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WAVELETS IN A FINITE DIMENSIONAL VECTOR SPACE AND DECOMPOSITION OF A SIGNAL

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Abstract: Wavelets in the vector space $V = [F_p(\sqrt{2}, \sqrt{m})]^N$ over the field $F = F_p(\sqrt{2}, \sqrt{m})$, where p is an odd prime and $N \equiv m(\text{mod } p)$ is constructed. This is used in expressing V as a direct sum of subspaces $V_j's$ based on the scales. An algorithm in decomposing a signal at various resolutions is also provided. The algorithm is implemented using *MATLAB* and its application in power engineering is mentioned.

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1. Introduction

In analysing frequency of a signal Fourier transforms have been used as a powerful tool. However, Fourier transforms are not very much suitable for those problems dealing with a signal which is localized in time or space. But wavelet transforms do serve the purpose in such cases. Unlike the Fourier transforms, these transforms make no assumptions regarding the periodicity of the data.

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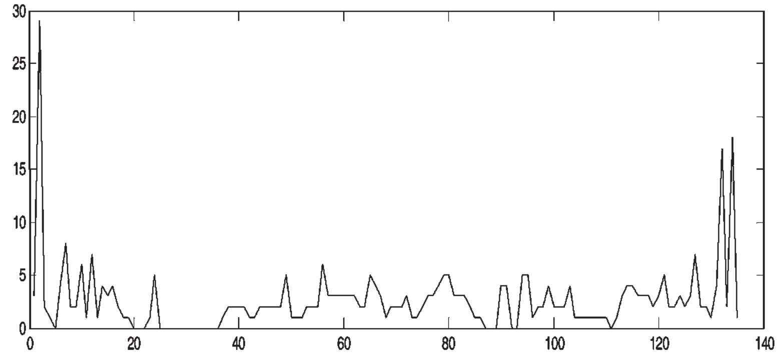

Figure 1: Original signal

Further one can construct wavelets having different characteristics which enables researchers to choose different wavelets based on their applications.

Mathematically, a wavelet is a function in $L^2(\mathbf{R})$ such that its translations and dilations form an orthonormal basis for $L^2(\mathbf{R})$, where $L^2(\mathbf{R})$ denotes the collection of all square integrable measurable functions on $\mathbf{R}$. There is a procedure due to Mallat [4] called multi resolution analysis in constructing such wavelets. This is defined as follows:

A multi resolution analysis of $L^2(\mathbf{R})$ is a sequence of closed subspaces $...V_{-1}, V_0, V_1, ...$ such that:

1. $V_n \subset V_{n-1}$, $n \in \mathbf{Z}$.
2. $\bigcup_{n=-\infty}^{\infty} V_n$ is dense in $L^2(\mathbf{R})$ and $\bigcap_{n=-\infty}^{\infty} V_n = \{0\}$.
3. $f(.) \in V_n \Leftrightarrow f(2.) \in V_{n-1}$.
4. $f(.) \in V_n \Leftrightarrow f(. - k) \in V_n$ for all $k \in \mathbf{Z}$.
5. There exists a function $\varphi \in V_0$ such that the collection $\{\varphi(. - k)/k \in \mathbf{Z}\}$ is an orthonormal basis for V_0.

If $\{\varphi(. - k)/k \in \mathbf{Z}\}$ is an orthonormal basis for V_0 then each $f \in V_0$ can be written as

$$f(x) = \sum_{k=-\infty}^{\infty} a_k \varphi(x - k).$$

As $\varphi \in V_0$, and $V_0 \subset V_{-1}$, we get $\varphi \in V_{-1}$. Then by property (3) we get $\varphi(\frac{1}{2}x) \in V_0$. Thus we have

$$\varphi(\frac{1}{2}x) = \sum_{n=-\infty}^{\infty} c_n \varphi(x - n), \qquad x \in \mathbf{R},$$

for some numbers $\{c_n\}$. Let $h_n = \frac{c_n}{\sqrt{2}}$. Then

$$\varphi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} h_n \varphi(2x - n), \qquad x \in \mathbf{R}.$$

This equation is called scaling identity or dilation equation . Again,we need a normalization $\int_{-\infty}^{\infty} \varphi(x)dx = 1$, which in turn gives $\sum_{n=-\infty}^{\infty} h_n = \sqrt{2}$.

Thus, given a φ with multiresolution analysis, we define the function ψ as

$$\psi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} g_n \varphi(2x - n), \quad x \in \mathbf{R}, \quad \text{where } g_n = (-1)^n h_{1-n}.$$

This ψ leads to an orthonormal basis in $L^2(\mathbf{R})$. The function φ is called scaling function and ψ is called the mother wavelet.

For a detailed study of wavelets, we refer to Daubechies [1].

