, *Math. Nachr.*, **253** (2003), 45-54.

[14] Z. Liu, S.M. Kang, Convergence and stability of perturbed three-step iterative algorithm for completely generalized nonlinear quasi-variational inequalities, *Appl. Math. Comput.*, **149** (2004), 245-258.

[15] Z. Liu, S.M. Kang, J.S. Ume, On general variational inclusions with noncompact valued mappings, *Adv. Nonlinear Var. Inequal.*, **5**, No. 2 (2002), 11-25.

[16] Z. Liu, S.M. Kang, J.S. Ume, Generalized variational inclusions for fuzzy mappings, *Adv. Nonlinear Var. Inequal.*, **6** (2003), 31-40.

[17] Z. Liu, S.M. Kang, J.S. Ume, The solvability of a class of quasi-variational inequalities, *Adv. Nonlinear Var. Inequal.*, **6**, No. 2 (2003), 69-78.

[18] Z. Liu, S.M. Kang, J.S. Ume, Completely generalized multivalued strongly quasivariational inequalities, *Publ. Math. Debrecen*, **62**, No. 1-2 (2003), 187-204.

[19] Z. Liu, J. Sheok, S.M. Kang, Stability of Noor iterations with errors for generalized nonlinear complementarity problems, *Acta Math. Acad. Paeda. Nytre.*, **20** (2004), 53-61.

[20] Z. Liu, J.S. Ume, S.M. Kang, General strongly nonlinear quasi-variational inequalities with relaxed Lipschitz and relaxed monotone mappings, *J. Optim. Theory Appl.*, **114**, No. 3 (2002), 639-656.

[21] Z. Liu, J.S. Ume, S.M. Kang, Resolvent equations technique for general variational inclusions, *Proc. Japan Acad., Ser. A*, **78**, No. 10 (2002), 188-193.

[22] Z. Liu, J.S. Ume, S.M. Kang, Nonlinear variational inequalities on reflexive Banach spaces and topological vector spaces, *Internat. J. Math. and Math. Sci.*, **2003**, No. 4 (2003), 199-207.

[23] Z. Liu, J.S. Ume, S.M. Kang, Completely generalized quasi-variational inequalities, *Adv. Nonlinear Var. Inequal.*, **7**, No. 1 (2004), 35-46.

[24] Z. Liu, J.S. Ume, S.M. Kang, General variational inclusions and general resolvent equations, *Bull. Korean Math. Soc.*, **41**, No. 2 (2004), 241-256.

[25] Z. Liu, J.S. Ume, S.M. Kang, Completely generalized quasivariational inequalities, *Internat. J. Pure Appl. Math.*, **16**, No. 1 (2004), 101-112.

[26] Z. Liu, L. Wang, S.M. Kang, J.S. Ume, Sensitivity analysis of solutions for generalized parametric nonlinear variational inequalities, *Internat. J. Pure Appl. Math.*, **6**, No. 1 (2003), 77-91.

[27] U. Mosco, *Implicit Variational Problems and Quasivariational Inequalities*, Lecture Notes in Math., Springer-Verlag, Berlin, Germany, **543** (1976).

[28] R.U. Verma, Generalized variational inequalities and associated nonlinear equations, *Czechoslovak Math. J.*, **48** (1998), 413-418.

[29] R.U. Verma, Generalized pseduo-contractions and nonlinear variational inequalities, *Publ. Math. Debrecen*, **53** (1998), 23-28.

[30] R.U. Verma, The solvability of a class of generalized nonlinear variational inequalities based on an iterative algorithm, *Appl. Math, Lett.*, **12** (1999), 51-53.