International Journal of Pure and Applied Mathematics
Volume 21 No. 1 2005, 51-57

FINITELY ⊕−SUPPLEMENTED MODULES

Hamza Çalışıcı[1 §], Ali Pancar[2]

[1]Department of Mathematics
Faculty of Education
Ondokuz Mayıs University
Amasya, 05189, TURKEY
e-mail: hcalisici@omu.edu.tr

[2]Department of Mathematics
Faculty of Arts and Sciences
Ondokuz Mayıs University
Samsun, 55139, TURKEY
e-mail: apancar@omu.edu.tr

Abstract: Let R be a ring and M a left R-module. M is called *finitely H-supplemented* if for every finitely generated submodule N of M, there exists a direct summand L of M such that $M = N + X$ holds if and only if $M = L + X$. M is called *finitely ⊕−supplemented* if every finitely generated submodule N of M has a supplement that is a direct summand of M. In this paper various properties of these modules are given. It is shown that a ring R is finitely semiperfect if and only if every finitely generated free R-module is finitely ⊕−supplemented.

AMS Subject Classification: 16D99
Key Words: semiperfect ring, supplement submodule, finitely ⊕− supplemented module

1. Introduction

Throughout this paper R is an associative ring with identity and all modules are

Received: March 23, 2005

§Correspondence author

unital left R-modules. Let M be an R-module. Rad$M$ will indicate Jacobson Radical of M. A submodule N of M is called *small* in M (notation $N \ll M$) if for every submodule $K \subset M$, the equality $N + K = M$ implies $K = M$. For a submodule $N$of M, a submodule L of M is called a *supplement* of N in M if L is a minimal element in the set of submodules K of M such that $M = N + K$. This is equivalent to $M = N + L$ and $N \cap L \ll L$. An R-module M is called *supplemented* if every submodule of M has a supplement in $M$and M is called *finitely supplemented* or briefly *f-supplemented* if every finitely generated submodule of M has a supplement in M (see [8]). Artinian modules are supplemented. Following [6], M is called *$\oplus-$supplemented* if every submodule of M has a supplement that is a direct summand of M. Clearly $\oplus-$supplemented modules are supplemented. Again following [6], M is called *H-supplemented* if for every submodule A of M, there exists a direct summand A' of M such that $M = A + X$ holds if and only if $M = A' + X$.

For basic properties of supplemented, $\oplus-$supplemented and H-supplemented modules we refer to [6] and [8].

2. Finitely $\oplus-$Supplemented Modules

We call a module M *finitely H-supplemented* if for every finitely generated submodule N of M, there exists a direct summand L of M such that $M = N + X$ holds if and only if $M = L + X$. Clearly every H-supplemented module is finitely H-supplemented. We call a module M *finitely $\oplus-$supplemented* or shortly *$f - \oplus-$supplemented* if every finitely generated submodule of M has a supplement that is a direct summand of M. For example, the module M with Rad$M = M$ is finitely $\oplus-$supplemented.

Proposition 2.1. *Every finitely H-supplemented R-module is finitely $\oplus-$supplemented.*

Proof. Let M be a finitely H-supplemented module and N be any finitely generated submodule of M. Then, there exists a direct summand L of M such that $M = N + X$ if and only if $M = L + X$. Let $M = L \oplus L'$. In particular, $M = N + L'$. If $N \cap L' + U = L'$ for a submodule U of L', then we have $N + L' = N + U = L + U = M = L \oplus L'$ hence $U = L'$. Thus $N \cap L' \ll L'$. □

Note that $\oplus-$supplemented modules are finitely $\oplus-$supplemented. Also, finitely $\oplus-$supplemented artinian modules are $\oplus-$supplemented, but it is not generally true that every finitely $\oplus-$supplemented module is $\oplus-$supplemented as the following example shows.

To show this we will consider von Neumann regular rings.

Recall that a ring R is called a *von Neumann regular* or shortly *regular* if for each $a \in R$ there is a $b \in R$ such that $aba = a$. Note that a regular ring R need not be semisimple (see [2, p. 176]).

Example. Let R be a regular ring not semisimple. Then the R-module ${}_RR$ is finitely $\oplus-$supplemented by [2, p. 176], but it is not $\oplus-$supplemented.

