International Journal of Pure and Applied Mathematics
Volume 21 No. 1 2005, 41-50

NUMEROV-TYPE P-STABLE LINEARLY IMPLICIT SCHEMES FOR SECOND ORDER INITIAL VALUE PROBLEMS

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Abstract: For the integration of *special* second order initial value problems with oscillatory solutions, P-stable schemes are necessarily implicit. For nonlinear problems, functional implicitness of these schemes requires the use of Newton's method at each time-step of integration. In the present paper, we propose linearized *linearly implicit* P-stable schemes which, for nonlinear problems, obviate the need to solve resulting nonlinear equations. We present two such schemes. First, we present a second order scheme and then a fourth order linearly implicit P-stable Numerov-type scheme. The obtained schemes are computationally illustrated for their order, accuracy and stability by considering two examples of practical interest.

AMS Subject Classification: 65L05
Key Words: second order initial value problems, oscillatory solutions, P-stability, linearly implicit schemes, second order method, Numerov-type method

1. Introduction

We consider *special* second order initial value problems with *oscillatory* solutions:

$$y'' = f\left(t, y\right), \quad y\left(t_0\right) = \eta_0, \; y'\left(t_0\right) = \eta_1. \tag{1.1}$$

Let $t_n = t_0 + nh$, $n = 0, 1, 2, ...$, $h =$ step-length, and let $y_n \approx y\left(t_n\right)$, $f_n =$

Received: March 17, 2005

$f(t_n, y_n)$, etc. For the integration of initial value problems (1.1), the order of a P-stable linear multistep method (LMM) cannot exceed two (Lambert and Watson [8]), and the second order two-step P-stable method is given by

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{4}\left[f_{n+1} + 2f_n + f_{n-1}\right] + \tau_n^{(2)}(h), \tag{1.2}$$

with the local truncation error given by

$$\tau_n^{(2)}(h) = -\frac{h^4}{6} y_n^{(4)} + O\left(h^6\right). \tag{1.3}$$

A well-known optimal two-step fourth order method is the Numerov method [9] described by

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{12}\left[f_{n+1} + 10f_n + f_{n-1}\right] + \tau_n^{(N)}(h), \tag{1.4}$$

with

$$\tau_n^{(N)}(h) = -\frac{h^6}{240} y_n^{(6)} + O\left(h^8\right). \tag{1.5}$$

However, Numerov method has only a finite interval of periodicity of size $H_p = \sqrt{6}$. In order to overcome the order barrier imposed by P-stability on LMMs, Chawla [3] and Cash [2] independently introduced symmetric hybrid two-steps P-stable methods of orders four and six. Later, Chawla [4] described a suitable modification of Numerov method which imparts P-stability to it, and the resulting Numerov-type P-stable method is given by

$$\begin{aligned} \overline{y}_n &= y_n - \alpha h^2 \left(f_{n+1} - 2f_n + f_{n-1}\right), \quad \overline{f}_n = f\left(t_n, \overline{y}_n\right), \\ y_{n+1} - 2y_n + y_{n-1} &= \frac{h^2}{12}\left[f_{n+1} + 10\overline{f}_n + f_{n-1}\right] + \tau_n^{(\alpha)}(h), \end{aligned} \tag{1.6}$$

with

$$\tau_n^{(\alpha)}(h) = -\frac{h^6}{240}\left[y_n^{(6)} - 200\alpha y_n^{(4)} f_n^y\right] + O\left(h^8\right). \tag{1.7}$$

This modified Numerov-type method is P-stable (equivalently, unconditionally stable in the terminology of Dahlquist [6]) for all $\alpha > 1/120$. Double-stride schemes for initial value problems with oscillatory solutions have been described by Chawla and Al-Zanaidi, see [5].

All the P-stable methods that have been given so far are necessarily implicit. Functional implicitness of these methods means that for nonlinear problems we need to use Newton's method to solve resulting nonlinear equations at each time-step of integration. The purpose of the present paper is to overcome functional implicitness of P-stable methods.

