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BEST APPROXIMATION IN A CLASS OF ORDERED NORMED SPACES WITHOUT A PRE-ORDER RELATION

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Abstract: We examine best approximation by closed sets in a class of ordered normed spaces with a relation which is not necessarily a pre-order relation. We use downward and upward sets as tools for examination of best approximation by closed sets. We first study best approximation properties of these sets.

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1. Introduction

The theory of best approximation by elements of convex sets in normed linear spaces, which has many important applications in mathematics and other sciences, is well-developed. However, convexity is sometimes a very restrictive assumption, and therefore arises the problem of finding best approximation by not necessarily convex sets. We need special tools in order to examine best

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approximation by non-convex sets. The aim of the present paper is to develop a theory of best approximation by elements of closed sets in a class of normed ordered spaces. It is assumed that these spaces are equipped with a relation, which is not a pre-order. Indeed, this relation is generated by a closed conic set which is a finite union of closed convex pointed cones K_i ($i \in I$, where I is a finite index set) such that the intersection $\cap_{i\in I}\text{int}K_i$ is not empty. In the special case $I = \{1\}$, this class of spaces contains Banach lattices such as the space $L^\infty(S, \Sigma, \mu)$ of all essentially bounded functions defined on a measure space (S, Σ, μ) and the space $C(Q)$ of all continuous functions defined on a compact topological space Q. We shall use the so-called downward and upward sets as a tool in the study of best approximations by closed sets. A non-empty subset U of the space X is called downward if $(u \in U,\ x \le u) \Longrightarrow x \in U$. Similarly, a non-empty subset V of the space X is called upward if $(v \in V,\ x \ge v) \Longrightarrow x \in V$. Downward and upward sets have a simple enough structure. If U is an arbitrary subset of X, we can consider its relatively downward hull $U_* = U - K_0$ and its relatively upward hull $U^* = U + K_0$, where $K_0 = \cap_{i\in I}K_i$. In case $I = \{1\}$, these sets coincide with usual downward hull and upward hull (see [7]). These relative hulls can be used for examination of approximation properties of U. More precisely, we define a continuous and homogeneous of the first degree function $s : X \longrightarrow \mathbb{R}$ and consider its level sets $Z_+ = \{x \in X : s(x) \ge 0\}$ and $Z_- = \{x \in X : s(x) \le 0\}$. Corresponding each pair (U, t), where $U \subseteq X$ and $t \in X$, we define two sets $U_t^+ = U \cap (t - Z_+)$ and $U_t^- = U \cap (t - Z_-)$. The examination of approximation properties of U can be reduced to examination of those of U_t^+ and U_t^-. Then we show that approximation properties of U_t^+ and U_t^- can be studied by means of the relatively downward hull $(U_t^+)_*$ and relatively upward hull $(U_t^-)^*$, respectively. Approximation properties of downward and upward sets play a crucial role in this paper. These properties have been studied in [3], [6], [2], [5], when X is a Banach lattice and in the finite dimensional case: $X = \mathbb{R}^n$. We show that some of results obtained in these papers are valid in much more general situation. The structure of this paper is as follows. In Section 2, we describe the class of spaces under consideration and recall some definitions. In Section 3, we establish some results related to downward sets and its corresponding results for upward sets. Approximation properties of downward and upward sets are described in Section 4. Strictly downward sets and strictly downward at a point sets are studied in Section 5. In this section we also introduce a notion of Chebyshev points and examine its relation to strictly downward sets. Some auxiliary sets are defined and studied in Section 6. Relatively downward hull and relatively upward hull of arbitrary sets are defined in Section 7. Best approximation by closed sets is examined in

Section 8.

2. Preliminaries

Let X be a normed space. Assume that X is equipped with a closed cone $K \subset X$ which is a finite union of closed convex pointed cones K_i (the latter means that $K_i \cap (-K_i) = \{0\}$) with the intersection $\cap_{i\in I} \text{int}\, K_i$ is non-empty, where I is a finite index set. The cone K generates a relation $\geq_K$. Putting $x \leq_K y$ whenever $y - x \in K$, we get $\leq$ is a relation on X. This relation is not an order relation if K is not a convex set. We say that x is greater than y and write $x >_K y$ if $x - y \in K \setminus \{0\}$. The cone K_i generates the order relation $\geq_{K_i}$. The relation $\geq_K$, which is generated by the cone K, admits the following interpretation:

$$x \geq_K y \quad \text{if and only if there exists } i \in I \text{ such that} \quad x \geq_{K_i} y. \tag{2.1}$$

Let $K = \cup_{i\in I} K_i$ and $\mathbf{1} \in \cap_{i\in I} \text{int}\, K_i$. Using $\mathbf{1}$, we can define the function $p_i : X \longrightarrow \mathbb{R}$ by

$$p_i(x) := \inf\{\lambda \in \mathbb{R} : x \leq_{K_i} \lambda\mathbf{1}\}, \qquad x \in X. \tag{2.2}$$

This function is defined and examined in [3]. Let $K_0 = \cap_{i\in I} K_i$. Clearly, K_0 is a closed convex pointed cone and $\mathbf{1} \in \text{int} K_0$. We need to define the function $p : X \longrightarrow \mathbb{R}$ by

$$p(x) := \inf\{\lambda \in \mathbb{R} : x \leq_{K_0} \lambda\mathbf{1}\}, \qquad x \in X. \tag{2.3}$$

Since $\mathbf{1} \in \text{int} K_0$, it follows that p is finite. We now describe some properties of the function p.

Proposition 2.1. *Let $p : X \longrightarrow \mathbb{R}$ be given by (2.3). Then:*
(1) *$p(\lambda x) = \lambda p(x)$ for all $x \in X$ and $\lambda > 0$.*
(2) *$p(x + \gamma\mathbf{1}) = p(x) + \gamma$ for all $x \in X$ and $\gamma \in \mathbb{R}$.*
(3) *$p(x) = \max_{i\in I} p_i(x)$ for all $x \in X$.*
(4) *p is increasing with respect to K_0 : $x \geq_{K_0} y \Longrightarrow p(x) \geq p(y)$.*
(5) *p is continuous.*
(6) *p is sublinear, that is, $p(x + y) \leq p(x) + p(y)$ for all $x,\ y \in X$.*

Proof. We only prove item (3) of the proposition. Fix $x \in X$. Let

$$\Lambda_x = \{\lambda \in \mathbb{R} : x \leq_{K_0} \lambda\mathbf{1}\},$$

and

$$\Lambda_x^i = \{\lambda \in \mathbb{R} : x \leq_{K_i} \lambda \mathbf{1}\}.$$

Obviously, $\Lambda_x \subseteq \Lambda_x^i$ for each $i \in I$. Therefore, $\max_{i\in I} p_i(x) \leq p(x)$. To prove the reverse inequality, note that by closedness of K_i, we have $x \leq_{K_i} p_i(x)\mathbf{1}$. Thus, $x \leq_{K_i} (\max_{i\in I} p_i(x))\mathbf{1}$ for each $i \in I$. It follows that $x \leq_{K_0} (\max_{i\in I} p_i(x))\mathbf{1}$. Hence, by (2.3), $p(x) \leq \max_{i\in I} p_i(x)$, which completes the proof. □

