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ASYMPTOTICS OF UPPER CRITICAL FIELD OF A SUPERCONDUCTOR IN APPLIED MAGNETIC FIELD VANISHING OF HIGHER ORDER

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Abstract: We consider the Ginzburg-Landau equations in the superconductivity theory. Our final aim is to get the asymptotic behavior of the upper critical field as the Ginzburg-Landau parameter is large in the case where the applied magnetic field vanishes of higher order in a submanifold of a bounded, simply connected region in $\mathbb{R}^2$. We also give the concentration of the order parameter in that case. This research is an improvement of the results in Pan and Kwek [10].

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1. Introduction

This paper is a continuation of the previous paper of the author Aramaki [1]. We shall improve the results of Pan and Kwek [10] on asymptotics of upper critical field of a superconductor. More precisely, we consider a cylindrical superconducting sample with cross section Ω in an applied magnetic field $\mathcal{H}$. It is well known that if the applied field is very strong, then the sample loses superconductivity property. When the magnitude of the applied field is reduced to the some value $\sigma^*(\kappa)$ called upper critical field, the nucleation of superconductivity occurs. Our purpose is to get the asymptotics of $\sigma^*(\kappa) = H_{c3}(\kappa)$ as the Ginzburg-Landau parameter κ increases.

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For a cylindrical sample of infinite height with cross section Ω which is a simply connected bounded domain in $\mathbb{R}^2$, we assume that a applied magnetic field is along the cylindrical axis which is chosen as the x_3 - axis. According to the Ginzburg-Landau theory, the sample is in the state such that the Ginzburg-Landau functional $\mathcal{E}$ takes the minimum there (cf. Gunzburger and Ockendon [4], Du, Gunzburger and Peterson [3]). We consider the nondimensionalized Ginzburg-Landau functional given by

$$\mathcal{F}(\psi, \mathcal{A}) = \int_\Omega \big\{|\nabla_{\kappa\mathcal{A}}\psi|^2 + \kappa^2|\mathrm{curl}\,\mathcal{A} - \mathcal{H}|^2 + \frac{\kappa^2}{2}(|\psi|^2-1)^2\big\}dx + \int_{\partial\Omega}\gamma|\psi|^2 ds.$$

Here ψ is a complex-valued function called order parameter, $\mathcal{A}$ is a real vector field called magnetic potential, $\mathcal{H}$ the applied magnetic field, $i = \sqrt{-1}$, κ Ginzburg-Landau parameter which is defined by the ratio of the London penetration depth and the coherence length of the superconductor and $\gamma \geq 0$ a constant.

If we take the Fréchet derivatives of $\mathcal{F}(\psi, \mathcal{A})$ with respect to ψ and $\mathcal{A}$, we get the Euler equations called the Ginzburg-Landau equations.

Throughout this paper we assume that

$$\mathcal{H}(x) = \sigma H_0(x)\,,$$

where $H_0(x)$ is a fixed applied field in $\overline{\Omega}$ and $\sigma > 0$ is a parameter. If we write $\mathcal{A} = \sigma\boldsymbol{A} = \sigma(A_1, A_2)$ and put $\mathcal{E}(\psi, \boldsymbol{A}) = \mathcal{F}(\psi, \sigma\boldsymbol{A})$, we can rewrite the Ginzburg-Landau functional $\mathcal{F}$ in the following

$$\mathcal{E}(\psi, \boldsymbol{A}) = \int_\Omega \big\{|\nabla_{\sigma\kappa\boldsymbol{A}}\psi|^2 + (\sigma\kappa)^2|\mathrm{curl}\,\boldsymbol{A} - H_0|^2 + \frac{\kappa^2}{2}(|\psi|^2-1)^2\big\}dx + \int_{\partial\Omega}\gamma|\psi|^2 ds. \quad (1.1)$$

Then the Ginzburg-Landau equations become

$$\begin{cases} -\nabla^2_{\sigma\kappa\boldsymbol{A}}\psi = \kappa^2(1-|\psi|^2)^2\psi, \\ \mathrm{curl}^2\boldsymbol{A} = -\dfrac{i}{2\kappa}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) - |\psi|^2\boldsymbol{A} + \mathrm{curl}\,H_0 \quad \text{in } \Omega\,. \end{cases} \quad (1.2)$$

Here and after here, we use the notations as follows:

$$\mathrm{curl}\,\boldsymbol{A} = \partial_1 A_2 - \partial_2 A_1, \quad \partial_j = \partial/\partial x_j \quad (j = 1, 2),$$

$$\operatorname{curl}^2 \boldsymbol{A} = (\partial_2(\operatorname{curl} \boldsymbol{A}), -\partial_1(\operatorname{curl} \boldsymbol{A})),$$
$$\nabla_{\sigma\kappa\boldsymbol{A}}\psi = \nabla\psi - i\sigma\kappa\boldsymbol{A}\psi,$$
$$\nabla^2_{\sigma\kappa\boldsymbol{A}}\psi = (\nabla - i\sigma\kappa\boldsymbol{A})^2\psi = \Delta\psi - i\sigma\kappa[2\boldsymbol{A}\cdot\nabla\psi + \psi \operatorname{div} \boldsymbol{A}] - (\sigma\kappa)^2|\boldsymbol{A}|^2\psi.$$

For two vectors $\boldsymbol{a}, \boldsymbol{b}$, $\boldsymbol{a}\cdot\boldsymbol{b}$ denotes the usual inner product.

The natural boundary conditions are the following:

$$\begin{cases} \dfrac{\partial\psi}{\partial\boldsymbol{\nu}} - i\sigma\kappa\boldsymbol{A}\cdot\boldsymbol{\nu} + \gamma\psi = 0, & \\ \operatorname{curl}\boldsymbol{A} - H_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\boldsymbol{\nu}$ denotes the unit out-normal vector at $\partial\Omega$.

We call a (global) minimizer $(\psi, \boldsymbol{A})$ of $\mathcal{E}$ a minimal solution of the Ginzburg-Landau equations (1.2) and (1.3) and ψ an order parameter if there exists a real vector field $\boldsymbol{A}$ such that $(\psi, \boldsymbol{A})$ is the minimal solution of (1.2) and (1.3).

We define

$$E(\kappa,\sigma) = \inf_{(\psi,\boldsymbol{A})\in\mathcal{W}} \mathcal{E}(\psi,\boldsymbol{A}), \tag{1.4}$$

where $\mathcal{W} = W^{1,2}(\Omega;\mathbb{C}) \times W^{1,2}(\Omega;\mathbb{R}^2)$, $W^{1,2}(\Omega;\mathbb{C})$ and $W^{1,2}(\Omega;\mathbb{R}^2)$ donote the usual Sobolev spaces with values in $\mathbb{C}$ and $\mathbb{R}^2$, respectively.