Suppose that ${}_RR$ is $\oplus-$supplemented and let N be any left ideal of R. Then there exists a submodule K of ${}_RR$ such that $R = N + K$ and $N \cap K \ll K$. Now $N \cap K \ll R$ and hence $N \cap K \leq \operatorname{Rad} R$. By [7, Theorem 3.3.18], $\operatorname{Rad} {}_RR = 0$ and so $N \cap K = 0$, that is, N is a direct summand of ${}_RR$. Hence ${}_RR$ is semisimple, which is a contradiction.

Let M be a R-module. A submodule K of M is *cofinite* in M if the factor module $\frac{M}{K}$ is finitely generated.

Proposition 2.2. *Let M be a finitely supplemented module such that every maximal submodule of M is a direct summand of M. Then M is finitely $\oplus-$ supplemented.*

Proof. Let N be a finitely generated submodule of M. Then by the assumption there exists a submodule K of M such that $M = N + K$ and $N \cap K \ll K$. Note that $\frac{M}{K} \cong \frac{N}{N \cap K}$ is finitely generated, so K is cofinite in M. By [1, Lemma 2.7], K is a direct summand of M. □

Lemma 2.3. (see [8], 41.3) *Let M be a finitely supplemented module such that* $\operatorname{Rad} M \ll M$. *Then every finitely generated submodule of $\frac{M}{\operatorname{Rad} M}$ is a direct summand.*

Corollary 2.4. *If M is a finitely supplemented module such that* $\operatorname{Rad} M \ll M$, *then $\frac{M}{\operatorname{Rad} M}$ is finitely $\oplus-$ supplemented.*

Now we show the relationship between finitely H-supplementation and the lifting property.

Proposition 2.5. *Let M be a module with* $\operatorname{Rad} M \ll M$. *Then M is finitely H-supplemented if and only if every finitely generated submodule of $\frac{M}{\operatorname{Rad} M}$ is a direct summand and every finitely generated direct summand of $\frac{M}{\operatorname{Rad} M}$ lifts to a direct summand of M.*

Proof. (⇒) The first statement follows from Lemma 2.3. Let $\bar{N} = \frac{N}{\operatorname{Rad} M}$ be a finitely generated submodule of $\frac{M}{\operatorname{Rad} M}$. Then N is of the form $N = K + \operatorname{Rad} M$, with K finitely generated. Since M is finitely H-supplemented, there exists a direct summand L of M such that $M = L \oplus L'$, and $M = K + X$ holds if and only if $M = L + X$. Then $M = K + L'$ and $K \cap L' \ll L'$. Since $\operatorname{Rad} M \ll M$ and $K \cap L' \leq \operatorname{Rad} M$, it follows that $\bar{M} = \bar{K} \oplus \bar{L}'$and hence $\bar{M} = \bar{N} \oplus \bar{L}'$. Now we show that $\bar{N} = \bar{L}$. Note that

$$\frac{K}{K \cap L'} \cong \frac{K + L'}{L'} = \frac{M}{L'} \cong L$$

is finitely generated and hence $\overline{K + L} = \overline{N + L}$ is finitely generated submodule of $\bar{M}$. Since every finitely generated submodule of $\bar{M}$ is a direct summand, there exists a direct summand $\bar{S}$ of $\bar{M}$ such that $\bar{M} = \overline{N + L} \oplus \bar{S}$. Hence $M = N + L + S$ and $(N + L) \cap S = \operatorname{Rad} M$. Then $M = N + S = L + S$. By modularity $N + L = (N + L) \cap (L + S) = L + S \cap (N + L) = L + \operatorname{Rad} M$ and $N + L = (N + L) \cap (N + S) = N + S \cap (N + L) = N$. Hence $N = L + \operatorname{Rad} M$, that is, $\bar{N} = \bar{L}$.

($\Leftarrow$)Let N be a finitely generated submodule of M. Then $\bar{N} = \frac{N + \operatorname{Rad} M}{\operatorname{Rad} M}$ is finitely generated submodule of $\bar{M}$. By assumption, $\bar{N}$ is a direct summand of $\bar{M}$ with $\bar{M} = \bar{N} \oplus \bar{K}$ and there exists a direct summand L of M such that $\bar{L} = \bar{N}$. Hence $N \cap K = \operatorname{Rad} M$. Since $\operatorname{Rad} M \ll M$, $\quad M = N + X$ holds if and only if $M = L + X$. Hence M is finitely H-supplemented. □

Let R be a ring and M an R-module. We consider the following condition.