Let

$$B_i = \{x \in X : \mathbf{1} \geq_{K_i} x \geq_{K_i} -\mathbf{1}\}, \qquad i \in I. \tag{2.4}$$

Since K_i is a closed convex pointed cone, it is easy to check that B_i can be considered as the unit ball of a norm $\|.\|_i$ $(i \in I)$ defined on X by

$$\|x\|_i := \max(p_i(x), p_i(-x)), \qquad x \in X. \tag{2.5}$$

Now, define a norm on X by

$$\|x\| = \max_{i\in I} \|x\|_i \qquad (x \in X;\ i \in I). \tag{2.6}$$

Therefore, $(X, \|.\|)$ is a normed linear space. It is easy to see that

$$\|x\| = \max(p(x), p(-x)), \qquad x \in X. \tag{2.7}$$

Due to (2.2) and (2.5), we have

$$B_i(x, r) := \{y \in X : \|y - x\|_i \leq r\} = \{y \in X : x + r\mathbf{1} \geq_{K_i} y \geq_{K_i} x - r\mathbf{1}\}, \tag{2.8}$$

where $x \in X$, $i \in I$ and $r > 0$. Now, define

$$B(x, r) := \{y \in X : \|y - x\| \leq r\} = \{y \in X : x + r\mathbf{1} \geq_{K_0} y \geq_{K_0} x - r\mathbf{1}\}, \tag{2.9}$$

where $x \in X$ and $r > 0$. $B(x, r)$ is called the closed ball with center at x and radius r. It follows from (2.1), (2.8) and (2.9) that

$$B(x, r) = \cap_{i\in I} B_i(x, r), \tag{2.10}$$

and

$$B(x, r) \subseteq \{y \in X : x + r\mathbf{1} \geq_K y \geq_K x - r\mathbf{1}\}. \tag{2.11}$$

We now present an example.

Example 2.1. Let $X = \mathbb{R}^2$. Consider the sets

$$\begin{aligned} A &= \{(x,y) \in X : x \geq 0 \text{ and } y \geq 2x\}, \\ B &= \{(x,y) \in X : x \leq 0 \text{ and } y \geq \frac{1}{2}x\}, \\ C &= \{(x,y) \in X : x \geq 0 \text{ and } y \geq -\frac{1}{2}x\}, \\ D &= \{(x,y) \in X : x \leq 0 \text{ and } y \geq -2x\}. \end{aligned}$$

Set $K_1 = A \cup B$, $K_2 = C \cup D$ and $K = K_1 \cup K_2$. It is easy to check that K is not convex and $K_0 = A \cup D$, which is convex and

$$\begin{aligned} p(\mathtt{x}) &= \max(y - 2x, y + 2x) \qquad \text{for all } \mathtt{x} = (x,y) \in X, \\ \|\mathtt{x}\| &= |y| + 2|x| \qquad \text{for all } \mathtt{x} = (x,y) \in X. \end{aligned}$$

We shall study in this paper downward sets. Recall (see [4], [7]) that a subset W of X is said to be *downward,* if $w \in W$ and $x \in X$ with $w \geq_K x$ implies that $x \in W$. Also, a subset V of X is called upward if $(v \in V,\ x \geq_K v) \Longrightarrow x \in V$. A function $f : X \longrightarrow \bar{\mathbb{R}} := [-\infty, +\infty]$ is called *topical* if this function is increasing $(x \geq_K y \implies f(x) \geq f(y))$ and plus-homogeneous in the following sense $f(x + \lambda\mathbf{1}) = f(x) + \lambda$ for all $x \in X$ and all $\lambda \in \mathbb{R}$. The definition of a topical function in finite dimensional case can be found in [6]. Also, for a function $f : X \longrightarrow \bar{\mathbb{R}}$ the lower level set $\mathcal{S}_\lambda(f)$ $(\lambda \in \mathbb{R})$ of f is defined by

$$\mathcal{S}_\lambda(f) = \{x \in X : f(x) \leq \lambda\}.$$

For any subset W of X, we shall denote by $\operatorname{int} W$, $\operatorname{cl} W$, and $\operatorname{bd} W$ the interior, the closure and the boundary of W, respectively. For a non-empty subset W of X and $x \in X$, define

$$d(x, W) := \inf_{w \in W} \|x - w\|.$$

Recall (see [1]) that an element $w_0 \in W$ is called a *best approximation* for $x \in X$ if

$$\|x - w_0\| = d(x, W).$$

If for each $x \in X$ there exists at least one best approximation $w_0 \in W$, then W is called a proximinal subset of X. If for each $x \in X$ there exists a unique best approximation $w_0 \in W$, then W is called a Chebyshev subset of X.

The set of all best approximations of x in W will be denoted by $P_W(x)$:

$$P_W(x) = \{w \in W : \|x - w\| = d(x, W)\}.$$

It is clear that $P_W(x)$ is a bounded subset of X. If W is a closed set, then $P_W(x)$ so is. If $x \notin W$, then $P_W(x)$ is located in the boundary of W. For $x \in X$ and a non-empty subset W of X, we also use the following notations:

$$d^i(x, W) := \inf_{w \in W} \|x - w\|_i, \qquad i \in I,$$

and

$$P^i_W(x) = \{w \in W : \|x - w\|_i = d^i(x, W)\}, \qquad i \in I.$$

3. Best Approximation by Downward and Upward Sets

In the remainder of the paper we assume that X is equipped with a closed cone K which is a finite union of closed convex pointed cones K_i with the intersection $\cap_{i \in I} \text{int} K_i$ is non-empty and the norm (2.7) coincides with the norm of the space X. We have the following proposition.

Proposition 3.1. *Let W be a downward subset of X and $x \in X$. Then the following are true:*

(1) *If $x \in W$, then $x - \varepsilon \mathbf{1} \in \text{int}\, W$ for all $\varepsilon > 0$.*

(2) *We have*

$$\text{int}\, W = \{x \in X : x + \varepsilon \mathbf{1} \in W \text{ for } \quad \text{some } \varepsilon > 0\}.$$

Proof. (1) Let $\varepsilon > 0$ be given and $x \in W$. Set

$$V = \{y \in X : \|y - (x - \varepsilon \mathbf{1})\| < \varepsilon\}.$$

It is clear that V is an open neighborhood of $(x - \varepsilon \mathbf{1})$. Then, by (2.11)

$$V \subset \{y \in X : x >_K y >_K x - 2\varepsilon \mathbf{1}\}.$$

Since W is a downward set and $x \in W$, it follows that $V \subset W$, and so $x - \varepsilon \mathbf{1} \in \text{int}\, W$.