Note that $\mathcal{E}$ has at least one minimizer in $\mathcal{W}$ and $\mathcal{E}$ is invariant under the gauge transformation, i.e., if $\phi \in W^{2,2}(\Omega;\mathbb{C})$, $\mathcal{E}(G_\phi(\psi,\boldsymbol{A})) = \mathcal{E}(\psi,\boldsymbol{A})$, where $G_\phi(\psi,\boldsymbol{A}) = (\psi e^{i\sigma\kappa\phi}, \boldsymbol{A} + \nabla\phi)$. Moreover, for any $(\psi,\boldsymbol{A}) \in \mathcal{W}$, if we choose a solution of the Neumann problem

$$\begin{cases} \Delta\phi = -\operatorname{div}\boldsymbol{A} & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\boldsymbol{\nu}} = -\boldsymbol{A}\cdot\boldsymbol{\nu} & \text{on } \partial\Omega, \end{cases}$$

and put $G_\phi(\psi,\boldsymbol{A}) = (\widetilde{\psi}, \widetilde{\boldsymbol{A}})$, we see that $\widetilde{\boldsymbol{A}}$ satisfies that $\operatorname{div}\widetilde{\boldsymbol{A}} = 0$ in Ω and $\widetilde{\boldsymbol{A}}\cdot\boldsymbol{\nu} = 0$ on $\partial\Omega$. Thus we always assume that the minimizer $(\psi,\boldsymbol{A})$ of $\mathcal{E}$ satisfies that $\operatorname{div}\boldsymbol{A} = 0$ in Ω and $\boldsymbol{A}\cdot\boldsymbol{\nu} = 0$ on $\partial\Omega$. For these facts, see [3], [4].

It is well known that for given smooth function $H_0(x)$ on $\overline{\Omega}$, there exists a uniquely smooth vector field $\boldsymbol{F}$ on $\overline{\Omega}$ such that

$$\begin{cases} \operatorname{curl}\boldsymbol{F} = H_0, & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{F} = 0, & \text{in } \Omega, \\ \boldsymbol{F}\cdot\boldsymbol{\nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

We can easily see that $(0,\boldsymbol{F})$ is a trivial critical value of $\mathcal{E}$ and for large $\sigma > 0$, $(0,\boldsymbol{F})$ is the only minimizer of $\mathcal{E}$. This means that if σ is large enough, the sample is in the normal state. Thus we define the upper critical field as

$$\sigma^*(\kappa) = \sigma^*(\kappa, H_0) := \inf\{\sigma > 0; (0,\boldsymbol{F}) \text{ is the only minimizer of } \mathcal{E}\}. \tag{1.6}$$

Since we consider the case where the applied field H_0 vanishes on a subset of $\overline{\Omega}$ in this paper, we define

$$\mathcal{Z}(H_0) = \{x \in \overline{\Omega}; H_0(x) = 0\}.$$

We assume that:

(H) $\mathcal{Z}(H_0)$ consists of a finite number of smooth curves and there exist an integer $k \geq 1$ and constants $C_1, C_2 > 0$ such that

$$C_1^{-1} d(x, \mathcal{Z}(H_0))^k \leq |H_0(x)| \leq C_1 d(x, \mathcal{Z}(H_0))^k \quad \text{if } d(x, \mathcal{Z}(H_0)) \leq C_2\,,$$

where $d(x, \mathcal{Z}(H_0))$ is the Euclidean metric from x to $\mathcal{Z}(H_0)$. Moreover, we define

$$\mathcal{Z}(H_0; \Omega) = \mathcal{Z}(H_0) \cap \Omega, \quad \mathcal{Z}(H_0; \partial\Omega) = \mathcal{Z}(H_0) \cap \partial\Omega.$$

We shall examine the asymptotics of $\sigma^*(\kappa)$ as $\kappa \to \infty$ and the concentration of the order parameter.

Let $\boldsymbol{n}(x) = (n_1(x), n_2(x))$ be the unit normal vector of $\mathcal{Z}(H_0)$ at $x \in \mathcal{Z}(H_0)$. By the hypothesis (H), we have $(\partial^j_{\boldsymbol{n}(x)} H_0)(x) = 0 \quad (0 \leq j \leq k-1)$ and $(\partial^k_{\boldsymbol{n}(x)} H_0)(x) \neq 0$ for $x \in \mathcal{Z}(H_0)$, where $\partial_{\boldsymbol{n}(x)} = \boldsymbol{n}(x) \cdot \nabla_x$. Thus in a neighborhood of $x^0 \in \mathcal{Z}(H_0)$, we can write

$$H_0(x) = \frac{1}{k!}\big((x - x^0) \cdot \boldsymbol{n}(x^0)\big)^k (\partial^k_{\boldsymbol{n}(x^0)} H_0)(x^0) + O(|x - x^0|^{k+1}) \tag{1.7}$$

as $x \to x^0$. If we define a vector field

$$\boldsymbol{F}_0(x) = \frac{1}{(k+1)!}\big((x - x^0) \cdot \boldsymbol{n}(x^0)\big)^{k+1} (\partial^k_{\boldsymbol{n}(x^0)} H_0)(x^0)(-n_2(x^0), n_1(x^0)), \tag{1.8}$$

we have

$$\operatorname{curl} \boldsymbol{F}_0 = \frac{1}{k!}\big((x - x^0) \cdot \boldsymbol{n}(x^0)\big)^k (\partial^k_{\boldsymbol{n}(x^0)} H_0)(x^0).$$

Therefore, the vector field $\boldsymbol{F}$ in (1.5) is written by $\operatorname{curl}(\boldsymbol{F} - \boldsymbol{F}_0) = O(|x - x^0|^{k+1})$ as $x \to x^0$. Then it follows from Helffer and Mohamed [6] that we may assume that

$$\boldsymbol{F} = \boldsymbol{F}_0 + \boldsymbol{F}_1, \quad \boldsymbol{F}_1 = O(|x - x^0|^{k+2}) \quad \text{as } x \to x^0. \tag{1.9}$$

For $x^0 \in \mathcal{Z}(H_0)$, we choose the angle $\theta(x^0)$ so that

$$\frac{(\partial^k_{\boldsymbol{n}(x^0)} H_0)(x^0)}{|(\partial^k_{\boldsymbol{n}(x^0)} H_0)(x^0)|}(n_2(x^0), -n_1(x^0)) = (\cos\theta(x^0), \sin\theta(x^0))$$

and define

$$\alpha_k(H_0) = \min\Big\{(\lambda^0)^{(k+2)/2} \inf_{x\in\mathcal{Z}(H_0;\Omega)} \frac{1}{k!}|\partial^k_{\boldsymbol{n}(x)} H_0(x)|, \min_{x\in\mathcal{Z}(H_0;\partial\Omega)} \lambda(\mathbb{R}^2_+;\theta(x))^{(k+2)/2}\frac{1}{k!}|\partial^k_{\boldsymbol{n}(x)} H_0(x)|\Big\}\,.$$

Here $\lambda^0, \lambda(\mathbb{R}^2_+, \theta(x))$ will be defined by (2.7) and (2.10), respectively. Now, we are in a position to state the main theorem.