Although wavelets in $L^2(\mathbf{R})$ is highly useful in various fields, it is better to study wavelets in a finite setting. For, the signals and images what we practically come across are only within the finite domain and the range. More specifically, the sample values of a signal or grey level values of an image are assumed to be taken from the set $\{0, 1, 2, \cdots, 2^n - 1\}$. Hence it is meaningful to construct wavelets in F^n over F, an appropriate finite field. In this paper, we wish to construct wavelets in $V = \left[F_p\left(\sqrt{2}, \sqrt{m}\right)\right]^N$ over $F = F_p(\sqrt{2}, \sqrt{m})$, where p is an odd prime and $N \equiv m \pmod{p}$ using Haar function. The motivation behind this is explained in Section 2.

It is important to observe that if we want to make use of wavelets in the study of signal or image processing problems, we consider them as functions in $L^2(\mathbf{R})$ or $L^2(\mathbf{R}^2)$. However in digital signal processing or digital image processing one deals with signals or images defined over a finite discrete set. For example, an image is mathematically a function $f : I_1 \times I_2 \to I$, where I_1, I_2 and I are intervals in $\mathbf{R}$. In particular, for a digital image the domain and range of f must be finite, discrete sets (see [2]). Further, for practical purposes, the sample values of a signal or grey level values of an image are assumed to be taken from the set $\{0, 1, 2, \cdots, 2^n - 1\}$. Moreover, in many of the signal processing problems, it is necessary to study the signals in various resolutions and express a signal as a sum of the signals obtained from various resolutions. If we regard the signal as a function in $L^2(\mathbf{R})$ and use wavelets in $L^2(\mathbf{R})$ to carry out such a decomposition and representation, the process becomes infinite. Further the representation of a signal from its components is only upto an approximation. Whereas here, as we work with finite dimensional vector space over a finite field, there is a definite level of decomposition (the number of scales are finite) and the representation is also exact. Thus it would be interesting if one can think of constructing wavelets in a finite dimensional vector space F^N over a finite field F instead of considering $L^2(\mathbf{R})$.

In this paper, we construct wavelets in $V = [F_p(\sqrt{2}, \sqrt{m})]^N$ over $F = F_p(\sqrt{2}, \sqrt{m})$, where p is an odd prime and m is an integer such that $N \equiv m(\text{mod } p)$. In other words, we assume that our functions (signals) are vectors $\overline{w} = (w^{(0)}, w^{(1)}, \cdots, w^{(N-1)})$, where each $w^{(i)} \in F_p(\sqrt{2}, \sqrt{m})$. The significance of adjoining $\sqrt{2}$ and $\sqrt{m}$ to F_p will be explained in due course. This wavelet basis helps us to write V as a direct sum of subspaces V_j's based on the scales, thus exhibiting a signal in various resolutions and again representing the signal as a sum of its decompositions. We also provide an algorithm and illustrate it with an example using *MATLAB*. We also give an outline of an application in power engineering.

A digital signal is denoted by a vector $\bar{w} = (w^{(0)}, w^{(1)} \cdots, w^{(N-1)})$, with each $w^{(i)} \in \mathbf{N}$. Now let $M_0 = \max_{0 \leq i \leq N-1}(w^{(i)})$. We choose an odd prime number p as follows. If M_0 is an odd prime number, we take $p = M_0$. Otherwise we choose a prime number p which is immediately larger than M_0. Since we deal with a practical signal, $M_0 < \infty$. Now define m such that $N \equiv m(\text{ mod } p)$. The problem is to decompose $\bar{w}$ based on its scales. We view $\bar{w} \in F^n$, where F is a field. Thus we decompose $\bar{w}$ as $\bar{w} = \bar{w}_1 + \bar{w}_2 + \cdots + \bar{w}_k$ for each scale $j = 1, 2, \cdots k$.

2. Modelling of the Problem

To obtain the part of a signal corresponding to a binary scale l, we do the following. If the given $N = 2^n$ for some n, we consider $\bar{w} \in F^N$. If N is not a power of 2, we take N' which is a power of 2 and immediately larger than N.