In the present paper, we propose linearized linearly implicit P-stable schemes which, for nonlinear problems, obviate the need to solve nonlinear equations. We present two such schemes. First, we present a second order scheme and then a fourth order linearly implicit P-stable Numerov-type scheme. The obtained schemes are computationally illustrated for their order, accuracy and stability by considering two examples of practical interest.

2. A Second Order Linearly Implicit Scheme

Consider a linearization of (1.2) of the form:

$$\begin{aligned} &\Delta y_n = y_{n+1} - y_n, \quad \widetilde{y}_n = y_n + \beta \Delta y_{n-1}, \\ &\left[1 - \frac{h^2}{4} f^y\left(t_{n+1}, \widetilde{y}_n\right)\right] \Delta y_n \\ &\quad = \Delta y_{n-1} + \frac{h^2}{4}\left[f_{n-1} + 2 f_n + f\left(t_{n+1}, y_n\right)\right] + t_n^{(2)}(h). \end{aligned} \tag{2.1}$$

Setting

$$g_n = f^y\left(t_{n+1}, \widetilde{y}_n\right), \quad w_n = f\left(t_{n+1}, y_n\right),$$

by Taylor expansion, it is easy to see that

$$\begin{aligned} t_n^{(2)}(h) = \frac{h^2}{4}\left(y_n'' - w_n\right) + \frac{h^3}{4}\left(y_n''' - y_n' g_n\right) - \frac{h^4}{24}\left(y_n^{(4)} + 3 y_n'' g_n\right) \\ + O\left(h^5\right). \end{aligned} \tag{2.2}$$

Since

$$w_n = f_n + h w_{n,1} + h^2 w_{n,2} + O\left(h^3\right),$$

with

$$w_{n,1} = f_n^t, \quad w_{n,2} = \frac{1}{2} f_n^{tt},$$

and

$$g_n = f_n^y + h g_{n,1} + O\left(h^2\right),$$

with

$$g_{n,1} = f_n^{ty} + \beta y_n' f_n^{yy},$$

from (2.2), it follows that

$$t_n^{(2)}(h) = \frac{h^3}{4}\left(y_n''' - y_n' f_n^y - w_{n,1}\right) - \frac{h^4}{24}\left(y_n^{(4)} + 3y_n'' f_n^y + 6w_{n,2} + 6y_n' g_{n,1}\right) + O\left(h^5\right). \quad (2.3)$$

Again, with the help of the formulas,

$$y''' = f^t + f^y y', \quad y^{(4)} = f^{tt} + 2f^{ty}y' + f^{yy}\left(y'\right)^2 + f^y f,$$

from (2.3) we obtain

$$t_n^{(2)}(h) = -\frac{h^4}{24}\left[4y_n^{(4)} + 3\left(1-2\beta\right) f^{yy}\left(y'\right)^2\right] + O\left(h^5\right).$$

The only order condition is

$$\beta = \frac{1}{2},$$

and then the local truncation error is given by

$$t_n^{(2)}(h) = -\frac{h^4}{6} y_n^{(4)} + O\left(h^5\right). \quad (2.4)$$

Thus, the resulting second order linearly implicit method is given by

$$\begin{aligned} \widetilde{y}_n &= y_n + \frac{1}{2}\Delta y_{n-1}, \\ \left[1 - \frac{h^2}{4} f^y\left(t_{n+1}, \widetilde{y}_n\right)\right]\Delta y_n &= \Delta y_{n-1} \\ &\quad + \frac{h^2}{4}\left[f_{n-1} + 2f_n + f\left(t_{n+1}, y_n\right)\right] + t_n^{(2)}(h). \end{aligned} \quad (2.5)$$

While we call the implicit second order method in (1.2) by M_2, we call the linearly implicit method (2.5) by $LI - M_2$. Note that $LI - M_2$ has the same principal local truncation error as the implicit method M_2.