Theorem 1.1. (i) (*Asymptotics of the upper critical field*)

$$\sigma^*(\kappa) = \sigma^*(\kappa, H_0) = \left(\frac{1}{\alpha_k(H_0)} + o(1)\right)\kappa^{k+1} \quad \text{as } \kappa \to \infty.$$

(ii) (*Concentration of the order parameter*) *Let $\{\kappa_n\}, \{\sigma_n\}$ be two sequences such that $\kappa_n, \sigma_n \to \infty$ as $n \to \infty, \sigma_n < \sigma^*(\kappa_n, H_0)$ and*

$$\lim_{n\to\infty}\frac{\kappa_n^{k+1}}{\sigma_n} = \alpha_k(H_0).$$

Moreover, let $(\psi_n, \boldsymbol{A}_n)$ be a non-trivial minimizer of $\mathcal{E}$ with $\kappa = \kappa_n, \sigma = \sigma_n$. Then there exist subsequences $\{\kappa_{n_l}\}, \{\sigma_{n_l}\}$ of $\{\kappa_n\}, \{\sigma_n\}$, respectively such that

$$\|\psi_{n_l}\|_{L^\infty(\Omega)} \to 0, \quad \operatorname{curl}\boldsymbol{A}_{n_l} \to H_0 \quad \text{in } C^\alpha(\Omega)$$

as $l \to \infty$ and

$$\frac{\psi_{n_l}(x)}{\|\psi_{n_l}\|_{L^\infty(\Omega)}} \to 0 \quad \text{for} \quad x \in \overline{\Omega}\setminus\widetilde{Z}(H_0)\,,$$

where

$$\widetilde{Z}(H_0) = \Big\{x\in\Omega; (\lambda^0)^{(k+2)/2}\frac{1}{k!}|\partial^k_{\boldsymbol{n}(x)}H_0(x)| = \alpha_k(H_0)\Big\} \cup \Big\{x\in\partial\Omega; \lambda(\mathbb{R}^2_+, \theta(x))^{(k+2)/2}\frac{1}{k!}|\partial^k_{\boldsymbol{n}(x)}H_0(x)| = \alpha_k(H_0)\Big\}.$$

Remark 1.2. Theorem 1.1 is an extension of [10] in which the case $k = 1$ is treated.

In order to prove Theorem 1.1, we consider a linear problem associated with (1.2) and (1.3). (cf. [1] and, Lu and Pan [7], [8] and [9]). Assume that the hypothesis (H) holds and that $\boldsymbol{F}$ is of the form in (1.9). Let $\mu = \mu(b\boldsymbol{F})$ be the lowest eigenvalue of the problem

$$\begin{cases} -\nabla^2_{b\boldsymbol{F}}\phi = \mu\phi & \text{in } \Omega, \\ (\nabla_{b\boldsymbol{F}}\phi)\cdot\boldsymbol{\nu} + \gamma\phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

That is to say,

$$\mu(b\boldsymbol{F}) = \inf_{\phi\in W^{1,2}(\Omega;\mathbb{C})} \frac{1}{\|\phi\|^2_{L^2(\Omega)}} \left\{ \int_\Omega |\nabla_{b\boldsymbol{F}}\phi|^2 dx + \int_{\partial\Omega} \gamma|\phi|^2 ds \right\}.$$

In the previous paper [1], we got the following theorem.

Theorem 1.3. (i) $\mu(b\boldsymbol{F}) = \big(\alpha_k(H_0)^{2/(k+2)} + o(1)\big)b^{2/(k+2)}$ *as* $b\to\infty$.
(ii) *If* ψ_b *be the eigenfunction of* (1.10) *associated with* $\mu(b\boldsymbol{F})$, *we have*

$$\frac{\psi_b(x)}{\|\psi_b\|_{L^\infty(\Omega)}} \to 0 \quad \textit{for} \quad x\in\overline{\Omega}\setminus \widetilde{Z}(H_0).$$

Remark 1.4. We are also interested in the estimate of the remainder term of Theorem 1.3 (i). We shall consider this problem elsewhere.

2. Preliminaries

In this section we shall review the results of the previous paper [1]. We treat the problem (1.10).

Let the magnetic potential be of the form

$$\boldsymbol{A} = -\frac{1}{k+1}(x_2\cos\theta - x_1\sin\theta)^{k+1}\boldsymbol{n}, \quad \boldsymbol{n} = (\cos\theta,\sin\theta). \tag{2.1}$$

According to $\mathcal{Z}(H_0;\Omega)\neq\emptyset$ or $\mathcal{Z}(H_0;\partial\Omega)\neq\emptyset$, we must consider the eigenvalue problems

$$-\nabla^2_{\boldsymbol{A}}\psi = \lambda\psi \quad \text{in } \mathbb{R}^2, \tag{2.2}$$

or

$$\begin{cases} -\nabla^2_{\boldsymbol{A}}\psi = \lambda\psi & \text{in } \mathbb{R}^2_+ = \{x=(x_1,x_2); x_2>0\}, \\ (\nabla_{\boldsymbol{A}}\psi)\cdot\boldsymbol{\nu} = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{2.3}$$

Here we say that the equations (2.2) or (2.3) has the eigenvalue λ if (2.2) or (2.3) has a non-trivial bounded solution in $\mathbb{R}^2$ or $\mathbb{R}^2_+$, respectively. By the rotation of the coordinate system

$$\begin{cases} x_1 = y_1\cos\theta - y_2\sin\theta, \\ x_2 = y_1\sin\theta + y_2\cos\theta, \end{cases} \tag{2.4}$$

we may assume that $\boldsymbol{n} = (1,0)$ and $\boldsymbol{A} = (-\frac{1}{k+1}y_2^{k+1}, 0)$.