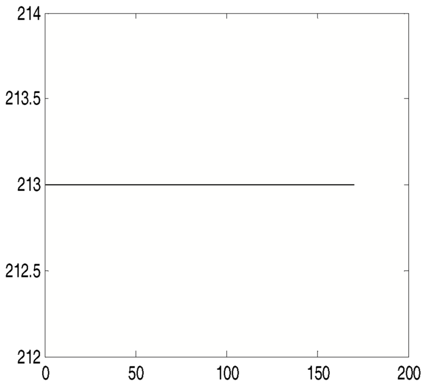

Figure 2.1: Resolution $j = 0$

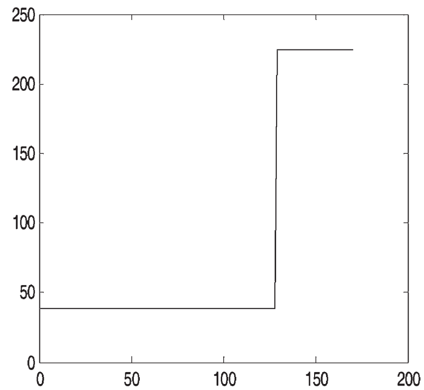

Figure 2.2: Resolution $j = 1$

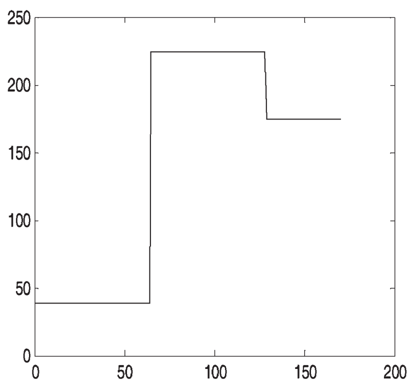

Figure 2.3: Resolution $j = 2$

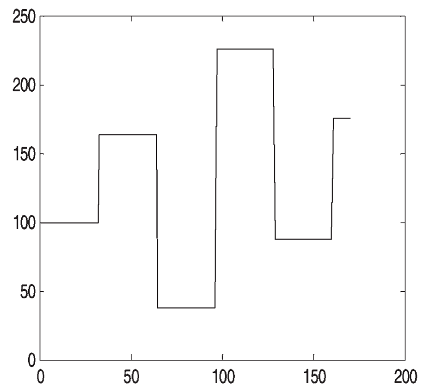

Figure 2.4: Resolution $j = 4$

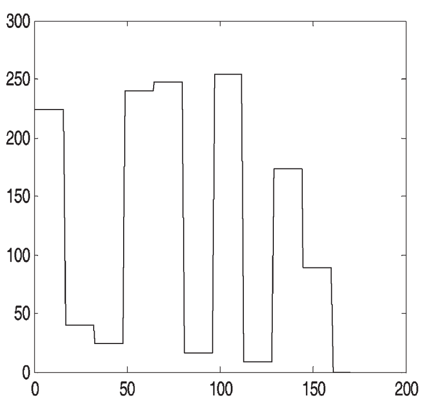

Figure 2.5: Resolution $j = 8$

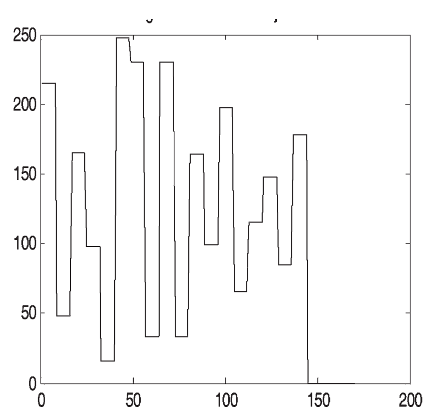

Figure 2.6: Resolution $j = 16$

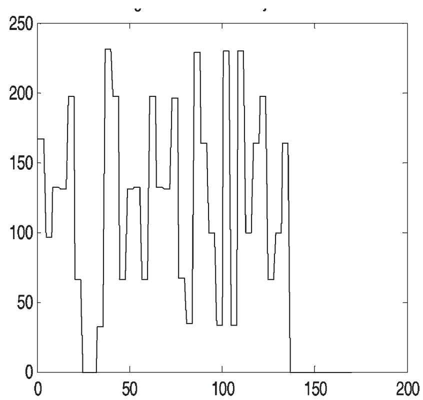

Figure 2.7: Resolution $j = 32$

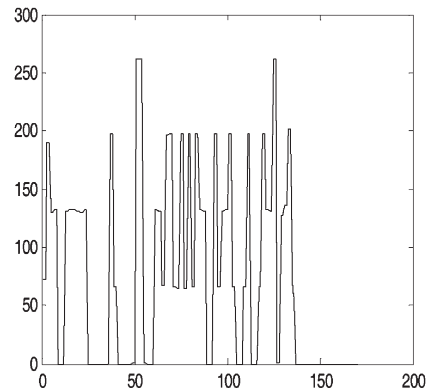

Figure 2.8: Resolution $j = 64$

To obtain the decomposition, we construct a wavelet basis for $V = F^N$, i.e., to find a $\psi \in V$ such that its translates and scales form a basis for V. We also require that the basis to be orthonormal in the sense of scalar product namely, a collection of vector $\{u_1, u_2, \cdots u_n\} \in F^N$ is called orthogonal if $u_i^T u_j = 0$ if $i \neq j$. An orthogonal set of vectors $\{u_1, u_2 \cdots, u_n\}$ is called orthonormal if in addition it satisfies the condition $u_i^T u_i = 1$. It is to be noticed that the operation of translation and dilatation must be taken in such a way that a vector in V after translation or dilation must belong to V (recall that $L^2(\mathbf{R})$ is translation and scaling invariant). Further the field F must be chosen in such a way that a vector in F^N after normalization (for the sake of orthonormality) must also belong of F^N (this is true in $L^2(\mathbf{R})$).