(2) Let $x \in \text{int}\, W$. Then there exists $\varepsilon_0 > 0$ such that the closed ball $B(x, \varepsilon_0) \subset W$. By (2.9), $x + \varepsilon_0 \mathbf{1} \in B(x, \varepsilon_0)$. Thus $x + \varepsilon_0 \mathbf{1} \in W$, that is

$$x \in \{x \in X : x + \varepsilon \mathbf{1} \in W \text{ for } \quad \text{some } \varepsilon > 0\}.$$

Conversely, suppose that there exists $\varepsilon > 0$ such that $x + \varepsilon \mathbf{1} \in W$. Then, by part (1), $x = (x + \varepsilon \mathbf{1}) - \varepsilon \mathbf{1} \in \text{int}\, W$, which completes the proof. ∎

Corollary 3.1. *Let W be a closed downward subset of X and $w \in W$. Then $w \in \operatorname{bd} W$ if and only if $\lambda \mathbf{1} + w \notin W$ for all $\lambda > 0$.*

Proposition 3.2. *Let W be a closed downward subset of X. Then W is proximinal in X.*

Proof. Let $x_0 \in X \setminus W$ be arbitrary and $r := d(x_0, W) = \inf_{w \in W} \|x_0 - w\| > 0$. This implies that for each $\varepsilon > 0$ there exists $w_\varepsilon \in W$ such that $\|x_0 - w_\varepsilon\| < r + \varepsilon$. Then, by (2.11), we have

$$w_\varepsilon \geq_K x_0 - (r + \varepsilon)\mathbf{1}.$$

Set $w_0 = x_0 - r\mathbf{1}$. Then, we have

$$\|x_0 - w_0\| = \|r\mathbf{1}\| = r = d(x_0, W)$$

and $w_\varepsilon \geq_K x_0 - r\mathbf{1} - \varepsilon\mathbf{1} = w_0 - \varepsilon\mathbf{1}$. Since W is a downward set and $w_\varepsilon \in W$, it follows that $w_0 - \varepsilon\mathbf{1} \in W$ for all $\varepsilon > 0$. This implies, by closedness of W, that $w_0 \in W$ and so $w_0 \in P_W(x_0)$. If $x_0 \in W$, then $w_0 = x_0$ and $P_W(x_0) = \{w_0\}$. Hence, W is proximinal in X. □

Theorem 3.1. *Let W be a closed downward subset of X and $x_0 \in X$. Then there exists the least element $w_0 := \min P_W(x_0)$ of the set $P_W(x_0)$, namely, $w_0 = x_0 - r\mathbf{1}$, where $r := d(x_0, W)$.*

Proof. If $x_0 \in W$, then the result holds. Suppose that $x_0 \notin W$ and $w_0 = x_0 - r\mathbf{1}$. Thus, by Proposition 3.2, $w_0 \in P_W(x_0)$. Since $\|x_0 - w_0\| = r$, in view of (2.11), we obtain

$$x \geq_K x_0 - r\mathbf{1} = w_0, \quad \forall\, x \in B(x_0, r).$$

This implies that w_0 is the least element of the closed ball $B(x_0, r)$. Now, let $w \in P_W(x_0)$ be arbitrary. Then, $\|x_0 - w\| = r$, and hence $w \in B(x_0, r)$. Thus, $w \geq_K w_0$. Hence, w_0 is the least element of the set $P_W(x_0)$. □

Corollary 3.2. *Let W be a closed downward subset of X, $x_0 \in X$ and $w_0 = \min P_W(x_0)$. Then, $x_0 \geq_K w_0$.*

Corollary 3.3. *Let W be a closed downward subset of X and $x \in X$ be arbitrary. Then*

$$d(x, W) = \min\{\lambda \geq 0 : x - \lambda\mathbf{1} \in W\}.$$

The results obtained demonstrate that for search of best approximation of an element x by a closed downward set W we need to solve the following optimization problem:

$$\text{Minimize } \lambda \quad \text{Subject to } \; x - \lambda\mathbf{1} \in W. \tag{3.1}$$

This is a one-dimensional optimization problem which is much easier than the original problem: minimize $\|x - w\|$ subject to $w \in W$. The problem (3.1) can be solved, for example, by usual binary procedure: First find numbers ρ_1 and σ_1 such that $x - \rho_1\mathbf{1} \in W$ and $x - \sigma_1\mathbf{1} \notin W$. Let $k \geq 1$. Assume that numbers ρ_k and σ_k are known such that $x - \rho_k\mathbf{1} \in W$ and $x - \sigma_k\mathbf{1} \notin W$. Then consider $\pi_k = \frac{1}{2}(\rho_k + \sigma_k)$. If $x - \pi_k\mathbf{1} \in W$, then we put $\rho_{k+1} = \pi_k,\ \sigma_{k+1} = \sigma_k$. If $x - \pi_k\mathbf{1} \notin W$, then put $\rho_{k+1} = \rho_k,\ \sigma_{k+1} = \pi_k$. The number $r = \lim_k \rho_k = \lim_k \sigma_k$ is the optimal value of (3.1). If r is known, $x - r\mathbf{1}$ is the desired best approximation.

We now turn to upward sets. Clearly a subset V of X is upward if and only if $-V$ is downward, so all results obtained for downward sets can be reformulated for upward sets.

Proposition 3.3. *A closed upward subset V of X is proximinal.*

Theorem 3.2. *Let V be a closed upward subset of X and $x_0 \in X$. Then there exists the greatest element $v_0 = \max P_V(x_0)$ of the set $P_V(x_0)$, namely $v_0 = x_0 + r\mathbf{1}$, where $r = d(x_0, V)$.*

Corollary 3.4. *Let V be a closed upward subset of X and $x \in X$. Then*

$$d(x, V) = \min\{\lambda \geq 0 : x + \lambda\mathbf{1} \in V\}.$$

4. Characterization of Best Approximations

Let $\varphi^i : X \times X \longrightarrow \mathbb{R}$ be a function defined by

$$\varphi^i(x, y) := \sup\{\lambda \in \mathbb{R} : x + y \geq_{K_i} \lambda\mathbf{1}\} \quad (x,\ y \in X;\ i \in I). \tag{4.1}$$

Since $\mathbf{1} \in \cap_{i \in I} \operatorname{int} K_i$, it follows that the set $\{\lambda \in \mathbb{R} : x + y \geq_{K_i} \lambda\mathbf{1}\}$ is non-empty and bounded from above (by the number $\|x + y\|_i$) ($i \in I$). Clearly this set is closed. For each $i \in I$, it follows from the definition of φ^i that the function φ^i enjoys the following properties:

$$-\infty < \varphi^i(x, y) \leq \|x + y\|_i \quad \text{for each} \quad x,\ y \in X, \tag{4.2}$$

$$x + y \geq_{K_i} \varphi^i(x,y)\mathbf{1} \quad \text{for all} \quad x,\ y \in X, \tag{4.3}$$

$$\varphi^i(x,y) = \varphi^i(y,x) \quad \text{for all} \quad x,\ y \in X, \tag{4.4}$$

$$\varphi^i(x,-x) = 0 \quad \text{for all} \quad x \in X, \tag{4.5}$$

$$\varphi^i(x, y + \gamma\mathbf{1}) = \varphi^i(x,y) + \gamma \quad \text{for all} \quad x,\ y \in X \text{ and } \gamma \in \mathbb{R}, \tag{4.6}$$

$$\varphi^i(x + \gamma\mathbf{1}, y) = \varphi^i(x,y) + \gamma \quad \text{for all} \quad x,\ y \in X \text{ and } \gamma \in \mathbb{R}. \tag{4.7}$$

For each $x,\ y \in X$, we define the functions $(\varphi^i)_x : X \longrightarrow \mathbb{R}$ and $(\varphi^i)_y : X \longrightarrow \mathbb{R}$ by

$$(\varphi^i)_x(t) = \varphi^i(x,t)\,, \qquad t \in X;\ i \in I, \tag{4.8}$$

and

$$(\varphi^i)_y(t) = \varphi^i(t,y)\,, \qquad t \in X;\ i \in I. \tag{4.9}$$

It is well-known (and easy to check) that the functions $(\varphi^i)_x$ and $(\varphi^i)_y$ are topical (with respect to the order induced by K_i) and Lipschitz continuous (see [3]), indeed, we have

$$|(\varphi^i)_x(y_1) - (\varphi^i)_x(y_2)| \leq \|y_1 - y_2\|_i\,, \qquad y_1, y_2 \in X;\ i \in I,$$

and

$$|(\varphi^i)_y(x_1) - (\varphi^i)_y(x_2)| \leq \|x_1 - x_2\|_i\,, \qquad x_1,\ x_2 \in X;\ i \in I.$$

Hence the function φ^i $(i \in I)$ is continuous.