We next discuss stability of the method $LI - M_2$. By applying the method to the test equation:

$$y'' = -\lambda^2 y, \quad \lambda > 0, \quad (2.6)$$

and setting $H = \lambda h$, we can write the resulting linear recurrence in the form:

$$A(H)\, y_{n+1} - 2B(H)\, y_n + A(H)\, y_{n-1} = 0, \quad (2.7)$$

where

$$A\left(H\right)=1+\frac{H^{2}}{4},\quad B\left(H\right)=1-\frac{H^{2}}{4}. \tag{2.8}$$

P-stability requires (see Lambert and Watson [8]) that the roots of the *characteristic equation* for (2.7):

$$A\left(H\right)\varsigma^{2}-2B\left(H\right)\varsigma+A\left(H\right)=0,$$

be complex conjugate and modulus one. This is equivalent to the requirements that

$$A\left(H\right)\pm B\left(H\right)>0. \tag{2.9}$$

It is readily verified that for the method $LI-M_2$, with $A\left(H\right)$ and $B\left(H\right)$ given by (2.8), both the conditions in (2.9) are satisfied for all $H>0$, and hence, the method $LI-M_2$ is a P-stable method.

3. A Fourth Order Numerov-Type Linearly Implicit Scheme

In order to obtain a fourth order linearly implicit scheme, we consider a linearization of (1.6) of the form:

$$\begin{aligned}
&\widetilde{y}_n=y_n+\alpha_1\Delta y_{n-1},\quad \widehat{y}_n=y_n+\alpha_2\Delta y_{n-1}+\beta_2h^2f_n,\\
&\overline{y}_n=y_n-\alpha h^2\left[f\left(t_{n+1},y_n\right)-2f_n+f_{n-1}\right],\\
&\left[1-\frac{h^2}{12}\left\{a_0f^y\left(t_{n+1},y_n\right)+a_1f^y\left(t_{n+1},\widetilde{y}_n\right)+a_2f^y\left(t_{n+1},\widehat{y}_n\right)\right\}+\frac{5\alpha}{6}h^4\left(f_n^y\right)^2\right]\\
&\qquad\times\Delta y_n=\Delta y_{n-1}+\frac{h^2}{12}\left[f_{n-1}+10f\left(t_n,\overline{y}_n\right)+f\left(t_{n+1},y_n\right)\right]+t_n^{(4)}\left(h\right).
\end{aligned} \tag{3.1}$$

The consistency condition is

$$a_0+a_1+a_2=1. \tag{3.2}$$

Setting

$$\begin{aligned}
&g_n=a_0f^y\left(t_{n+1},y_n\right)+a_1f^y\left(t_{n+1},\widetilde{y}_n\right)+a_2f^y\left(t_{n+1},\widehat{y}_n\right),\quad w_n=f\left(t_n,\overline{y}_n\right),\\
&u_n=f\left(t_{n+1},y_n\right),\quad v_n=f\left(t_{n+1},y_n\right)-2f_n+f_{n-1},
\end{aligned}$$

by Taylor expansion, we obtain

$$\begin{aligned}
t_n^{(4)}\left(h\right)=&\frac{h^2}{12}\left(11y_n''-10w_n-u_n\right)+\frac{h^3}{12}\left(y_n'''-y_n'g_n\right)+\frac{h^4}{24}\left(y_n^{(4)}-y_n''g_n\right)\\
&+\frac{h^5}{72}\left(y_n^{(5)}-y_n'''g_n+60\alpha y_n'\left(f_n^y\right)^2\right)+O\left(h^6\right).
\end{aligned} \tag{3.3}$$

