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SOME REMARKS ON THE WAVE EQUATION WITH POTENTIAL TYPE DAMPING COEFFICIENTS

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Abstract: We shall present a new type of finite propagation speed property to linear wave equations with a potential type damping term. In this case, we do not assume any compactness of the support on the initial data. The Todorova-Yordanov [5] method is effectively applied.

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1. Introduction

Let $\Omega \subset \mathbf{R}^N (N \geq 2)$ be an exterior domain with a compact smooth boundary $\partial\Omega$. Without loss of generality we may assume $0 \notin \bar{\Omega}$. In the case when $N = 1$, we take $\Omega = (0, +\infty)$. In this paper we are concerned with the following mixed problem:

$$u_{tt}(t,x) - \Delta u(t,x) + a(x)u_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \Omega, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u|_{\partial\Omega} = 0, \quad t > 0, \tag{1.3}$$

where

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$$a(x) = \frac{\delta}{|x|^{\sigma}}, \quad \delta > 0, \quad \sigma \geq 0. \tag{1.4}$$

First, let us introduce some notations used throughout this paper. $\|\cdot\|_p$ means the usual $L^p(\Omega)$-norm ($1 \leq p \leq +\infty$), and especially we set $\|\cdot\| = \|\cdot\|_2$. The L^2-inner product is defined by (as usual)

$$(f, g) = \int_{\Omega} f(x)g(x)\,dx \quad \text{for } f,\ g \in L^2(\Omega).$$

The total energy $E(t)$ to the equation (1.1) is defined by

$$E(t) = \frac{1}{2}\{\|u_t(t,\cdot)\|^2 + \|\nabla u(t,\cdot)\|^2\}.$$

Before stating our new results we shall mention the well-posedness to the problem (1.1)-(1.3) (cf. Lions-Strauss [3]).

Proposition 1.1. *Let $N \geq 1$, $\delta = 1$ and $\sigma \geq 0$. For each $[u_0, u_1] \in H_0^1(\Omega)\times L^2(\Omega)$, there exists a unique solution $u \in C([0,\infty); H_0^1(\Omega))\cap C^1([0,\infty); L^2(\Omega))$ to problem* (1.1)-(1.4) *satisfying*

$$E(t) + \int_0^t \int_{\Omega} a(x)|u_t(t,x)|^2 dxdt = E(0), \tag{1.5}$$

for all $t \geq 0$.

Let us consider the case $\delta > 0$. Mochizuki-Nakazawa [4] have recently observed that the exponent $\sigma = 1$ in (1.4) is critical to problem (1.1)-(1.3) in the sense that if $\sigma \leq 1$, then the total energy decays at a certain rate, while in the case when $\sigma > 1$ the total energy does not decay in general (in fact, they dealt with a more general damping coefficient $a(x)$ replaced by $a(t,x)$, which depends also on t).

The purpose of this paper is to observe that the exponent $\sigma = 1$ is also critical through a different information from other known results concerning how to spread of the local energy in the $t - x$ plane. For this purpose we shall use the weighted energy method due to Todorova-Yordanov [5], which was recently applied to the study of some damped wave equations (i.e., $\delta = 1$ and $\sigma = 0$ in (1.4)) with nonlinear term $|u|^p$ successfully. Todorova-Yordanov used a weight function $\psi(t,x)$, which depends on the support of the solution. After them [5], Ikehata-Tanizawa [2] improved the weight function of theirs in order to remove the compactness assumption on the support of the solution in [5], and applied it to the nonlinear critical exponent problem. In this paper, we also use another type of weight function, which is explicitly constructed in obedience to the shape of (1.4) in order to estimate the weighted energy of the solution (see (2.1) below).

Our result reads as follows.

Proposition 1.2. *Let $N \geq 1$, $\delta = 1$ and $\sigma \geq 0$ with $\sigma \neq 2$. If the initial data $[u_0, u_1] \in H^1_0(\Omega) \times L^2(\Omega)$ further satisfy*

$$K_0 = \int_\Omega \exp(\frac{2|x|^{2-\sigma}}{(2-\sigma)^2})(|u_1(x)|^2 + |\nabla u_0(x)|^2)dx < +\infty, \tag{1.6}$$

then the unique solution $u(t,x)$ to problem (1.1)-(1.4) *as in Proposition* 1.1 *satisfies*

$$\int_\Omega \exp(\frac{2|x|^{2-\sigma}}{(2-\sigma)^2(1+t)})(|u_t(t,x)|^2 + |\nabla u(t,x)|^2)dx \leq K_0. \tag{1.7}$$

Remark 1.1. By Proposition 1.2 we see that only the case $\sigma = 2$ is excluded. In this sense the $\sigma = 2$ case seems critical.

As consequences of Proposition 1.2 one has the following local energy decay results.

Theorem 1.1. *Let $N \geq 1$, $\sigma \in [0, 1)$, and $\eta \in (0, (1-\sigma)/2)$. Then under the same assumptions as in Proposition 1.2, it is true that for large $t > 1$,*

$$\int_{\{x\in\Omega:|x|\geq t^{(1+2\eta)/(2-\sigma)}\}} (|u_t|^2 + |\nabla u|^2)dx \leq \exp(-\frac{t^{2\eta}}{(2-\sigma)^2})K_0$$

with $(1 + 2\eta)/(2-\sigma) < 1$.