At first, we consider the equation (2.2). By the above transformation of the coodinate system, we can rewrite (2.2) into the form

$$-\Delta\psi - i\frac{2}{k+1}y_2^{k+1}\partial_1\psi + \frac{1}{(k+1)^2}y_2^{2(k+1)}\psi = \lambda\psi \quad \text{in } \mathbb{R}^2. \tag{2.5}$$

The lowest eigenvalue $\lambda(\mathbb{R}^2, \boldsymbol{A})$ is defined by

$$\lambda(\mathbb{R}^2, \boldsymbol{A}) = \inf_{0\neq\psi\in\mathcal{W}^{1,2}(\mathbb{R}^2,\boldsymbol{A})} \frac{1}{\|\psi\|^2_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} |\nabla_{\boldsymbol{A}}\psi|^2 dx\,, \tag{2.6}$$

where $\mathcal{W}^{1,2}(\mathbb{R}^2, \boldsymbol{A}) = \{\psi \in L^2(\mathbb{R}^2); |\nabla - i\boldsymbol{A}\psi| \in L^2(\mathbb{R}^2)\}$. By the partial Fourier transformation in y_1, we see that the lowest eigenvalue $\lambda(\mathbb{R}^2; \boldsymbol{A})$ of (2.2) is equal to

$$\lambda_0 := \inf_{-\infty<\tau<\infty} \lambda(\tau)\,, \tag{2.7}$$

where $\lambda(\tau)$ is the lowest eigenvalue of the ordinary differential equation in $L^2(\mathbb{R})$:

$$-y'' + q(t,\tau)y = \lambda y, \quad (-\infty < t < \infty)\,, \tag{2.8}$$

where

$$q(t,\tau) = \frac{1}{(k+1)^2}(t^{k+1} + (k+1)\tau)^2.$$

That is to say,

$$\lambda(\tau) = \inf_{0\neq u\in\mathcal{W}^{1,2}(\mathbb{R},q)} \frac{1}{\|u\|^2_{L^2(\mathbb{R})}} \int_{-\infty}^{\infty} \{|u'(t)|^2 + q(t,\tau)|u(t)|^2 dx\,, \tag{2.9}$$

where $\mathcal{W}^{1,2}(\mathbb{R}, q) = \{u \in W^{1,2}(\mathbb{R}); \sqrt{q}u \in L^2(\mathbb{R})\}$. We note that $\lambda(\tau)$ is achieved in $\mathcal{W}^{1,2}(\mathbb{R}, q)$. Then we have the following proposition.

Proposition 2.1. (i) *When k is an even integer, there exist at most finitely many τ_j $(j = 1, 2, \ldots, N)$ such that*

$$\lambda(\tau_j) = \lambda^0 := \inf_{-\infty<\tau<\infty} \lambda(\tau) \quad \textit{for } j = 1, 2, \ldots, N.$$

When k is an odd integer, there exists a unique $\tau_0 < 0$ such that

$$\lambda(\tau_0) = \lambda^0 = \inf_{-\infty<\tau<\infty} \lambda(\tau).$$

(ii) *When k is an even integer, the eigenfunctions associated with $\lambda(\mathbb{R}^2, \boldsymbol{A})$ are given by*

$$\psi = \sum_{j=1}^{N} c_j e^{i\tau_j y_1} u_{\tau_j}(y_2)\,,$$

where u_{τ_j} is the positive eigenfunction of (2.8) *associated with $\tau = \tau_j, \lambda = \lambda(\tau_j)$ for $j = 1, 2, \ldots, N$ and when k is an odd integer, the eigenfunctions associated with $\lambda(\mathbb{R}^2; \boldsymbol{A})$ are given by $\psi = ce^{i\tau_0 y_1} u_{\tau_0}(y_2)$, where $u_{\tau_0}(t)$ is the positive eigenfunction of* (2.8) *associated with $\lambda^0 = \lambda(\tau_0)$.*

Remark 2.2. (1) If k be a positive even integer, then it is easy to see that $\lambda(\tau) > 0$ and $\lambda(\tau)$ is an even fucntion in τ since $q(t, -\tau) = q(-t, \tau)$. In this case, we can prove that $\lim_{|\tau|\to\infty} \lambda(\tau) = \infty$. Since $\lambda(\tau)$ is an analytic function in τ, the set

$$E_{\lambda_0} = \{\tau \in \mathbb{R}; \lambda(\tau) = \lambda_0 = \inf_{-\infty<\tau<\infty} \lambda(\tau)\}$$

is at most finite set. Therefore, we can get the associated eigenfunction of type $\phi = \sum_{j=0}^{N} c_j e^{i\tau_j y_1} u_j(y_2)$ in (i) of the above theorem. We do not know if the minimum point is unique or not in the case where k is a positive even integer. Note that $\lambda(\tau) \equiv 0$ in the case where $k = 0$.

(2) Note that the absolute value $|\psi|$ of the order parameter ψ is constant on an infinite set $\{y_2 = 0\}$.

Next, if $\mathcal{Z}(H_0, \partial\Omega) \neq \emptyset$, we must consider the equation (2.3). We define the lowest eigenvalue of (2.3) as $\lambda(\mathbb{R}^2_+; \boldsymbol{A}) = \lambda(\mathbb{R}^2_+, \theta)$. That is to say,

$$\lambda(\mathbb{R}^2_+, \theta) = \inf_{\phi \in \mathcal{W}^{1,2}(\mathbb{R}^2_+, \boldsymbol{A})} \frac{1}{\|\phi\|^2_{L^2(\mathbb{R}^2_+)}} \int_{\mathbb{R}^2_+} |\nabla_{\boldsymbol{A}} \phi|^2 dx\,, \tag{2.10}$$

where $\mathcal{W}^{1,2}(\mathbb{R}^2_+, \boldsymbol{A}) = \{\phi \in L^2(\mathbb{R}^2_+); |\nabla\phi - i\boldsymbol{A}\phi| \in L^2(\mathbb{R}^2_+)\}$. Since we can easily see that $\lambda(\mathbb{R}^2_+, \pi + \theta) = \lambda(\mathbb{R}^2_+, \theta)$, we may assume that $0 \le \theta < \pi$. When $\theta = 0$, (2.3) is equivalent to the problem

$$\begin{cases} -\Delta\phi - i\frac{1}{k+1}x_2^{k+1}\partial_1\phi + \frac{1}{(k+1)^2}x_2^{2(k+1)}\phi = \lambda\phi & \text{in } \mathbb{R}^2_+, \\ \partial_2\phi = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{2.11}$$

By the similar argument as above, we consider the Neumann problem of the ordinary differential equation in $L^2(\mathbb{R}_+)$:

$$\begin{cases} -u'' + q(t,\tau)u = \lambda u, \quad t > 0, \\ u'(0) = 0\,. \end{cases} \tag{2.12}$$

In the following, the lowest eigenvalue of (2.12) is written by $\lambda_+(\tau)$. Note that if k is odd, since $q(t,\tau)$ is a even function in t, we can see that $\lambda(\tau) = \lambda_+(\tau)$ (cf. Dauge and Helffer [2]).