Now, it can be seen that

$$u_n = f_n + hu_{n,1} + h^2 u_{n,2} + h^3 u_{n,3} + O\left(h^4\right),$$

with

$$u_{n,1} = f_n^t, \quad u_{n,2} = \frac{1}{2} f_n^{tt}, \quad u_{n,3} = \frac{1}{6} D_t^3 f_n, \quad D_t = \frac{\partial}{\partial t},$$
$$v_n = hv_{n,1} + O\left(h^2\right), \quad v_{n,1} = f_n^t - y_n''',$$
$$w_n = f_n - h^3 w_{n,3} + O\left(h^4\right), \quad w_{n,3} = \alpha v_{n,1} f_n^y,$$

and

$$g_n = f_n^y + hg_{n,1} + h^2 g_{n,2} + O\left(h^3\right),$$

with

$$\begin{aligned}
g_{n,1} &= f_n^{yt} + (a_1\alpha_1 + a_2\alpha_2)\, y_n' f_n^{yy}, \\
g_{n,2} &= \frac{1}{2} f_n^{ytt} + \left\{ a_2\beta_2 - \frac{1}{2}(a_1\alpha_1 + a_2\alpha_2) \right\} f_n f_n^{yy} + (a_1\alpha_1 + a_2\alpha_2)\, y_n' f_n^{yyt} \\
&\quad + \frac{1}{2}\left(a_1\alpha_1^2 + a_2\alpha_2^2\right)\left(y_n'\right)^2 f_n^{yyy}.
\end{aligned}$$

With these results, from (3.3) we get

$$\begin{aligned}
t_n^{(4)}(h) &= -\frac{h^3}{12}\left(u_{n,1} - y_n''' + y_n' f_n^y\right) - \frac{h^4}{24}\left(2u_{n,2} + 2y_n' g_{n,1} - y_n^{(4)} + y_n'' f_n^y\right) \\
&+ \frac{h^5}{72}\left(60 w_{n,3} - 6u_{n,3} - 6y_n' g_{n,2} - 3y_n'' g_{n,1} + y_n^{(5)} - y_n''' f_n^y + 60\alpha y_n' \left(f_n^y\right)^2\right) \\
&+ O\left(h^6\right). \quad (3.4)
\end{aligned}$$

Again, with the formulas for third and fourth derivatives of y given above, and with the formula:

$$\begin{aligned}
y^{(5)} &= D_t^3 f + 3f^{tty} y' + 3f^{yyt}\left(y'\right)^2 + 3f^{ty} f + f^{yyy}\left(y'\right)^3 \\
&\quad + 3f^{yy} y' f + f^y f^t + \left(f^y\right)^2 y',
\end{aligned}$$

from (3.4) we obtain

$$\begin{aligned}
t_n^{(4)}(h) &= -\frac{h^4}{12}\left(a_1\alpha_1 + a_2\alpha_2 - \frac{1}{2}\right) \\
&- \frac{h^5}{72}\left[\begin{array}{c} 3(2a_2\beta_2 - 1) f_n^{yy} f_n y_n' + 3\left\{2(a_1\alpha_1 + a_2\alpha_2) - 1\right\} f_n^{yyt}\left(y_n'\right)^2 \\ + \left\{3\left(a_1\alpha_1^2 + a_2\alpha_2^2\right) - 1\right\} f_n^{yyy}\left(y_n'\right)^3 \end{array} \right] \\
&+ O\left(h^6\right). \quad (3.5)
\end{aligned}$$

From (3.5) it follows that conditions for the method to be order four are given by

$$\left.\begin{aligned} a_1\alpha_1 + a_2\alpha_2 &= \frac{1}{2}, \\ a_2\beta_2 &= \frac{1}{2}, \\ a_1\alpha_1^2 + a_2\alpha_2^2 &= \frac{1}{3}, \end{aligned}\right\} \tag{3.6}$$

in addition to the consistency condition (3.2). It follows that $a_2 \neq 0$, $\beta_2 \neq 0$. A solution of the order conditions is given by

$$\begin{aligned} a_0 &= \frac{6\alpha_1\alpha_2 - 3(\alpha_1+\alpha_2)+2}{6\alpha_1\alpha_2}, \quad a_1 = \frac{3\alpha_2 - 2}{6\alpha_1(\alpha_2-\alpha_1)}, \\ a_2 &= \frac{2-3\alpha_1}{6\alpha_2(\alpha_2-\alpha_1)}, \quad \beta_2 = \frac{3\alpha_2(\alpha_2-\alpha_1)}{2-3\alpha_1}, \end{aligned}$$

provided that $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\alpha_1 \neq 2/3$.