Now, define the function $\varphi : X \times X \longrightarrow \mathbb{R}$ by

$$\varphi(x,y) := \sup\{\lambda \in \mathbb{R} : x + y \geq_K \lambda\mathbf{1}\} \quad (x,\ y \in X). \tag{4.10}$$

Since $\mathbf{1} \in \operatorname{int} K$, it follows that the set $\{\lambda \in \mathbb{R} : x + y \geq_K \lambda\mathbf{1}\}$ is non-empty and bounded from above (by the number $\|x+y\|$). Clearly this set is closed. It follows from the definition of φ that the function φ has the following properties:

$$-\infty < \varphi(x,y) \leq \|x+y\| \quad \text{for each} \quad x,\ y \in X, \tag{4.11}$$

$$x + y \geq_K \varphi(x,y)\mathbf{1} \quad \text{for all} \quad x,\ y \in X, \tag{4.12}$$

$$\varphi(x,y) = \varphi(y,x) \quad \text{for all} \quad x,\ y \in X, \tag{4.13}$$

$$\varphi(x,-x) = \sup\{\lambda \in \mathbb{R} : 0 = x - x \geq_K \lambda\mathbf{1}\} = 0 \quad \text{for all} \quad x \in X, \tag{4.14}$$

$$\varphi(x, y + \lambda\mathbf{1}) = \varphi(x,y) + \lambda \quad \text{for all} \quad x,\ y \in X \quad \text{and} \quad \lambda \in \mathbb{R}, \tag{4.15}$$

$$\varphi(x + \lambda\mathbf{1}, y) = \varphi(x,y) + \lambda \quad \text{for all} \quad x,\ y \in X \quad \text{and} \quad \lambda \in \mathbb{R}, \tag{4.16}$$

$$\varphi(\gamma x, \gamma y) = \gamma\varphi(x,y) \quad \text{for all} \quad x,\ y \in X \quad \text{and} \quad \gamma > 0. \tag{4.17}$$

Proposition 4.1. *Let φ^i and φ be the functions defined by* (4.1) *and* (4.10)*, respectively. Then*

$$\varphi(x,y) = \max_{i\in I} \varphi^i(x,y) \quad (x,\ y \in X). \tag{4.18}$$

Proof. For each $x,\ y \in X$, let

$$\Lambda_{xy} = \{\lambda \in \mathbb{R} : x + y \geq_K \lambda \mathbf{1}\},$$

and

$$\Lambda^i_{xy} = \{\lambda \in \mathbb{R} : x + y \geq_{K_i} \lambda \mathbf{1}\}.$$

In view of (2.1), we have $\Lambda_{xy} = \cup_{i\in I} \Lambda^i_{xy}$, and so

$$\sup \Lambda_{xy} = \max_{i\in I} \sup \Lambda^i_{xy}.$$

Hence, by (4.1) and (4.10), we obtain (4.18). □

Now, consider $x,\ y \in X$. We define the functions $\varphi_x : X \longrightarrow \mathbb{R}$ and $\varphi_y : X \longrightarrow \mathbb{R}$ by

$$\varphi_x(t) = \varphi(x,t)\,, \qquad t \in X, \tag{4.19}$$

and

$$\varphi_y(t) = \varphi(t,y)\,, \qquad t \in X. \tag{4.20}$$

Note that φ_x and φ_y are not increasing functions with respect to the order of X. We have the following result.

Corollary 4.1. *For each $x,\ y \in X$, the functions defined by* (4.19) *and* (4.20) *are Lipschitz continuous.*

Proof. This is an immediate consequence of Lipschitz continuity of $(\varphi^i)_x$ and $(\varphi^i)_y$ $(i \in I)$ and Proposition 4.1. □

Corollary 4.2. *The function φ is defined by* (4.10) *is continuous. It follows from Corollary 4.1.*

Proposition 4.2. *Let φ be the function defined by* (4.10) *and set*

$$\Lambda(y,\alpha) = \{x \in X : \varphi(x,y) \geq \alpha\} \quad (y \in X;\ \alpha \in \mathbb{R}).$$

Then, $\Lambda(y,\alpha) = K + \alpha\mathbf{1} - y$ for all $y \in X$ and all $\alpha \in \mathbb{R}$.

Proof. Fix $y \in X$ and $\alpha \in \mathbb{R}$. Let $x \in \Lambda(y,\alpha)$. Then, $\varphi(x,y) \geq \alpha$. Due to Proposition 4.1, there exists $j \in I$ such that $\varphi^j(x,y) \geq \alpha$. By (4.3), we have $x+y \geq_{K_j} \varphi^j(x,y)\mathbf{1} \geq_{K_j} \alpha\mathbf{1}$, and so $x+y \geq_{K_j} \alpha\mathbf{1}$. Thus we have $x+y \geq_K \alpha\mathbf{1}$. This implies that $x+y-\alpha\mathbf{1} \in K$, and hence $x \in K+\alpha\mathbf{1}-y$. Conversely, Let $x \in K+\alpha\mathbf{1}-y$. Then there exists $k \in K$ such that $x+y-\alpha\mathbf{1} = k \in K$. Thus, $\alpha \in \Lambda_{xy}$ (see Proposition 4.1). This yields that $\alpha \leq \varphi(x,y)$. Hence, $x \in \Lambda(y,\alpha)$, which completes the proof. □

Corollary 4.3. *Under the hypotheses of Proposition* 4.2, *we have*

$$\varphi(x,y) \geq \alpha \quad \textit{if and only if} \quad x+y \geq_K \alpha\mathbf{1} \quad (x,\ y \in X;\ \alpha \in \mathbb{R}).$$

Lemma 4.1. *Let W be a closed downward subset of X, $y_0 \in \operatorname{bd} W$ and φ be the function defined by* (4.10). *Then*

$$\varphi(w,-y_0) \leq 0 = \varphi(y_0,-y_0) = \varphi^i(y_0,-y_0)\,, \qquad \forall\ w \in W,\ \forall\ i \in I. \tag{4.21}$$

Proof. The proof is similar to the proof of Lemma 4.3 in [3]. □

Lemma 4.2. *Let W be a closed downward subset of X, $x \in X \setminus W$, $r > 0$ and $i \in I$. Then, $r = d^i(x,W)$ if and only if $x - r\mathbf{1} \in W$ and $\varphi^i(w, r\mathbf{1}-x) \leq 0$ for all $w \in W$.*