Theorem 1.2. *Let $N \geq 1$, $\sigma \in [1, 2)$, and $\eta > 0$. Then under the same assumptions as in Proposition 1.2, one has the same conclusion as in Theorem 1.2 except for the condition $(1 + 2\eta)/(2-\sigma) > 1$.*

Remark 1.2. From Theorem 1.1 and Theorem 1.2 we can observe an interesting property of solutions to problem (1.1)-(1.4) with some weighted initial data. That is to say, in the case when $\sigma \in [0, 1)$ the total energy concentrates in a ball much smaller than the usual forward light cone $\{|x| \leq t\}$, while in the case when $\sigma \in [1, 2)$ the total energy may concentrate in a ball much larger than the forward light cone $\{|x| \leq t\}$. Note that in the present case we do not assume any compactness of the support on the initial data. In this sense $\sigma = 1$ is a critical exponent. In the case when $\sigma = 0$ similar results to our theorems have already been pointed out in Todorova-Yordanov [5] under the assumption that the initial data have a compact support. Our results generalize theirs to the general exponent σ and a non-compactly supported solution case.

Remark 1.3. In the case when $\sigma > 2$ we can not find a remarkable property like Theorem 1.1 and Theorem 1.2 by now, and this case in $\mathbf{R}^N$ is a future theme.

2. Proof of the Results

The proof of Proposition 1.2 can be done by use of the (modified) Todorova-Yordanov method [5]. Their method was originally applied to the damped wave equations (i.e., (1.1) with $\sigma = 0$). The next lemma tells us that the Todorova-Yordanov method is also applicable to the potential type damped wave equations if we choose a weight function $\psi(t,x)$ appropriately. In this paper it is essential how we choose $\psi(t,x)$. In fact, for $\sigma \neq 2$ we take a weight function

$$\psi(t,x) = \frac{|x|^{2-\sigma}}{(2-\sigma)^2(1+t)}. \tag{2.1}$$

It is easily verified that the function (2.1) satisfies

$$a(x)\psi_t(t,x) + |\nabla\psi(t,x)|^2 = 0, \tag{2.2}$$

$$\psi_t(t,x) < 0. \tag{2.3}$$

Remark 2.1. In the case when $\sigma = 0$, the weight function has already been found in Ikehata-Tanizawa [2]. In the case when $\delta = 0$ in (1.1) and (1.4), the weight function is chosen in Ikehata [1] like $\psi(t,x) = |x| - t$, and this function played an essential role in deriving the local energy decay for the free wave equation.

Now one can prove Proposition 1.2.

Proof of Proposition 1.2. The proof follows from the new weighted energy method due to Todorova-Yordanov [5]. Since we are dealing with a weak solution, by density argument we may assume that the initial data and the corresponding solution are sufficiently smooth and vanish as $|x| \to +\infty$. Set

$$E(t,x) = \frac{1}{2}(|u_t(t,x)|^2 + |\nabla u(t,x)|^2).$$

We multiply both sides of (1.1) by $e^{2\psi(t,x)}u_t(t,x)$. Then one has an identity:

$$0 = e^{2\psi}u_t(u_{tt} - \Delta u + a(x)u_t) = \frac{d}{dt}(e^{2\psi}E(t,x)) - \mathrm{div}(e^{2\psi}u_t\nabla u)$$

$$-\frac{e^{2\psi}}{\psi_t}|\psi_t\nabla u - u_t\nabla\psi|^2 + \frac{e^{2\psi}}{\psi_t}u_t^2(|\nabla\psi|^2 + a(x)\psi_t) - e^{2\psi}\psi_t u_t^2.$$

Now, since the function $\psi = \psi(t,x)$ satisfies (2.2) and (2.3) one has

$$0 \geq \frac{d}{dt}(e^{2\psi}E(t,x)) - \mathrm{div}(e^{2\psi}u_t\nabla u).$$

Integrating the above inequality over $[0,t]\times\Omega$ we see

$$\int_0^t\int_\Omega \mathrm{div}(e^{2\psi}u_t\nabla u)dxdt \geq \int_\Omega e^{2\psi}E(t,x)dx - \int_\Omega e^{2\psi(0,x)}E(0,x)dx.$$

It follows from the divergence formula that

$$\int_0^t\int_\Omega \mathrm{div}(e^{2\psi}u_t\nabla u)dxdt = \int_0^t ds\int_{\partial\Omega} e^{2\psi(s,r)}u_s(s,r)\frac{\partial u}{\partial\nu}(s,r)dr = 0,$$

where $\nu = \nu(r)$ is the unit outward normal vector at $r \in \partial\Omega$. In the above caluculation we have used the facts

$$u_s(s,r) = 0, \quad r \in \partial\Omega,$$
$$|\frac{\partial u}{\partial\nu}(s,r)| \leq |\nabla u(s,r)| < +\infty, \quad r \in \partial\Omega.$$

This implies the desired estimate. □

Proof of Theorem 1.1. It follows from (1.7) that for large $t > 1$,

$$K_0 \geq \int_\Omega \exp(\frac{|x|^{2-\sigma}}{(2-\sigma)^2 t})E(t,x)dx$$
$$\geq \int_{\{x\in\Omega:|x|^{2-\sigma}\geq t^{1+2\eta}\}} \exp(\frac{t^{2\eta}}{(2-\sigma)^2})E(t,x)dx,$$

which implies the desired estimate. □

The proof of Theorem 1.2 can be done along the same way, so that we shall leave it to the reader's exercise.

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