(D3) If M_1 and M_2 are direct summands of M such that $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

If M is $\oplus-$supplemented module with (D3) then M is completely $\oplus-$-supplemented (i.e. every direct summand of M is $\oplus-$supplemented)(see [4, Proposition 2.3]). Now we show that the same is true for the property finitely $\oplus-$supplemented.

Theorem 2.6. *Let M be a finitely $\oplus-$ supplemented module with (D3). Then every direct summand of M is finitely $\oplus-$supplemented.*

Proof. Let N be a direct summand of M and let K be a finitely generated submodule of N. By the assumption, there exists a direct summand L of M such that $M = K + L$ and $K \cap L \ll L$. Then $N = K + L \cap N$. Since M has (D3), $L \cap N$ is a direct summand of M. Note that $L \cap N$ is also a direct summand of N. Then $K \cap (L \cap N) = K \cap L$ is small in $L \cap N$ by [8, 19.3]. Hence N is finitely $\oplus-$supplemented. □

Corollary 2.7. *Let M be a quasi-projective module. Then M is finitely $\oplus-$supplemented if and only if every direct summand of M is finitely $\oplus-$supplemented.*

Proof. Sufficiency is clear. Conversely, let M be a quasi-projective module. By [6, Lemma 4.6 and Proposition 4.38], M has (D3). □

Note that it is unknown that the finite direct sum of finitely ⊕−supplemented module is finitely ⊕−supplemented, but we can show the following theorem. Firstly we need the following technical lemma. □

Lemma 2.8. *Let M be an R-module and U, M_1 be submodules of M such that U is finitely generated, M_1 finitely supplemented. If $M_1 + U$ has a supplement X in M such that $M_1 \cap (U + X)$ is finitely generated, then $M_1 \cap (U + X)$ has a supplement Y in M_1, and $X + Y$ is a supplement of U in M.*

Proof. Let X be a supplement of $M_1 + U$ in M. Then $M = (M_1 + U) + X$ and $(M_1 + U) \cap X$ is small in X. By the assumption, $M_1 \cap (U + X)$ is finitely generated submodule of M_1. Since M_1 is finitely supplemented, $M_1 \cap (U + X)$ has a supplement Y in M_1. Note that $M_1 = M_1 \cap (U + X) + Y$ and $M_1 \cap (U + X) \cap Y = (U + X) \cap Y$ is small in Y. Then $M = (M_1 + U) + X = [M_1 \cap (U + X) + Y] + U + X = U + X + Y$ and by [8, 19.3],

$$\begin{aligned} U \cap (X + Y) &\leq X \cap (U + Y) + Y \cap (U + X) \\ &\leq X \cap (U + M_1) + Y \cap (U + X) \ll X + Y. \end{aligned}$$

Therefore $X + Y$ is a supplement of U in M. □

Theorem 2.9. *Let M be an R-module and $M = M_1 \oplus M_2$ such that M_1, M_2 are finitely generated and finitely ⊕−supplemented. Assume that:*

(i) M is coherent, or

(ii) M is quasi-projective.

Then M is finitely ⊕−supplemented.

Proof. Let U be a finitely generated submodule of M.

(i) Clearly $M = M_1 + M_2 + U$ has the trivial supplement 0 in M. Since M is coherent, by [8, 26.1], $M_2 \cap (M_1 + U + 0)$, as an intersection of finitely generated submodules, is finitely generated. Since M_2 is finitely ⊕−supplemented, $M_2 \cap (M_1 + U)$ has a supplement X in M_2 such that X is a direct summand of M_2. By Lemma 2.8, X is a supplement of $M_1 + U$ in M, that is, $M = M_1 + U + X$ and $(M_1 + U) \cap X \ll X$. Now

$$\frac{M}{M_1 + U} \cong \frac{X}{(M_1 + U) \cap X}$$

is finitely generated, and so X is finitely generated by [8, 19.6]. Hence $M_1 \cap (X + U)$ is also finitely generated by [8, 26.1]. Since M_1 is finitely ⊕−supplemented, $M_1 \cap (X + U)$ has a supplement Y in M_1 such that Y is a direct summand of M_1. Applying again Lemma 2.8, we have that $X + Y$ is a supplement of U in M. Since every M_i $(i = 1, 2)$ is a direct summand of M, it follows that $X + Y = X \oplus Y$ is a direct summand of M.