Proof. Let $r = d^i(x,W)$. In a manner analogous to the proof of Proposition 3.2, one can prove that $x - r\mathbf{1} \in P^i_W(x) \subset W$. Since $P^i_W(x) \subseteq \operatorname{bd} W$, it follows from Lemma 4.1 that $\varphi^i(w, r\mathbf{1}-x) \leq 0$ for all $w \in W$. Conversely, suppose that $x - r\mathbf{1} \in W$ and $\varphi^i(w, r\mathbf{1}-x) \leq 0$ for all $w \in W$. Let $w \in W$ be arbitrary. Since $\varphi^i(w,.)$ is topical and Lipschitz continuous, we have

$$\|x-w\|_i \geq \varphi^i(x,-x) - \varphi^i(w,-x) \geq r.$$

Since $\|x-(x-r\mathbf{1})\|_i = r$ and $x - r\mathbf{1} \in W$, we conclude that $r = d^i(x,W)$. □

Lemma 4.3. *Let W be a closed downward subset of X, $x \in X \setminus W$ and $r > 0$. Then $r = d(x,W)$ if and only if $x - r\mathbf{1} \in W$ and for some $i \in I$, $\varphi^i(w, r\mathbf{1}-x) \leq 0$ for all $w \in W$.*

Proof. Let $r = d(x,W)$. By Proposition 3.2, we have $x - r\mathbf{1} \in P_W(x) \subseteq \operatorname{bd} W$. Then, it follows from Lemma 4.1 that $\varphi(w, r\mathbf{1}-x) \leq 0$ for all $w \in W$. In view of (4.18), $\varphi^i(w, r\mathbf{1}-x) \leq 0$ for all $w \in W$ and all $i \in I$. Conversely, suppose that $x - r\mathbf{1} \in W$ and for some $i \in I$, $\varphi^i(w, r\mathbf{1}-x) \leq 0$ for all $w \in W$. Consider $w \in W$. Since $\varphi^i(w,.)$ is topical and Lipschitz continuous, we have

$$\|x-w\| \geq \|x-w\|_i \geq \varphi^i(x,-x) - \varphi^i(w,-x) \geq r.$$

Since $r = \|x - (x - r\mathbf{1})\|$ and $x - r\mathbf{1} \in W$, thus one has $r = d(x, W)$. □

The following result is an immediate consequence of Lemma 4.2 and Lemma 4.3.

Corollary 4.4. *Let W be a closed downward subset of X, $x \in X \setminus W$. Then*

$$d(x, W) = d^i(x, W) \qquad \text{for all } i \in I. \tag{4.22}$$

Corollary 4.5. *Let W be a closed downward subset of X, $x \in X \setminus W$ and $w_0 \in W$. Then, $w_0 \in P_W(x)$ if and only if $w_0 \in P_W^i(x)$ for each $i \in I$.*

Proof. Let $w_0 \in P_W(x)$. Then, $\|x - w_0\| = d(x, W)$. In view of (2.6) and (4.22), we have $\|x - w_0\|_i = d^i(x, W)$ for each $i \in I$. Therefore, $w_0 \in P_W^i(x)$ for each $i \in I$. Conversely, Let $w_0 \in P_W^i(x)$ for each $i \in I$. Then, $\|x - w_0\|_i = d^i(x, W)$ for each $i \in I$. Hence, by (4.22), we get $\|x - w_0\| = \max_{i \in I} \|x - w_0\|_i = d(x, W)$, that is, $w_0 \in P_W(x)$. □

Theorem 4.1. *Let W be a closed downward subset of X, $x_0 \in X \setminus W$, $y_0 \in W$ and $r_0 := \|x_0 - y_0\|$. Assume that φ is the function defined by (4.10). Then the following assertions are equivalent:*

(1) $y_0 \in P_W(x_0)$.

(2) *There exists $l \in X$ such that*

$$\varphi(w, l) \leq 0 \leq \varphi(y, l), \quad \forall\, w \in W,\ y \in B(x_0, r_0). \tag{4.23}$$

Moreover, if (4.23) holds with $l = -y_0$, then, $y_0 = w_0 = \min P_W(x_0)$, where $w_0 = x_0 - r\mathbf{1}$ is the least element of the set $P_W(x_0)$ and $r := d(x_0, W)$.

Proof. (1) $\Longrightarrow$ (2) Suppose that $y_0 \in P_W(x_0)$. Then, $r_0 = \|x_0 - y_0\| = d(x_0, W) = r$. Since W is a closed downward subset of X, it follows from Theorem 3.1 that the least element $w_0 = x_0 - r_0\mathbf{1}$ of the set $P_W(x_0)$ exists. Let $l = -w_0$ and $y \in B(x_0, r_0)$ be arbitrary. Then, by (2.11), we have $y \geq_K -l$ or $y + l \geq_K 0$. Hence, in view of Corollary 4.3, we have $\varphi(y, l) \geq 0$. On the other hand, since $w_0 \in P_W(x_0)$, it follows that $w_0 \in \operatorname{bd} W$. Hence, by Lemma 4.1, we have $\varphi(w, l) \leq 0$ for all $w \in W$.

(2) $\Longrightarrow$ (1) Assume that (2) holds. By (2.9), it is clear that $x_0 - r_0\mathbf{1} \in B(x_0, r_0)$. Therefore, by (4.23), we have $\varphi(x_0 - r_0\mathbf{1}, l) \geq 0$. Due to Corollary 4.3, we get $x_0 - r_0\mathbf{1} + l \geq_K 0$, and so $l - r_0\mathbf{1} \geq_K -x_0$. Hence there exists $j \in I$ such that

$$l - r_0\mathbf{1} \geq_{K_j} -x_0. \tag{4.24}$$

Now, let $w \in W$ be arbitrary. Since $\varphi^j(w,.)$ is topical and that (4.18) and (4.23) hold, it follows from (4.24) that

$$\varphi^j(w,-x_0) \leq \varphi^j(w, l - r_0\mathbf{1}) = \varphi^j(w,l) - r_0 \leq 0 - r_0 = -r_0.$$

By (2.6), (4.5) and Lipschitz continuity of $\varphi^j(.,-x_0)$, we have

$$\begin{aligned} r_0 \leq |\varphi^j(w,-x_0)| &= |\varphi^j(x_0,-x_0) - \varphi^j(w,-x_0)| \\ &\leq \|x_0 - w\|_j \\ &\leq \|x_0 - w\| \quad \text{for all} \quad w \in W. \end{aligned}$$

Thus, $\|x_0 - y_0\| = d(x_0, W)$. Consequently, $y_0 \in P_W(x_0)$. Finally, suppose that (4.23) holds with $l = -y_0$. Then, by the implication (2) $\Longrightarrow$ (1), we have $y_0 \in P_W(x_0)$, and so $r_0 = \|x_0 - y_0\| = d(x_0, W)$ and $y_0 \geq_K w_0$, where $w_0 = x_0 - r\mathbf{1}$ is the least element of the set $P_W(x_0)$ and $r := d(x_0, W)$. Now, let $w \in P_W(x_0)$ be arbitrary. Then $\|x_0 - w\| = d(x_0, W) = r_0$, that is, $w \in B(x_0, r_0)$. It follows from (4.23) that $\varphi(w, -y_0) \geq 0$. In view of Corollary 4.3, we have $w - y_0 \geq_K 0$, and so $w \geq_K y_0$. This means that $y_0 = \min P_W(x_0) = w_0$. This completes the proof. □

5. Strictly Downward Sets and Their Best Approximation Properties

We start with the following definitions which were introduced in [3] for downward subsets of a Banach lattice.