3. Background

3.1. Fields

Given a field F, an extension field K of F is a field containing F as its subfield. If $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_n \notin F$, then $K = F(\alpha_1, \alpha_2, \cdots \alpha_n)$ obtained by adjoining $\alpha_1, \alpha_2, \cdots \alpha_n$ to F denotes the collection of elements of the form $a_0 + a_1\alpha_1 + a_2\alpha_2 \cdots + a_n\alpha_n$, where $a_0, a_1 \cdots a_n \in F$. This K is a vector space over F. The degree of field extension denoted by $[K : F]$ is defined to be the dimension of K over F. A field having finitely many elements is called a finite field. The ring of integers modulo p, namely $\mathbf{Z}_p$ is a field, where p is a prime number. In general, any finite field K consists of p^r elements for a prime p and $r \geq 1$. Further any two finite fields of the same order are isomorphic to each other.

3.2. Quadratic Residue

Let p an odd prime and $\gcd(a, p) = 1$. If the congruence $x^2 \equiv a(\text{mod } p)$ has a solution, then 'a' is said to be a quadratic residue of p, otherwise 'a' is called a quadratic nonresidue of p.

3.3. Legendre Symbol

Let p be an odd prime and $\gcd(a, p) = 1$. The Legendre symbol (a/p) is defined by

$$(a/p) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue of } p\,, \\ -1 & \text{if } a \text{ is a quadratic nonresidue of } p\,. \end{cases}$$

If $a \equiv 0(\text{mod } p)$, then, $(a/p) = 0$. Further Legendre symbol is completely multiplicative. In other words $(ab/p) = (a/p)(b/p)$. The following theorem is well known.

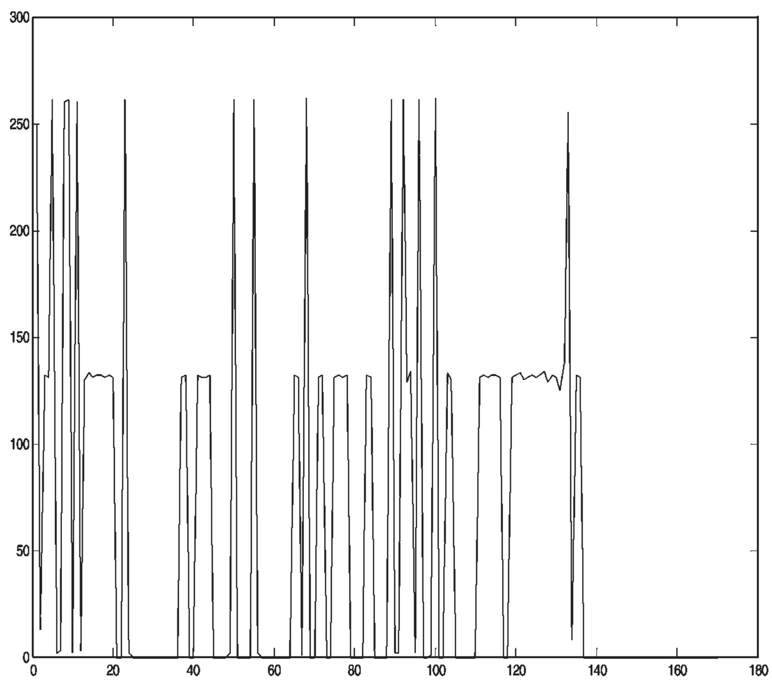

Figure 2.9: Resolution $j = 128$

Theorem 3.1. *If p is an odd prime, then*

$$(2/p) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 8 \text{ or } p \equiv 7 \pmod 8\,, \\ -1 & \text{if } p \equiv 3 \pmod 8 \text{ or } p \equiv 5 \pmod 8\,. \end{cases}$$

We say that an element $\sqrt{m} \in \mathbf{Z}_p$ if $x^2 \equiv m (\mathrm{mod}\ p)$ has a solution.

4. The Main Result

Definition 4.1. A vector $\bar{\psi} = (\ \psi^{(0)},\ \psi^{(1)} \cdots\ \psi^{(N-1)}) \in V$ is called a wavelet if $\bar{\psi}_{j,k} = (\ \psi^{(0j+k)}, \psi^{(1j+k)} \cdots\ \psi^{((n-1)j+k)})$ form an orthogonal set in V.