Then we have the following result.

Proposition 2.3. (i) *There exists a unique $\tau_+^0 < 0$ such that*

$$\lambda_+(\tau_+^0) = \lambda_+^0 := \inf_{-\infty<\tau<\infty} \lambda_+(\tau).$$

In the particular case where k is an odd integer, $\lambda_+(\tau) = \lambda(\tau)$, where $\lambda(\tau)$ is as in Proposition 2.1, *so $\tau_+^0 = \tau_0$.*

(ii) *The eigenfunctions associated with $\lambda(\mathbb{R}^2_+, \boldsymbol{A})$ are given by*

$$\psi = ce^{i\tau_0 y_1}u_{+,\tau_0}(y_2)\,,$$

where $u_{+,\tau_0}(t)$ is the positive eigenfunction of (2.12) *associated with $\lambda_+^0 = \lambda_+(\tau_+^0) = \lambda^0 = \lambda(\tau_0)$.*

Next, we give some known results which are necessary in the later section. (cf. Helffer [5] and [8]).

Lemma 2.4. (i) *If $\mu(\sigma\kappa\boldsymbol{F}) < \kappa^2$, $\mathcal{E}$ has a non-trivial minimizer.*

(ii) *If $\mathcal{E}$ has a non-trivial minimizer $(\psi, \boldsymbol{A})$, it follows that $\mu(\sigma\kappa\boldsymbol{A}) < \kappa^2$.*

Now we define

$$\sigma_*(\kappa, H_0) = \min\{\sigma > 0; \mu(\sigma\kappa\boldsymbol{F}) = \kappa^2\}. \tag{2.13}$$

Note that if $0 \le \sigma < \sigma_*(\kappa, H_0)$, $\mathcal{E}$ has a non-trivial minimizer and that $\sigma^*(\kappa, H_0) \ge \sigma_*(\kappa, H_0)$.

3. Proof of Theorem 1.1

In this section, we shall give a proof of Theorem 1.1.

By Theorem 1.3 with $b = \sigma\kappa$, we have

$$\lim_{\kappa\to\infty} \frac{1}{(\sigma\kappa)^{2/(k+2)}}\mu(\sigma\kappa \boldsymbol{F}) = \alpha_k(H_0)^{2/(k+2)}. \tag{3.1}$$

For this equality, we get

$$\sigma_*(\kappa; H_0) = \left(\frac{1}{\alpha_k(H_0)} + o(1)\right)\kappa^{k+1} \quad \text{as } \kappa\to\infty\,,$$

where $\sigma_*(\kappa; H_0)$ was defined in (2.13). We note that it is clear to show that $\sigma^*(\kappa; H_0) \geq \sigma_*(\kappa; H_0)$. Let two arbitrary sequences $\{\kappa\}$ and $\{\sigma\}$ satisfy the following

$$\begin{cases} \kappa, \sigma \to \infty, \\ \sigma < \sigma^*(\kappa; H_0), \\ \lim_{\kappa\to\infty} \frac{\kappa^{k+1}}{\sigma} = a, \\ 0 \leq a \leq \alpha_k(H_0). \end{cases} \tag{3.2}$$

To see that the asymptotics of $\sigma^*(\kappa; H_0)$ and $\sigma_*(\kappa; H_0)$ are identical modulo $o(\kappa^{k+1})$, it suffices to show $a \geq \alpha_k(H_0)$. It is convenient to put $\epsilon = 1/(\sigma\kappa)^{1/(k+2)}$ and rewrite the functional $\mathcal{E}$ as

$$\mathcal{E}_\epsilon(\psi, \boldsymbol{A}) = \int_\Omega \big\{|\nabla_{\frac{1}{\epsilon^{k+2}}\boldsymbol{A}}\psi|^2 + \frac{1}{\epsilon^{2(k+2)}}|\mathrm{curl}\,\boldsymbol{A} - H_0|^2 + \frac{\kappa^2}{2}(|\psi|^2-1)^2\big\}dx$$

and put

$$E(\epsilon) = \inf_{(\psi,\boldsymbol{A})\in\mathcal{W}} \mathcal{E}_\epsilon(\psi, \boldsymbol{A}).$$

If we write the minimizer of $\mathcal{E}_\epsilon$ by $(\psi^\epsilon, \boldsymbol{A}^\epsilon)$, $(\psi^\epsilon, \boldsymbol{A}^\epsilon)$ satisfies that

$$\begin{cases} -\nabla^2_{\frac{1}{\epsilon^{k+2}}\boldsymbol{A}}\psi = \kappa^2(1-|\psi|^2)\psi, \\ \mathrm{curl}^2(\boldsymbol{A}-\boldsymbol{F}) = \epsilon^{k+2}\Im(\overline{\psi}\nabla_{\frac{1}{\epsilon^{k+2}}\boldsymbol{A}}\psi) \quad \text{in } \Omega \end{cases} \tag{3.3}$$

and that

$$\begin{cases} (\nabla_{\frac{1}{\epsilon^{k+2}}\boldsymbol{A}}\psi)\cdot\boldsymbol{\nu} + \gamma\psi = 0, \\ \mathrm{curl}\,(\boldsymbol{A}-\boldsymbol{F}) = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

By a gauge transformation, we may assume that $\mathrm{div}\,\boldsymbol{A}^\epsilon = 0$ in Ω and $\boldsymbol{A}^\epsilon\cdot\boldsymbol{\nu} = 0$ on $\partial\Omega$. We call the minimizer $(\psi^\epsilon, \boldsymbol{A}^\epsilon)$ of $\mathcal{E}_\epsilon$ the minimal solution of (3.3) and (3.4).

We must use the following elliptic estimates (cf. [8]).