(ii) Let M be quasi-projective module. Then by [8, 18.1],

$$\frac{M_1+U}{M_2\cap(M_1+U)}\cong\frac{M_1+M_2+U}{M_2}=\frac{M}{M_2}\cong M_1$$

is $M-$projective and by [2, 16.12], $(M_1+U)-$projective. Hence $M_2\cap(M_1+U)$ is a direct summand of M_1+U and so it is finitely generated. Since M_2 is finitely $\oplus-$supplemented, $M_2\cap(M_1+U)$ has a supplement X in M_2 such that X is a direct summand of M_2. By Lemma 2.8, X is a supplement of M_1+U in M. From the proof of (i), X is finitely generated.

Now

$$\frac{X+U}{M_1\cap(X+U)}\cong\frac{M_1+X+U}{M_1}=\frac{M}{M_1}\cong M_2$$

is $M-$projective by [8, 18.1] and by [2, 16.12], it is $(X+U)-$projective. Hence $M_1\cap(X+U)$ is a direct summand of $X+U$ and so it is finitely generated. Since M_1 is finitely $\oplus-$supplemented, $M_1\cap(X+U)$ has a supplement Y in M_1 such that Y is a direct summand of M_1. Applying again Lemma 2.8, we have that $X+Y$ is a supplement of U in M such that $X+Y=X\oplus Y$ is a direct summand of M. □

It was shown in [5, Theorem 2.1] that R is semiperfect if and only if every finitely generated free R-module is $\oplus-$supplemented. Now we will give an analogous characterization for finitely semiperfect rings.

Let M be a R-module. M is called *finitely semiperfect* if for every finitely generated submodule N of M, the factor module $\frac{M}{N}$ has a projective cover.

Proposition 2.10. *Let M be a projective module. Then M is finitely semiperfect if and only if M is finitely $\oplus-$supplemented.*

Proof. By [3, Proposition 1.4]. □

Theorem 2.11. *For a ring R, the following statements are equivalent*:
(i) R is finitely semiperfect.
(ii) ${}_RR$is finitely $\oplus-$supplemented.
(iii) Every finitely generated free R- module is finitely $\oplus-$supplemented.

Proof. $(i)\Leftrightarrow(ii)$ By Proposition 2.10.

$(ii)\Rightarrow(iii)$ Let M be a finitely generated free R-module. Then, it is well known that there exists elements a_i of M such that $M=\overset{n}{\underset{i=1}{\oplus}}Ra_i$ and $R\cong Ra_i$ for all $i=1,2,...,n$. By assumption, every cyclic R-module Ra_i is finitely $\oplus-$supplemented and M is finitely $\oplus-$supplemented by Theorem 2.9.

$(iii)\Rightarrow(ii)$ Clear. □

In the beginning of this section we gave an example of module, which is finitely ⊕−supplemented but not ⊕−supplemented. Finally, we give another example of module, which holds this condition.

Example. Let R be a finitely semiperfect ring not semiperfect. Then, for a finite index set Λ, the R-module $R^{(\Lambda)}$ is finitely ⊕−supplemented by Theorem 2.11 but not ⊕−supplemented by [5, Theorem 2.1].

3. Acknowledgments

We would like to thank Refail Alizade for his interest and helpful suggestions.

References

[1] R. Alizade, G. Bilhan, P.F. Smith, Modules whose maximal submodules have supplements, *Comm. Algebra*, **29**, No. 6 (2001), 2389-2405.

[2] F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, Springer, New York (1992).

[3] G. Azumaya, F-semi-perfect modules, *J. Algebra*, **136** (1991), 73-85.

[4] A. Harmancı, D. Keskin, P.F. Smith, On ⊕−supplemented modules, *Acta Math. Hungar.*, **83**, No. 1-2 (1999), 161-169.

[5] D. Keskin, P.F. Smith, W. Xue, Rings whose modules are ⊕−supplemented, *J. Algebra*, **218** (1999), 470-487.

[6] S.H. Mohamed, B.J. Müller, *Continuous and Discrete Modules*, London Math. Soc. LNS, **147**, Cambridge Univ. Press, Cambridge (1990).

[7] P. Ribenboim, *Rings and Modules*, John Wiley and Sons, New York (1967).

[8] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Philadelphia (1991).