Haar Function. $\bar{\psi} = (\ \psi^{(0)},\ \psi^{(1)} \cdots\ \psi^{(N-1)})$, where

$$\psi^{(i)} = \begin{cases} 1 & \text{if } 0 \le \frac{i}{N} < \frac{1}{2}\,, \\ p-1 & \text{if } \frac{1}{2} \le \frac{i}{N} < 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Theorem 4.1. *Let $N = 2^n$. Let $V = \mathbf{Z}_p^N$. Then the collection of vectors $\{(1, 1 \cdots 1), \{\ \bar{\psi}_{j,((\frac{Nk}{j}) \bmod N)} : j = 1, 2, 2^2, \cdots 2^{n-1},\ k = 0, 1, \cdots j-1\}$ forms an orthogonal basis for V.*

Proof. It is easy to verify that the number of vectors in the set $\{\ \bar{\psi}_{j,((\frac{Nk}{j}) \bmod N)} : j = 1, 2, 2^2, \cdots 2^n,\ k = 0, 1, \cdots j-1\}$ is $1+2+2^2+\cdots 2^{n-1} = 2^n - 1$. This along with $\bar{\psi}_0 = \{1, 1, \cdots 1\}$ forms a set of 2^n vectors. Further orthogonality of a set of vectors in the sense of scalar product implies that those vectors are linearly independent. Since $\dim V = \dim F^N = N = 2^n$, these vectors form a basis for V. Thus it remains to show that the vectors are orthogonal. To prove that the scalar product $\bar{\psi}_{j,k}\ \bar{\psi}_{j',k'} \equiv 0 (\mathrm{mod}\ p)$ for $j \neq j'$ or $k \neq k'$. We shall establish this in three steps.

Step 1. Assume $j \neq j', k = k' = 0$.

Without loss of generality let us assume $j > j'$. It follows from our definition of Haar function that

$$\psi_{j,0}^{(i)} = \begin{cases} 1 & \text{if } 0 \le i < \frac{N}{2j}\,, \\ p-1 & \text{if } \frac{N}{2j} \le i < \frac{N}{j}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Similar definition holds for j' also. If $2j' > j$, we have $j' < j < 2j'$. But this is not possible as j, j' are divisors of 2^{n-1}. Thus we can assume $2j' \leq j$. This implies that $\frac{N}{2j'} \geq \frac{N}{j}$. Then

$$\bar{\psi}_{j,o}\,\bar{\psi}_{j',o} = \sum_{i=0}^{N-1} \psi_{j,o}^{(i)}\ \psi_{j',o}^{(i)}$$
$$= \sum_{i=0}^{\frac{N}{2j}-1} \psi_{j,0}^{(i)}\,\psi_{j',0}^{(i)} + \sum_{i=\frac{N}{2j}}^{\frac{N}{j}-1} \psi_{j,0}^{(i)}\,\psi_{j',0}^{(i)} + \sum_{i=\frac{N}{j}}^{N-1} \psi_{j,0}^{(i)}\,\psi_{j',0}^{(i)} = S_1 + S_2 + S_3 \quad \text{(say)}.$$

Consider S_1. Here $0 \leq i < \frac{N}{2j}$ and hence $\psi_{j,0}^{(i)} \equiv 1 (\text{mod } p)$ for all i. Since $\frac{N}{2j} < \frac{N}{2j'}$, $\psi_{j',0}^{(i)} \equiv 1 (\text{mod } p)$ for all i. Thus $S_1 = \frac{N}{2j} (\text{mod } p)$. Consider S_2. Here $\frac{N}{2j} \leq i < \frac{N}{j}$ and hence $\psi_{j,0}^{(i)} \equiv (p-1)\ (\text{mod } p) \equiv (-1)\ (\text{mod } p)$. Since $\frac{N}{j} \leq \frac{N}{2j'}$, $\psi_{j',0}^{(i)} \equiv 1 (\text{mod } p)$. Thus $S_2 = \frac{N}{2j}(-1)(\text{mod } p)$. Consider S_3. Here $\frac{N}{j} \leq i < N$. But $\psi_{j,0}^{(i)} = 0$ for all $i \geq \frac{N}{j}$, Thus $S_3 = 0$. Hence it follows that $\bar{\psi}_{j,0}\bar{\psi}_{j',0} = S_1 + S_2 + S_3 \equiv 0 (\text{mod } p)$ proving orthogonality.

Step 2. Let $j \neq j', k \neq 0$ we prove that $\psi_{j,0}$ and $\psi_{j',k}$ are orthogonal. Consider

$$\bar{\psi}_{j,0}\,\bar{\psi}_{j',k} = \sum_{i=0}^{N-1} \psi_{j,0}^{(i)}\,\psi_{j',k}^{(i)} = \sum_{i=0}^{N-1} \psi_{j,0}^{(i)}\,\psi_{j',0}^{(i+(\frac{Nk}{j'})\ \text{mod}\ N)}$$
$$= \sum_{i=0}^{N-1} \psi_{j,0}^{(i)}\,\psi_{j',0}^{(i+r)} \qquad \text{for some r}$$
$$= \sum_{i=0}^{N-1} \psi_{j,0}^{(i)}\,\psi_{j',0}^{(l)} \qquad \text{for some } l \text{ such that } 0 \leq l \leq N-1.$$