Definition 5.1. A downward subset W of X is called strictly downward if for each boundary point w_0 of W, the inequality $w >_K w_0$ implies $w \notin W$.

Definition 5.2. Let W be a downward subset of X. We say that W is strictly downward at a point $w' \in \operatorname{bd} W$ if for all $w_0 \in \operatorname{bd} W$ with $w' \geq_K w_0$, the inequality $w >_K w_0$ implies $w \notin W$.

The following lemmas have been proved in [3], however those proofs hold for the case under consideration.

Lemma 5.1. *Let $f : X \longrightarrow \mathbb{R}$ be a continuous strictly increasing function. Then all non-empty level sets $\mathcal{S}_c(f)$ $(c \in \mathbb{R})$ of f are strictly downward.*

Lemma 5.2. *Let W be a closed downward subset of X. Then W is strictly downward at $w' \in \operatorname{bd} W$ if and only if:*

(*i*) $w >_K w' \implies w \notin W$;

(*ii*) $(w' \geq_K w_0, w_0 \in \mathrm{bd}\, W) \implies w_0 = w'$.

Lemma 5.3. *Let W be a closed downward subset of X. Then W is strictly downward if and only if W is strictly downward at each its boundary point.*

Lemma 5.4. *Let φ be the function defined by (4.10) and W be a closed downward subset of X that is strictly downward at a point $w' \in \mathrm{bd}\, W$. Then there exists unique $l \in X$ such that*

$$\varphi(w, l) \leq 0 = \varphi(w', l), \quad \forall\, w \in W.$$

Theorem 5.1. *Let φ be the function defined by (4.10). Then for a closed downward subset W of X the following assertions are equivalent:*
(1) *W is strictly downward.*
(2) *For each $w_0 \in \mathrm{bd}\, W$ there exists unique $l \in X$ such that*

$$\varphi(w, l) \leq 0 = \varphi(w_0, l), \quad \forall\, w \in W.$$

Proof. The implication (1) $\implies$ (2) follows from Lemma 5.4. We now prove the implication (2) $\implies$ (1). Assume that for each $w_0 \in \mathrm{bd}\, W$ there exists unique $l \in X$ such that

$$\varphi(w, l) \leq 0 = \varphi(w_0, l), \quad \forall\, w \in W.$$

Let $w_0 \in \mathrm{bd}\, W$ and $y \in X$ with $y >_K w_0$. Assume that $y \in W$. We claim that $y + \lambda \mathbf{1} \notin W$ for all $\lambda > 0$. Suppose that there exists $\lambda_0 > 0$ such that $y + \lambda_0 \mathbf{1} \in W$. Since $y + \lambda_0 \mathbf{1} >_K w_0 + \lambda_0 \mathbf{1}$ and W is a downward set, we have $w_0 + \lambda_0 \mathbf{1} \in W$. In view of Corollary 3.1, it contradicts with $w_0 \in \mathrm{bd}\, W$, and so the claim is true. Then, by Corollary 3.1, we have $y \in \mathrm{bd}\, W$. Let $l = -y$. It follows from Lemma 4.1 that

$$\varphi(w, l) \leq 0 = \varphi(y, l), \quad \forall\, w \in W. \tag{5.1}$$

On the other hand, applying Lemma 4.1 to the point w_0 we have for $l' = -w_0$:

$$\varphi(w, l') \leq 0 = \varphi(w_0, l'), \quad \forall\, w \in W. \tag{5.2}$$

Since $y >_{K_i} w_0$ for some $i \in I$ and $\varphi^i(., l')$ is increasing, it follows from (4.5), (5.2) and Proposition 4.1 that $0 = \varphi^i(w_0, l') \leq \varphi^i(y, l') \leq \varphi(y, l') \leq 0$. This, together with (5.2) imply that

$$\varphi(w, l') \leq 0 = \varphi(y, l'), \quad \forall\, w \in W. \tag{5.3}$$

Since $w_0 \neq y$, it follows that $l' \neq l$. Hence (5.1) and (5.3) contradict the uniqueness of l. We have demonstrated that the assumption $y \in W$ leads to a contradiction. Thus $y \notin W$. This means that W is strictly downward. □

Corollary 5.1. *Let $f : X \longrightarrow \mathbb{R}$ be a continuous strictly increasing function and φ be the function defined by* (4.10)*. Then for each $x \in X$ there exists unique $l = -x$ such that*

$$\varphi(w, l) \leq 0 = \varphi(x, l)\,, \quad \forall\, w \in \mathcal{S}_c(f),$$

where $c = f(x)$.

Proof. This is an immediate consequence of Lemma 5.1 and Theorem 5.1. □

Definition 5.3. Let W be a downward subset of X. A point $w' \in \operatorname{bd} W$ is said to be a Chebyshev point if for each $w_0 \in \operatorname{bd} W$ with $w' \geq_K w_0$ and for each $x_0 \notin W$ such that $w_0 \in P_W(x_0)$ it follows that $P_W(x_0) = \{w_0\}$, that is, best approximation of x_0 is unique.

Definition 5.3 was introduced in [3] for a downward subset of a Banach lattice.

Definition 5.4. Let W be a downward subset of X. A point $w' \in \operatorname{bd} W$ is said to be a Chebyshev point of W with respect to each K_i $(i \in I)$, if for each $w_0 \in \operatorname{bd} W$ with $w' \geq_K w_0$ and for each $x_0 \notin W$ such that $w_0 \in {P^i}_W(x_0)$ for each $i \in I$, it follows that ${P^i}_W(x_0) = \{w_0\}$ for each $i \in I$.

Theorem 5.2. *Let W be a closed downward subset of X and $w' \in \operatorname{bd} W$. If w' is a Chebyshev point of W with respect to each K_i $(i \in I)$, Then W is a strictly downward set at w'.*

Proof. Suppose that w' is a Chebyshev point of W with respect to each K_i $(i \in I)$. Assume if possible that W is not strictly downward at w'. Then we can find $w_0 \in \operatorname{bd} W$ and $w \in W$ such that $w' \geq_K w_0$ and $w >_K w_0$. Let $r \geq \|w - w_0\| > 0$. It follows from (2.10) that

$$r\mathbf{1} \geq_{K_i} w - w_0\,, \qquad \forall\, i \in I.$$

Thus, $w_0 + r\mathbf{1} \geq_{K_i} w$ for all $i \in I$. Set $x_0 = w_0 + r\mathbf{1} \in X$. Since $w_0 \in \operatorname{bd} W$, by Lemma 4.1, we have $\varphi(y, -w_0) \leq 0$ for all $y \in W$. Also, $x_0 - r\mathbf{1} = w_0 \in W$. Thus, by Proposition 4.1 and Lemma 4.3, we get $r = d(x_0, W)$. Since $\|x_0 - w_0\|_i = \|r\mathbf{1}\|_i = r$ for all $i \in I$, it follows from (2.6) that $\|x_0 - w_0\| = r$, and hence $w_0 \in P_W(x_0)$. In view of Corollary 4.5, we obtain $w_0 \in P^i_W(x_0)$ for all $i \in I$.