Proposition 3.1. *Let $(\psi^\epsilon, \boldsymbol{A}^\epsilon)$ be the minimal solution of* (3.3) *and* (3.4). *Then there exists a constant $C > 0$ such that*

$$\begin{aligned} \|\psi^\epsilon\|_{L^\infty(\overline{\Omega})} &\leq 1, \\ \|\nabla_{\frac{1}{\epsilon^{k+2}}\boldsymbol{A}^\epsilon}\psi^\epsilon\|_{L^2(\Omega)} &\leq C/\epsilon, \\ \|\nabla_{\frac{1}{\epsilon^{k+2}}\boldsymbol{A}^\epsilon}\psi^\epsilon\|_{W^{1,2}(\Omega;\mathbb{C})} &\leq C/\epsilon^2. \end{aligned}$$

Moereover for any $1 < p < \infty$ and $0 < \alpha < 1$, there exist constants $C(p), C(\alpha) > 0$ such that

$$\begin{aligned} \|\boldsymbol{A}^\epsilon\|_{W^{2,p}(\Omega;\mathbb{R}^2)} &\leq C(p), \\ \|\boldsymbol{A}^\epsilon\|_{C^{1,\alpha}(\Omega;\mathbb{R}^2)} &\leq C(\alpha). \end{aligned}$$

Thus passing to a subsequence of $\{\epsilon\}$, we may assume that $\boldsymbol{A}^\epsilon \to \boldsymbol{A}^0$ in $C^{1,\alpha}(\Omega;\mathbb{R}^2)$ as $\epsilon \to 0$ and $\operatorname{curl}\boldsymbol{A}^0 = \operatorname{curl}\boldsymbol{F} = H_0$ in Ω.

Now we consider an eigenvalue problem:

$$\begin{cases} -\nabla_{\frac{1}{\epsilon^{k+2}}\boldsymbol{A}^\epsilon}\phi = \mu\phi & \text{in } \Omega, \\ (\nabla_{\frac{1}{\epsilon^{k+2}}\boldsymbol{A}^\epsilon}\phi)\cdot\boldsymbol{\nu} + \gamma\phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

If we denote the lowest eigenvalue by $\mu\big(\frac{1}{\epsilon^{k+2}}\boldsymbol{A}^\epsilon\big)$, we get from (3.1) and Proposition 3.1,

$$\begin{aligned} \lim_{\epsilon\to 0}\epsilon^{4/(k+2)}\mu\big(\frac{1}{\epsilon^{k+2}}\boldsymbol{A}^\epsilon\big) &= \lim_{\epsilon\to 0}\epsilon^{4/(k+2)}\mu\big(\frac{1}{\epsilon^{k+2}}\boldsymbol{A}^0\big) \\ &= \lim_{\epsilon\to 0}\epsilon^{4/(k+2)}\mu\big(\frac{1}{\epsilon^{k+2}}\boldsymbol{F}\big) \\ &= \alpha_k(H_0)^{2/(k+2)}. \end{aligned}$$

Moreover, by Lemma 2.4 and (3.2),

$$\epsilon^{4/(k+2)}\mu\big(\frac{1}{\epsilon^{k+2}}\boldsymbol{A}^\epsilon\big) < \epsilon^{4/(k+2)}\kappa^2 = \left(\frac{\kappa^{k+2}}{\sigma}\right)^{2/(k+2)}.$$

Thus we have

$$\begin{aligned} \alpha_k(H_0)^{2/(k+2)} &\geq a^{2/(k+2)} = \lim_{\kappa\to\infty}\left(\frac{\kappa^{k+2}}{\sigma}\right)^{2/(k+2)} \\ &\geq \lim_{\kappa\to\infty}\mu\big(\frac{1}{\epsilon^{k+2}}\boldsymbol{A}^\epsilon\big) \end{aligned}$$

$$= \quad \alpha_k(H_0)^{2/(k+2)}.$$

Therefore (i) of Theorem 1.1 holds.

To show (ii) of Theorem 1.1, choose $x^\epsilon \in \overline{\Omega}$ such that $\lambda_\epsilon = \|\psi^\epsilon\|_{L^\infty(\overline{\Omega})} = |\psi^\epsilon(x^\epsilon)|$. Since Ω is bounded, passing to a subsequence and using a translation, we may assume that $x^\epsilon \to x^0 = 0 \in \overline{\Omega}$.

We consider the following problems:

$$-\nabla^2_{\boldsymbol{A}}\psi = \lambda(1-|\psi|^2)\psi \quad \text{in } \mathbb{R}^2 \tag{3.6}$$

and

$$\begin{cases} -\nabla^2_{\boldsymbol{A}}\psi = \lambda(1-|\psi|^2)\psi & \text{in } \mathbb{R}^2, \\ (\nabla_{\boldsymbol{A}}\psi)\cdot\boldsymbol{\nu} = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{3.7}$$

Then we have the following result.

Proposition 3.2. (i) *When* $0 \le \lambda \le \lambda^0$, *any bounded solution of* (3.6) *is identically zero.*

(ii) *When* $0 \le \lambda \le \lambda(\mathbb{R}^2_+, \theta)$, *any bounded solution of* (3.7) *is identically zero.*

Proof. We only to prove (i). At first, we note that

$$\int |\nabla_{\boldsymbol{A}}\varphi|^2 dx \ge \lambda^0 \int |\varphi|^2 dx \quad \text{for } \varphi \in W^{1,2}(\mathbb{R}^2, \boldsymbol{A}).$$

We claim that for $0 \le \lambda \le \lambda^0$, if $\psi \ne 0$ is a bounded solution of (3.6), then $\psi \in L^2(\mathbb{R}^2)$.

In fact, let η be a smooth cut-off function such that $\operatorname{supp}\eta \subset B_{2R}$, $\eta = 1$ on B_R and $|\nabla\eta| \le 2/R$. Then we have

$$\begin{aligned} &(\lambda^0-\lambda)\int_{\mathbb{R}^2}|\eta\psi|^2 + \lambda\int_{\mathbb{R}^2}|\eta|^2|\psi|^4 dx \\ &\le \int_{\mathbb{R}^2}|\nabla\eta|^2|\psi|^2 dx \\ &\le \frac{4}{R^2}\int_{B_{2R}\setminus B_R}|\psi|^2 dx \le 16\pi^2 M^2\,, \end{aligned}$$

where $M = \sup_{x\in\mathbb{R}^2}|\psi|$. Therefore, if $\lambda < \lambda^0$,

$$\int_{\mathbb{R}^2}|\eta\psi|^2 dx \le \frac{16\pi^2 M^2}{\lambda^0-\lambda}.$$

Letting $R \to \infty$, we see that $\psi \in L^2(\mathbb{R}^2)$. If $\lambda = \lambda^0$, since $\|\psi\|^4_{L^4(\mathbb{R}^2)} \leq 16\pi^2 M^2/\lambda^0$, it follows from the Hölder inequality that

$$\left(\int_{B_R} |\psi|^2 dx\right)^2 \leq \frac{4\pi}{\lambda^0}\left\{\int_{B_{2R}} |\psi|^2 dx - \int_{B_R} |\psi|^2 dx\right\}.$$