If $i = l$, the result follows from step 1. Assume $i \neq l$ then

$$\sum_{i=0}^{N-1} \psi_{j,0}^{(i)}\,\psi_{j',0}^{(l)} = \sum_{i=0}^{\frac{N}{2j}-1} \psi_{j,0}^{(i)}\,\psi_{j,0}^{(l)} + \sum_{i=\frac{N}{2j}}^{\frac{N}{j}-1} \psi_{j,0}^{(i)}\ \psi_{j',0}^{(l)} + \sum_{i=\frac{N}{j}}^{N-1} \psi_{j,0}^{(i)}\,\psi_{j',0}^{(l)}$$
$$= S_1 + S_2 + S_3 \quad \text{(say)}.$$

Consider S_1. Here $0 \leq i < \frac{N}{2j}$ and hence $\psi_{j,0}^{(i)} \equiv 1 (\text{mod } p)$. $\psi_{j',0}^{(l)} \equiv 1 (\text{mod } p)$ if $0 \leq l < \frac{N}{2j'}$ and $\psi_{j',0}^{(l)} \equiv (p-1)\ (\text{mod}\, p)$ if $\frac{N}{2j'} \leq l < \frac{N}{j'}$ and $\psi_{j,0}^{(l)} \equiv (p-1)\ (\text{mod}\, p)$

if $\frac{N}{2j'} \leq l < \frac{N}{j'}$. Thus $S_1 = \frac{N}{2j}$ if $0 \leq l < \frac{N}{2j'}$ and $S_1 = \frac{N}{2j}(p-1) \equiv \frac{-N}{2j}$ (mod p) if $\frac{N}{2j'} \leq l < \frac{N}{j}$. Further $S_1 = 0 \ \ \forall\, l > \frac{N}{2j'}$.

Consider S_2. Here $\frac{N}{2j} \leq j < \frac{N}{j}$ and hence $\psi_{j,0}^{(i)} \equiv (p-1)$ (mod p) $\equiv (-1)$ (mod p). Then $S_2 = (p-1)\frac{N}{2j} \equiv \frac{-N}{2j}$ (mod p) if $0 \leq l < \frac{N}{2j'}$ and $S_2 = (p-1)^2 \frac{N}{2j} \equiv \frac{N}{2j}$ (mod p)if $\frac{N}{2j'} \leq l < \frac{N}{j'}$.

Further $S_2 = 0$ if $l > \frac{N}{2j'}$.

Consider S_3. Here $\frac{N}{j} \leq i < N$, but $\psi_{j,0} \equiv 0$ for all $i \geq \frac{N}{j}$. Hence it follows that $\bar{\psi}_{j,0}\ \bar{\psi}_{j',k} = S_1 + S_2 + S_3 \equiv 0($mod $p)$, proving orthogonality.

Step 3. Let $j \neq j', k \neq 0$.

We prove that $\psi_{j,k}$ and $\psi_{j',k}$ are orthogonal. Consider

$$\begin{aligned}
\bar{\psi}_{j,k}\ \bar{\psi}_{j',k} &= \sum_{i=0}^{N-1} \psi_{j,k}^{(i)}\ \psi_{j',k}^{(i)} = \sum_{i=0}^{N-1} \psi_{j,0}^{(i+m)}\ \psi_{j',0}^{i+m'} \qquad \text{for some } m, m' \\
&= \sum_{i'=m}^{N-1+m} \psi_{j,o}^{(i')}\ \psi_{j',0}^{(i'+m'-m)} = \sum_{i'=m}^{N-1+m} \psi_{j,0}^{(i')}\ \psi_{j',0}^{(i'+k')} \quad \text{for some } k' \\
&= \sum_{i'=m}^{N-1+m} \psi_{j,o}^{(i')}\ \psi_{j',0}^{(l)} \qquad \text{for some } l\,. \qquad (1)
\end{aligned}$$

But observe that the sum ranging over m to $N-1+m$ is the sum ranging over 0 and $N-1$ in some order.

For example, take $m = 4, N = 8$. Then i' ranges from 4 to 11 mod N which implies that i' ranges from 0 to 7. It follows that the range always falls between 0 and $N-1$. Hence by (1), we have $\psi_{j,k}\ \psi_{j',k'} = \sum_{j=0}^{N-1} \psi_{j,0}^{(i)}\ \psi_{j',0}^{(l)}$ for some l whose orthogonality has already been established in Step 2.

Thus, in general, one can establish by similar arguments as in Step 2 and Step 3 that $\bar{\psi}_{jk}$ and $\bar{\psi}_{j',k'}$ are orthogonal if $j \neq j'$ or $k \neq k'$.

Now, in order to establish the orthonormality of vectors, we observe the following: by definition of Haar function, it is easy to note that $\psi_{j,k}$ consists of exactly $\frac{N}{2j}$1's and $\frac{N}{2j}(p-1)$'s and remaining 0's.