On the other hand, we have $x_0 = w_0 + r\mathbf{1} \geq_{K_i} w$ for all $i \in I$. Since $w >_K w_0$, we conclude that there exists $j \in I$ such that $w >_{K_j} w_0$. It follows that $r\mathbf{1} = x_0 - w_0 >_{K_j} x_0 - w \geq_{K_j} 0$. Hence

$$\|x_0 - w\|_j \leq \|r\mathbf{1}\|_j = r = d^j(x_0, W) \leq \|x_0 - w\|_j.$$

Thus, $\|x_0 - w\|_j = d^j(x_0, W)$, and so $w \in P^j{}_W(x_0)$ with $w \neq w_0$. Whence there exist a point $x_0 \in X \setminus W$ and a point $w_0 \in \mathrm{bd}\, W$ with $w' \geq_K w_0$ such that $w_0 \in P^i{}_W(x_0)$ for each $i \in I$ and $P^j{}_W(x_0)$ contains at least one point different from w_0. This is a contradiction because w' is a Chebyshev point of W with respect to each K_i $(i \in I)$, which completes the proof. □

Proposition 5.1. *Let W be a closed downward subset of X and $w' \in \mathrm{bd}\, W$. If W is a strictly downward set at w', then w' is a Chebyshev point of W.*

Proof. The proof is similar to the proof of Theorem 4.2 (the implication (2) $\Longrightarrow$ (1)) in [3]. □

Corollary 5.2. *Let $f : X \longrightarrow \mathbb{R}$ be a continuous strictly increasing function. Then $\mathcal{S}_c(f)$ $(c \in \mathbb{R})$ is a Chebyshev subset of X.*

Proof. This is an immediate consequence of Lemma 5.1 and Proposition 5.1. □

6. Sets Z_+ and Z_-

Consider the function s defined on X by

$$s(x) = \frac{1}{2}[p(x) - p(-x)].$$

It is not difficult to show that the function s has the following properties:

(1) s is homogeneous of the first degree, that is, $s(\lambda x) = \lambda s(x)$ for all $x \in X$ and all $\lambda \in \mathbb{R}$.

(2) s is increasing with respect to the order induced by K_0, that is,

$$x \leq_{K_0} y \Longrightarrow s(x) \leq s(y).$$

(3) s is plus-homogeneous, that is, $s(x + \mu\mathbf{1}) = s(x) + \mu$ for all $x \in X$ and $\mu \in \mathbb{R}$.

(4) s is continuous.

We will investigate the level sets:

$$Z_+ = \{x \in X : s(x) \geq 0\} \quad \text{and} \quad Z_- = \{x \in X : s(x) \leq 0\}$$

of the function s. Clearly, $Z_+ \cup Z_- = X$. Also we have

$$x \in Z_+ \Longleftrightarrow p(x) \geq p(-x) \Longleftrightarrow p(x) = \|x\|, \tag{6.1}$$

and

$$x \in Z_- \iff p(x) \leq p(-x) \iff p(-x) = \|x\|. \tag{6.2}$$

Since s is homogeneous, we have $Z_- = -Z_+$. Let $Z_0 = \{x : s(x) = 0\}$. Then

$$Z_+ \cap Z_- = Z_0.$$

Since s is continuous, it follows that Z_+ and Z_- are closed sets in X. Note that both Z_+ and Z_- are conic sets. (Recall that a set $C \subseteq X$ is called conic if $x \in C$ and $\lambda > 0$ implies that $\lambda x \in C$).

Definition 6.1. A subset U of the space X is called relatively downward if it is downward with respect to the order induced by K_0. Similarly, a subset V of the space X is called relatively upward if it is upward with respect to the order induced by K_0.

It is clear that every downward set is relatively downward. It follows from the monotonicity of s that Z_+ is relatively upward and Z_- is relatively downward. We now give an example of the sets Z_+, Z_- and Z_0.

Example 6.1. Let X and K be as in Example 2.1. It is easy to check that

$$\begin{aligned} Z_+ &= \{(x, y) \in X : y \geq 0\}, \\ Z_- &= \{(x, y) \in X : y \leq 0\}, \\ Z_0 &= \{(x, y) \in X : y = 0\}, \\ s(x, y) &= y \quad \text{for all } (x, y) \in X. \end{aligned}$$

7. Relatively Downward Hull and Relatively Upward Hull

Let U be a subset of X. The intersection U_* of all relatively downward sets containing U is called the relatively downward hull of U. Since the intersection of an arbitrary family of relatively downward sets is relatively downward, it follows that U_* is relatively downward. Clearly U_* is the least (by inclusion) relatively downward set which contains U. The intersection U^* of all relatively upward sets containing U is called the relatively upward hull of U. The set U^* is relatively upward and is the least (by inclusion) relatively upward set containing U.

Proposition 7.1. (see [7], Proposition 2.3) *Let $U \subseteq X$. Then*

$$U_* = U - K_0 = \{u - v : u \in U,\ v \in K_0\}, \tag{7.1}$$

and

$$U^* = U + K_0 = \{u + v : u \in U,\ v \in K_0\}. \tag{7.2}$$

We shall use the relatively downward hull U_* and the relatively upward hull U^* of the closed set U for examination of best approximation of a certain element t by U.

Proposition 7.2. *Consider a subset U of X.*
(1) *Let $t \in X$ be an element such that $t - U \subset Z_+$. Then $d(t, U) = d(t, U_*)$.*
(2) *Let $t \in X$ be an element such that $t - U \subset Z_-$. Then $d(t, U) = d(t, U^*)$.*

Proof. We shall prove only the first part of the proposition. The second part can be proved in a similar way. Let $r = d(t, U_*)$. Since $U \subseteq U_*$, we have $r \leq d(t, U)$. So we need only to check the reverse inequality. Let $u_* \in U_*$ be arbitrary. By Proposition 7.1, there exists $u \in U$ and $v \in K_0$ such that $u_* = u - v$. Since $t - u \in Z_+$, by (2.7) and part (4) of Proposition 2.1, we have

$$\|t - u\| = p(t - u) \leq p(t - u + v) = p(t - u_*) \leq \|t - u_*\|.$$

Thus, for each $u_* \in U_*$, there exists $u \in U$ such that $\|t - u_*\| \geq \|t - u\|$. This yields that $r \geq d(t, U)$. Hence, we prove that $r = d(t, U)$. □

We shall use the following definitions. Consider $U \subseteq X$ and $t \in X$. The pair (U, t) is called proximinal, if there exists a best approximation of t by U. The set U is called proximinal, if the pair (U, t) is proximnal for all $t \in X$. U is called locally compact, if the set $U_r = \{u \in U : \|u\| \leq r\}$ is compact for each $r > 0$. Each locally compact set is closed. Clearly each locally compact set is proximinal.