If we put $L(R) = \int_{B_R} |\psi|^2 dx$, we have

$$L(R)^2 \leq \frac{4\pi}{\lambda^0}(L(2R) - L(R)) \leq \frac{4\pi}{\lambda^0} L(2R).$$

Thus we have $L(R) \leq \sqrt{4\pi L(2R)/\lambda^0}$. Moreover, if we put $\rho(R) = \log L(R)$, we see $\rho(R) \leq \frac{1}{2}\rho(2R) + C$, where $C = (\log(4\pi)/\lambda^0)/2$. Therefore, we have

$$\rho(R) \leq \frac{1}{2^n}\rho(2^n R) + C\big(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\big).$$

On the other hand, we have

$$L(R) \leq |(\pi R^2)^{1/2} \|\psi\|^2_{L^4(B^2_R)} \leq 4(\lambda^0)^{-1/2}\pi M R.$$

Then we see that

$$\rho(R) \leq C\sum_{l=1}^{\infty}\frac{1}{2^l} = 2C = \log\big(\frac{4\pi}{\lambda^0}\big)$$

for all $R > 0$. Thus $L(R) \leq e^{2C}$ for all $R > 0$. That is to say,

$$\int_{B_R} |\psi|^2 dx \leq e^{2C}$$

for all $R > 0$. So this implies that $\psi \in L^2(\mathbb{R}^2)$.

By the above claim, if $\psi \neq 0$ is a bounded solution of (3.6), we have

$$\begin{aligned} \lambda^0 \int_{\mathbb{R}^2} |\psi|^2 dx &\leq \int_{\mathbb{R}^2} |\nabla_{\boldsymbol{A}}\psi|^2 dx \\ &= \lambda \int_{\mathbb{R}^2} (1 - |\psi|^2)|\psi|^2 dx \\ &< \lambda \int_{\mathbb{R}^2} |\psi|^2 dx. \end{aligned}$$

Therefore, we get $\lambda^0 < \lambda$. Thus if $\lambda \leq \lambda^0$, any bounded solution of (3.6) is identically zero. $\square$

We continue the proof of Theorem 1.1.

When $x^0 = 0 \in \mathcal{Z}(H_0;\Omega)$, we define

$$\begin{aligned} \Omega_\epsilon &= (\Omega - x^\epsilon)/\epsilon, \\ \boldsymbol{A}_\epsilon(y) &= \frac{1}{\epsilon^{k+1}}\boldsymbol{A}^\epsilon(x^\epsilon + \epsilon y), \\ \boldsymbol{F}_\epsilon(y) &= \frac{1}{\epsilon^{k+1}}\boldsymbol{F}^\epsilon(x^\epsilon + \epsilon y), \\ \varphi_\epsilon(y) &= \psi^\epsilon(x^\epsilon + \epsilon y). \end{aligned}$$

Then $(\varphi_\epsilon, \boldsymbol{A}_\epsilon)$ satisfies

$$\begin{cases} -\nabla^2_{\boldsymbol{A}_\epsilon}\varphi_\epsilon = \epsilon^2\kappa^2(1-|\varphi_\epsilon|^2)\varphi_\epsilon & \text{in } \Omega_\epsilon, \\ (\nabla_{\boldsymbol{A}_\epsilon}\varphi_\epsilon)\cdot\boldsymbol{\nu} + \gamma\epsilon\varphi_\epsilon = 0 & \text{on } \partial\Omega_\epsilon. \end{cases} \tag{3.8}$$

Since we may assume that $\boldsymbol{A}_\epsilon \to \boldsymbol{F}$ in $C^{1,\alpha}_{\rm loc}(\mathbb{R}^2)$ and $\varphi_\epsilon \to \varphi_0$ in $C^{2,\alpha}_{\rm loc}(\mathbb{R}^2)$, we have

$$-\nabla^2_{\boldsymbol{F}}\varphi_0 = \lim_{\kappa\to\infty}\left(\frac{\kappa^{k+1}}{\sigma}\right)^{2/(k+2)}\varphi_0 = a^{2/(k+2)}\varphi_0 \quad \text{in } \mathbb{R}^2. \tag{3.9}$$

If we put $\phi(x) = \rho^{1/(k+2)}\varphi(\rho^{1/(k+2)}x)$, ϕ satisfies

$$-\nabla^2_{\boldsymbol{F}}\phi = \rho^{2/(k+2)}\lambda(1-|\phi|^2)\phi \quad \text{in } \mathbb{R}^2.$$

By the hypothesis, $a^{2/(k+2)} \le \alpha_k(H_0)^{2/(k+2)} \le \lambda^0\rho^{2/(k+2)}$ and so we obtain that $\rho^{-2/(k+2)}a^{2/(k+2)} \le \lambda^0$. Therefore, by Proposition 3.2, $\varphi_0 = 0$. That is to say, $\varphi_\epsilon \to 0$ in $C^{2,\alpha}_{\rm loc}(\mathbb{R}^2)$. Thus we have $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. If we define $\phi_\epsilon(x) = \varphi_\epsilon(x)/\lambda_\epsilon$, (3.8) becomes

$$\begin{cases} -\nabla^2_{\boldsymbol{A}_\epsilon}\phi_\epsilon = \epsilon^2\kappa^2(1-\lambda_\epsilon|\phi_\epsilon|^2)\phi_\epsilon & \text{in } \Omega_\epsilon, \\ (\nabla_{\boldsymbol{A}_\epsilon}\phi_\epsilon)\cdot\boldsymbol{\nu} + \gamma\epsilon\phi_\epsilon = 0 & \text{on } \partial\Omega_\epsilon. \end{cases}$$

Since $\lambda_\epsilon \to 0$ as $\epsilon \to 0$, we have

$$-\nabla^2_{\boldsymbol{F}}\phi_0 = a^{2/(k+2)}\phi_0 \quad \text{in } \mathbb{R}^2.$$

Here according to $|\phi_0(0)| = \lim_{\epsilon\to 0}|\phi_\epsilon(0)| = 1$, $\phi_0 \ne 0$. Since the lowest eigenvalue of $-\nabla^2_{\boldsymbol{F}}$ is equal to $\rho^{2/(k+2)}\lambda^0$, we have $a \ge (\lambda^0)^{(k+2)/2}\rho \ge \alpha_k(H_0)$. Thus we get $a = \alpha_k(H_0)$. In particular, $x^0 \in \widetilde{\mathcal{Z}}(H_0;\Omega)$.