Since $N = 2^n, j = 2^r$ for $0 \leq r \leq N-1$, some n $\psi_{j,k}$ consists of $2^{n-r-1}(p-1)$'s and 1's. Thus

$$\begin{aligned}
\bar{\psi}_{j,k}\ \bar{\psi}_{j,k} = \sum_{i=0}^{N-1} \psi_{j,k}^{(i)}\ \psi_{j,k}^{(i)} = 2^{n-r-1}[1^2 + (p-1)^2] &= 2^{n-r-1}(2)(\text{mod } p) \\
&= 2^{n-r-1}(\text{mod } p).
\end{aligned}$$

Thus to get orthonormality, we must have $\bar{\psi}_{j,k}\ \bar{\psi}_{j,k} \equiv 1 \pmod{p}$. Hence we take the normalised vector as $\frac{\bar{\psi}_{j,k}}{\sqrt{2^{n-r}}}$. However, as mentioned earlier, $\frac{\bar{\psi}_{j,k}}{\sqrt{2^{n-r}}}$ should belong to our vector space F^N. In particular $\frac{\psi^{(i)}_{j,k}}{\sqrt{2}}, \frac{\psi^{(i)}_{j,k}}{2\sqrt{2}} \cdots \frac{\psi^{(i)}_{j,k}}{2^{n/2}}$ should belong to our field F.

If $F = \mathbf{Z}_p$ and if $p \equiv \pm 1 \pmod{8}$, then by Theorem 4.1, $x^2 \equiv 2 \pmod{p}$ has a solution. In other words, $\sqrt{2}$ or $\frac{1}{\sqrt{2}}$ belong to F. In all other cases we need to extend our field $F = \mathbf{Z}_p$ by adjoining $\sqrt{2}$. Thus our field under consideration would be $\mathbf{Z}_p(\sqrt{2})$.

Further, $\bar{\psi}_0\ \bar{\psi}_0 = N$. But in order to preserve orthonormality, we need to have $\bar{\psi}_0\ \bar{\psi}_0 \equiv 1 \pmod{p}$. Hence we find an integer m such that $N \equiv m \pmod{p}$ and requiring that $\sqrt{m}$ (hence $\frac{1}{\sqrt{m}}$) belong to our field F. Thus, we extend the field $F = \mathbf{Z}_p(\sqrt{2})$ by adjoining $\sqrt{m}$. Hence we assume $F = \mathbf{Z}_p(\sqrt{2}, \sqrt{m})$. These facts lead us to the following result.

Corollary 4.2. *If $V = [\mathbf{Z}_p(\sqrt{2}, \sqrt{m})]^N$ and $N = 2^n$, then the set of vectors in Theorem 4.1 form an orthonormal basis for V.*

If N is not a power of 2, we then take an integer N' which is a power of 2 and immediately larger than N and we obtain the orthonormal basis for $V = F^{N'}$ instead of $V = F^N$. In particular a signal $(w^{(0)}, w^{(1)}, \cdots w^{(N-1)})$ is received as $(w^{(0)}, w^{(1)}, \cdots w^{(N-1)}, 0, 0 \cdots 0)(N' - 1\ 0\text{'s})$ in the vector space $V = F^{N'}$.

Our intention is to decompose a given signal at various resolutions. For this we do the following. The given vector space V is expressed as $V = V_1 \oplus V_2 \oplus \cdots \oplus V_l$, where V_1 is the subspace spanned by the signal vector $\bar{\psi}_0$. Let V_j denote the subspace V spanned by the vectors $\{\bar{\psi}_{j,k} : k = 0, 1, \cdots j-1\}$. Thus, a signal $\bar{w} = (w^{(0)}, w^{(1)}, \cdots, w^{(N-1)}) \in V$ is expressed *uniquely* as $w_1 + w_2 + \cdots + w_l$, where w_j denote the j-th resolution of $\bar{w}$ in the binary scale.

Now, we provide an algorithm for obtaining various resolutions of a given signal.

5. Algorithm and Illustration

Let w denote the input vector (signal) $(w^{(0)}, \cdots w^{(N-1)})$. Let p denote an odd prime.

Let j denote the scaling parameter and k denote the translation parameter.

Step 1. Input N, P, W

Step 2. Define $\psi^{(i)}$ of the Haar function ψ by

$$\psi^{(i)} = \begin{cases} 1 & \text{if } 0 \leq \frac{1}{n} < \frac{1}{2}\,, \\ p-1 & \text{if } \frac{1}{2} \leq \frac{1}{N} < 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Step 3. Generate scaled versions of ψ for all j belonging to divisors of $\frac{N}{2}$ by $\psi_{j,0}^{(i)} = \psi^{(ji)}$, for i ranging form 0 to $N-1$.