Thus, we have obtained a two-parameter family of fourth order linearly implicit methods (3.1) with α_1 and α_2 as parameters. Since these parameters do not affect stability, we choose the parameters to obtain a 'simpler' method. With the choice $\alpha_2 = 2/3$, $a_1 = 0$, and choosing $\alpha_1 = 0$, we obtain the following method:

$$\begin{aligned} &\widehat{y}_n = y_n + \frac{2}{3}\Delta y_{n-1} + \frac{2}{3}h^2 f_n, \\ &\overline{y}_n = y_n - \alpha h^2\left[f(t_{n+1}, y_n) - 2f_n + f_{n-1}\right], \\ &\left[1 - \frac{h^2}{48}\left\{f^y(t_{n+1}, y_n) + 3f^y(t_{n+1}, \widehat{y}_n)\right\} + \frac{5\alpha}{6}h^4\left(f_n^y\right)^2\right]\Delta y_n \\ &= \Delta y_{n-1} + \frac{h^2}{12}\left[f_{n-1} + 10f(t_n, \overline{y}_n) + f(t_{n+1}, y_n)\right]. \end{aligned} \tag{3.7}$$

While we call the functionally implicit method (1.6) by M_4, we call the present fourth order linearly implicit method (3.7) by $LI - M_4$.

For stability of the method $LI - M_4$, by applying the method to the test equation in (2.6) we obtain (2.7), where now

$$A(H) = 1 + \frac{1}{12}H^2 + \frac{5\alpha}{6}H^4, \quad B(H) = 1 - \frac{5}{12}H^2 + \frac{5\alpha}{6}H^4. \tag{3.8}$$

With $A(H)$ and $B(H)$ given by (3.8), it is easily verified that the two conditions in (2.9) are satisfied for $\alpha > 1/120$. Thus, the fourth order linearly implicit method $LI - M_4$ described in (3.7) is P-stable for all $\alpha > 1/120$.

4. Numerical Illustrations

We illustrate computationally the order, accuracy and stability of the obtained linearly implicit methods $LI-M_2$ of second order and $LI-M_4$ of fourth order. The performance of these methods is compared, respectively, with the functionally implicit methods M_2 and M_4. For numerical illustration, we consider the following two problems of practical interest, including a *non-autonomous* problem.

Problem 1. Consider the equation of motion of a one mass-spring system in which the spring with *material nonlinearity* provides the restoring force:

$$my'' + k_0 y\left(1+\varepsilon y^2\right) = 0, \quad y\left(0\right)=1, \; y'\left(0\right)=0. \tag{4.1}$$

Note that when $\varepsilon > 0$, the spring becomes stiffer with the displacement (see, e.g. Fried [7], p. 207]). We computed the solution of (4.1) for $\varepsilon = 1$, $k_0 = 1$, $m = 1$ for the time-range $0 \le t \le 20$. The absolute errors in computing the solution at $t = 20$ are shown in Table 1. It is clear that the linearly implicit methods $LI-M_2$ and $LI-M_4$ do provide second and fourth order accuracy, respectively, comparable with that provided by the functionally implicit methods. The approximations computed for fourth order methods with $h = \frac{1}{5}$ are displayed in Figure 1 to graphically illustrate the accuracy provided by the linearly implicit method $LI-M_4$.