Fix $\overline{x} \in \widetilde{\mathcal{Z}}(H_0;\Omega)$. If we choose $x_\epsilon = \overline{x}$, as above arguments, we get

$$-\nabla^2_{\boldsymbol{F}}\phi_0 = a^{2/(k+2)}\phi_0 = \mu(\boldsymbol{F})\phi_0$$

and ϕ_0 is an eigenfunction. By Theorem 1.3, $\lim \psi_\epsilon(x) = 0$ for $x \in \Omega\setminus\widetilde{\mathcal{Z}}(H_0;\Omega)$.

When $x^0 \in \partial\Omega$, we need the following lemma.

Lemma 3.3. *If* $\operatorname{dist}(x^\epsilon, \partial\Omega) \le C\epsilon$, *then we have* $x^0 \in \mathcal{Z}(H_0; \partial\Omega)$, $\operatorname{dist}(x^\epsilon, \partial\Omega) = o(\epsilon)$ *as* $\epsilon \to 0$. *And moreover,*

$$\begin{aligned} \operatorname{curl} \boldsymbol{A}^\epsilon &\to H_0 \quad \text{in } C^\alpha(\Omega), \\ \|\psi^\epsilon\|_{L^\infty(\overline{\Omega})} &\to 0, \\ \frac{\psi^\epsilon(x)}{\|\psi^\epsilon\|_{L^\infty(\overline{\Omega})}} &\to 0 \quad \text{if } x \in \partial\Omega \setminus \widetilde{\mathcal{Z}}(H_0; \partial\Omega) \quad \text{as } \epsilon \to 0. \end{aligned}$$

Choose $\overline{x^\epsilon} \in \partial\Omega$ such that $\operatorname{dist}(\overline{x^\epsilon}, \partial\Omega)$ and let $\mathcal{F}_\epsilon$ be the diffeomorphism which straighten a portion of the boundary arround $\overline{x^\epsilon}$ so that $\mathcal{F}_\epsilon(0) = \overline{x^\epsilon}$. If we write $y^\epsilon = \mathcal{F}_\epsilon^{-1}(x^\epsilon)$, $z^\epsilon := y^\epsilon/\epsilon$ is bounded. Passing to a subsequence, we may assume that $z^\epsilon \to z^0 \in \overline{\Omega}$. Since the domain of $\mathcal{F}_\epsilon$ contains a ball B_{R_0} independent of ϵ and $\mathcal{F}_\epsilon$, $D\mathcal{F}_\epsilon$ are uniformly smooth on B_{R_0}. If we denote the associated vector field wth $\boldsymbol{A}^\epsilon$ by $\boldsymbol{a}^\epsilon$, we have

$$\boldsymbol{a}^\epsilon = [g\boldsymbol{A}^\epsilon \cdot \boldsymbol{e}_1]\boldsymbol{e}_1 + [\boldsymbol{A}^\epsilon \cdot \boldsymbol{e}_2]\boldsymbol{e}_2.$$

Then we see that $\boldsymbol{a}^\epsilon(0) = \boldsymbol{A}^\epsilon(\overline{x}^\epsilon)$ and $\boldsymbol{a}^\epsilon \cdot \boldsymbol{e}_2 = 0$. Here we define $\widetilde{\varphi}_\epsilon(y) = \frac{1}{\boldsymbol{u}(\epsilon y)}\psi^\epsilon(\epsilon y)$, it follows from the elliptic estimates that

$$\begin{aligned} \|\widetilde{\varphi}_\epsilon\|_{C^{2,\alpha}(B_R^+)} &\le C(R), \\ \|\boldsymbol{u}_\epsilon\|_{C^{2,\alpha}(B_R^+)} &\le C(R) \quad \text{for} \quad 0 < R < R_0. \end{aligned}$$

Passing to a subsequence, we may assume that $\boldsymbol{u}_\epsilon \to \boldsymbol{u}_0 \in C^{2,\alpha}(\mathbb{R}^2_+)$ as $\epsilon \to 0$. Since $\operatorname{curl} \boldsymbol{u}_0 = 0$ and $\operatorname{div} \boldsymbol{u}_0 = 0$, we get $\boldsymbol{u}_0$ is a constant. Thus we may assume that $\boldsymbol{a}_\epsilon \to \boldsymbol{f}$ and $\widetilde{\varphi}_\epsilon \to \widetilde{\varphi}_0 \in C^{2,\alpha}(\mathbb{R}^2_+)$ in $C^{2,\alpha}(\mathbb{R}^2_+)$. Then we have

$$\begin{cases} -\nabla_{\boldsymbol{F}}^2 \widetilde{\varphi}_0 = a^{2/(k+2)}\widetilde{\varphi}_0 & \text{in } \mathbb{R}^2_+, \\ (\nabla_{\boldsymbol{F}} \widetilde{\varphi}_0) \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

By Proposition 3.2, we see that $\widetilde{\varphi}_0 = 0$. Thus we have

$$|\widetilde{\varphi}_0(z^0)| = \lim_{\epsilon\to 0} |\widetilde{\varphi}_0(z^\epsilon)| = \frac{1}{w(x^0)} \lim_{\epsilon \to 0} \|\psi^\epsilon\|_{L^\infty(\overline{\Omega})}.$$

Therefore, we get $\|\psi^\epsilon\|_{L^\infty(\overline{\Omega})} \to 0$ as $\epsilon \to 0$.

If we define $\widetilde{\phi}_\epsilon(y) = \widetilde{\varphi}_\epsilon(y)/\|\psi^\epsilon\|_{L^\infty(\overline{\Omega})}$, then $\widetilde{\phi}_\epsilon \to \widetilde{\phi}_0$ as $\epsilon \to 0$. Since $|\widetilde{\phi}_0(z^0)| = 1$, $\widetilde{\phi}_0$ satisfies

$$\begin{cases} -\nabla_{\boldsymbol{F}}^2 \phi = a^{2/(k+2)}\phi & \text{in } \mathbb{R}^2_+, \\ (\nabla_{\boldsymbol{F}} \phi) \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

As similar arguments of the proof of (i), we get $a = \alpha_k(H_0)$. Therefore, $x^0 \in \widetilde{\mathcal{Z}}(H_0; \partial\Omega)$ and $\lim_{\epsilon\to 0} \widetilde{\phi}^\epsilon(y) = 0$ for $y \in \partial\Omega \setminus \mathcal{Z}(H_0; \partial\Omega)$. This completes the proof of Theorem 1.1.

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