Step 4. Generate translated version of ψ_{j0} by using $\psi_{jk}^{(i)} = \psi_{j,0}^{(i+\frac{NK}{j}(\text{mod}N))}$ for i ranging form 0 to $N-1$.

Step 5. To obtain the normalizing factor $N.F.$ Compute the number of 1's in each vector and denote it by count.

Define $r = \dfrac{\log(\text{count})}{\log 2}$ and $N.F.$=2^{r+1}.

Step 6. Compute the inverse of 2^{r+1} in $\mathbf{Z}_p$ and denote it by $\text{inv}_{j,k}$.

Step 7. Define a new set of vectors $b_{j,k}$ by $b_{j,k}^{(i)}$ =$\text{inv}_{j,k}\ \psi_{j,k}^{(i)}$.

Step 8. To compute the coefficients C_{jk} in the orthonormal expansion of w: Define $C_{jk} = \sum_{i=0}^{N-1} w^{(i)} b_{jk}^{(i)}$.

Set $C_{jk} = c_{jk}$ % p.

Step 9. The resolutions (scales) of w are obtained by the formula: $S_j^{(i)} = \sum_{k=0}^{j-1} C_{jk}\psi_{j,k}^{(i)}$, for i ranging form $0 \cdots N-1$.

Remark 5.1. It can be verified that

$$\bar{w} = (w^{(0)}, \cdots w^{(N-1)}) = \sum_{j/(N/2)} (s_j^{(0)}, s_j^{(1)} \cdots s_j^{(N-1)}).$$

The algorithm is implemented for a signal mentioned in Figure 1. The corresponding resolutions of the signal and the reconstructed signal from the scales are shown in Figure 2.

6. Application to Power Engineering

Power disturbances occur due to changes in the electrical configuration of a power circuit. As the industrial and residential loads vary, the disturbances

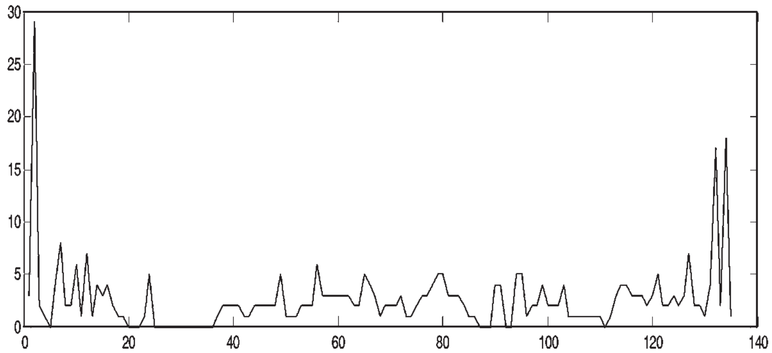

Figure 2.10: Reconstructed signal

follow. However in order to determine the cause and sources of disturbances, one must be able to detect and localize these disturbances.

A novel approach to detection and localization of various power disturbances such as fast voltage fluctuations, short and long duration voltage variation, etc. is based on multi-resolution analysis. Fast and short transient disturbances are detected at lower scales because the mother wavelet is mostly localized in time whereas slow and long transient disturbances are detected at higher scales.

This methodology permits online estimation of the system dynamic performance and may also work as a power disturbance recorder to detect, localize and classify different disturbances. For a more detailed study of applications of wavelet in power engineering we refer to [3].

The identification phase relates to their decomposition into fundamental components and their representation as a sum of wavelet basis functions. Robertson et al [5, 6] suggested the use of wavelet analysis for identification of power system transients.

For example, the signal in Figure 1 is decomposed based on different scales namely $0, 2^0, 2^1, \cdots 2^7$ showing the fundamental components (refer Figure 2.1 to Figure 2.9) based on our method. The reconstructed signal (Figure 2.10) as the sum of these fundamental components is *exactly the same* as that the original signal. It is also to be noted that not only the harmonics can be detected but also the time of their appearance can be determined.

If wavelets in $L^2(\mathbf{R})$ is used in obtaining the fundamental components of a signal, the process in infinite. Mathematically, this means writing $L^2(\mathbf{R}) = \oplus_{j=-\infty}^{\infty} W_j$, where W_j is a closed subspace generated by the vector $\{\psi_{j,k} | j, k \in z\}$ and $\psi_{jk}(x) = 2^{-j/2}\psi(2^{-j}x - k)$. Hence the representation as a sum of wavelets basis functions is only to upto an approximation in the finite level. Also a priori, one is not sure upto which resolutions, one has to find the fundamental components of the given signal. Whereas here, we work with finite dimensional vector spaces F^N, the number of scales (j) and hence the number of fundamental components are finite and the representation as a sum of their fundamental components is also exact.

6.1. Future Study

This theory can be developed further by making use of a function possessing smoothness and regularity instead of Haar function. In these cases one may be able to obtain the detections in various resultions more clearly.

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