h	M_2	$LI-M_2$	order	M_4	$LI-M_4$	order
$\frac{1}{5}$	1.2(-1)	1.9(-1)		1.9(-3)	2.8(-3)	
$\frac{1}{10}$	3.1(-2)	4.0(-2)	2.25	1.2(-4)	1.7(-4)	4.04
$\frac{1}{20}$	7.9(-3)	9.0(-3)	2.15	7.3(-6)	1.0(-5)	4.09
$\frac{1}{40}$	1.9(-3)	2.0(-3)	2.17	4.3(-7)	5.8(-7)	4.12

Table 1: Absolute errors at $t = 20$

Problem 2. We consider a Painlevé equation (see Borrelli et al [1]):

$$y'' = y^2 - t, \;\; 0 < t < 20, \quad y\left(0\right)=0, \; y'\left(0\right)=\eta_1, \tag{4.2}$$

with $-5 \le \eta_1 \le 1$. Some solutions appear to "blow up" in finite time, while others display typical oscillatory (but non-periodic) behavior of a Painlevé solution. We computed the solution of (4.2) with $\eta_1 = 0$. The absolute errors in computing the solution at $t = 20$ are shown in Table 2. It is clear that the linearly implicit methods $LI-M_2$ and $LI-M_4$ do provide second order and fourth order accuracy, respectively, with comparable accuracy as provided by the respective order implicit methods. The approximations computed with second order methods with $h = \frac{1}{20}$ are displayed in Figure 2.

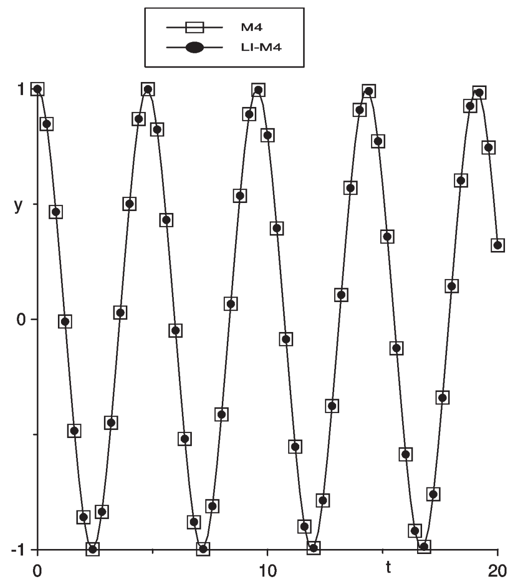

Figure 1: Fourth order methods for Problem 1

h	M_2	$LI-M_2$	order	M_4	$LI-M_4$	order
$\frac{1}{5}$	4.8(-1)	4.8(-1)		7.0(-3)	6.8(-3)	
$\frac{1}{10}$	1.1(-1)	1.1(-1)	2.13	4.4(-4)	4.3(-4)	3.98
$\frac{1}{20}$	2.5(-2)	2.5(-2)	2.14	2.9(-5)	2.8(-5)	3.94
$\frac{1}{40}$	5.8(-3)	5.8(-3)	2.11	2.1(-6)	2.1(-6)	3.74

Table 2: Absolute errors at $t = 20$

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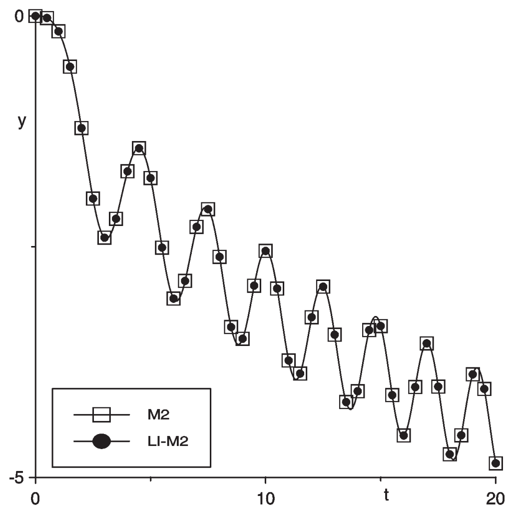

Figure 2: Second order methods for Problem 2

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