Proposition 7.3. (1) *Let $t \in X$ be an element such that $t - U \subset Z_+$ and U_* be a closed set. Then (U, t) is a proximinal pair.*

(2) *Let $t \in X$ be an element such that $t - U \subset Z_-$ and U^* be a closed set. Then (U, t) is a proximinal pair.*

Proof. We shall prove only the first part of proposition. Since U_* is a closed and relatively downward set, we conclude, in a manner analogous to the proof of Proposition 3.2, that the least element u_0 of $P_{U_*}(t)$ exists and $u_0 = t - r\mathbf{1}$, where $r = d(t, U_*)$. By Proposition 7.2, we have $r = d(t, U)$. Since $u_0 \in U_*$, there exist $u \in U$ and $v \in K_0$ such that $u_0 = t - r\mathbf{1} = u - v$. By hypothesis, $t - u \in Z_+$. Hence, in view of (2.7), we have

$$\|t - u\| = p(t - u) = p(r\mathbf{1} - v) \leq p(r\mathbf{1}) = r\,.$$

On the other hand, $\|t-u\| \geq d(t,U) = r$. Thus $\|t-u\| = r$ and so $u \in P_U(t)$, which completes the proof. □

The proof of the next proposition is similar to that of Proposition 7.3. We omit its easy proof.

Proposition 7.4. (1) *Let $t-U \subset Z_+$ and U_* be a closed. Let $r = d(t,U_*)$. Then there exists $u_0 \in U$ such that $u_0 \geq_{K_0} t - r\mathbf{1}$ (and hence $u_0 \geq_K t - r\mathbf{1}$); each element u_0 with this property is a best approximation of t by U.*

(2) *Let $t - U \subset Z_-$ and U^* be a closed set. Let $r = d(t,U^*)$. Then there exist $u_0 \in U$ such that $u_0 \leq_{K_0} t + r\mathbf{1}$ (and hence $u_0 \leq_K t + r\mathbf{1}$); each element u_0 with this property is a best approximation of t by U.*

Theorem 7.1. *Let $U \subset X$, $t \in X$ be such that $t - U \subset Z_+$, φ be the function defined by (4.10) and $r \geq 0$. Then the following assertions are equivalent:*

(1) $r = d(t,U)$.

(2) *There exists $m \in X$ such that*

$$\varphi(u,m) \leq 0 \leq \varphi(y,m)\,, \qquad \forall u \in U,\ y \in B(t,r), \tag{7.3}$$

and there exists a sequence $u_n \in U$ such that $\|u_n - t\| \longrightarrow r$.

Proof. (1) $\Longrightarrow$ (2) Let $r = d(t,U)$. Then there exists a sequence $u_n \in U$ such that $\|u_n - t\| \longrightarrow r$. By Proposition 7.2, we have $r = d(t,U) = d(t,U_*) = d(t, \mathrm{cl}U_*)$. Since $\mathrm{cl}U_*$ is a closed relatively downward set, by a similar argument as in the proof of Proposition 3.2, we have $t - r\mathbf{1} \in P_{\mathrm{cl}U_*}(t) \subset \mathrm{bd}\,\mathrm{cl}U_* = \mathrm{bd}\,U_*$. Set $m = r\mathbf{1} - t$. By Lemma 4.1, we get $\varphi(u,m) \leq 0$ for all $u \in U$. On the other hand, $t - r\mathbf{1} \in B(t,r)$. Thus, by (2.11), for any $y \in B(t,r)$ we have $y \geq_K t - r\mathbf{1}$, that is, $y + m \geq_K 0$, and by Corollary 4.3, we obtain $\varphi(y,m) \geq 0$. Hence (7.3) holds.

(2) $\Longrightarrow$ (1) Suppose that (2) holds. Since $t - r\mathbf{1} \in B(t,r)$, we have $\varphi(t - r\mathbf{1}, m) \geq 0$. Therefore, due to Corollary 4.3, we have $t - r\mathbf{1} + m \geq_K 0$ or $m \geq_K r\mathbf{1} - t$. In view of the existence a sequence $u_n \in U$ such that $\|u_n - t\| \longrightarrow r$, in a manner analogous to the proof of Theorem 4.1 (the implication (2) $\Longrightarrow$ (1)), we conclude that $r = d(t,U)$. □

The proof of the next theorem is similar to that of Theorem 7.1.

Theorem 7.2. *Let $U \subset X$, $t \in X$ be such that $t - U \subseteq Z_-$, φ be the function defined by (4.10) and $r \geq 0$. Then the following assertions are equivalent:*

(1) $r = d(t,U)$.

(2) *There exists* $m \in X$ *such that*

$$\varphi(-u,m) \leq 0 \leq \varphi(-y,m), \qquad \forall\, u \in U,\ y \in B(t,r), \tag{7.4}$$

and there exists a sequence $u_n \in U$ *such that* $\|u_n - t\| \longrightarrow r$.

8. Best Approximation by a Closed Set

Relatively downward and relatively upward sets can be used for examination of best approximation by closed sets (it is assumed that best approximation exists). We start with the following assertion.

Proposition 8.1. *Let* U *be closed subset of* X *and* $t \in X$. *Consider the following sets*

$$U_t^+ = U \cap (t - Z_+) \quad \text{and} \quad U_t^- = U \cap (t - Z_-).$$

Then:
(1) $t - U_t^+ \subseteq Z_+, \quad t - U_t^- \subseteq Z_-$.
(2) $U_t^+ \cup U_t^- = U$.
(3) $U_t^+ \cap U_t^- = U \cap (t - Z_0)$, *where* $Z_0 = \{x \in X : s(x) = 0\}$.
(4) U_t^+ *and* U_t^- *are closed.*

In the rest of this section, we consider a fixed proximinal pair (U,t). Since $U_t^+ \cup U_t^- = U$, it follows that

$$\inf_{u \in U} \|t-u\| = \min\,(\inf_{u^+ \in U_t^+} \|t-u^+\|\ ,\ \inf_{u^- \in U_t^-} \|t-u^-\|). \tag{8.1}$$

Therefore at least one of the pairs (U_t^+,t) and (U_t^-,t) is proximinal and a best approximation of t by U coincides with a best approximation of t by at least one of the sets U_t^+ or U_t^-. Let

$$r_+ = d(t,U_t^+), \quad r_- = d(t,U_t^-), \quad r = \min(r_-,r_+).$$

For examination of best approximation of t by elements of U we need to find numbers r_+ and r_-. The numbers r_+ and r_- can be found by solving a one-dimensional optimization problem of the form (3.1). If $r_+ < r_-$, then best approximation of t by U is reduced to the best approximation of t by U_t^+. If relatively downward hull of $(U_t^+)_*$ of the set U_t^+ is closed, by part (1) of Proposition 8.1 and Proposition 7.4, we have

$$P_U(t) = P_{U_t^+}(t) = \{u \in U_t^+ : u \geq_{K_0} t - r\mathbf{1}\}.$$

If $r_- < r_+$, then best approximation of t by U is reduced to the best approximation of t by U_t^-. If the set $(U_t^-)^*$ is closed, we can assert that

$$P_U(t) = P_{U_t^-}(t) = \{u \in U_t^- : u \leq_{K_0} t + r\mathbf{1}\}.$$

If $r_+ = r_-$, then $r = r_+ = r_-$ and we can use each of them. In the remainder of this section, we assume that both pair (U_t^+, t) and (U_t^-, t) are proximinal. In particular, if U is a locally compact, then these pairs are proximinal for any $t \in X$. We are now interested in best approximations u of t by U such that $s(u - t) = 0$. We introduce the following definition.

Definition 8.1. A pair (U, t) with $U \subseteq X$ and $t \in X$ is called strongly proximinal, if $s(t - u) = 0$ for each best approximation u of t by U.

Clearly $s(t - u) = 0$ if and only if $t - u \in Z_0$.

Proposition 8.2. *A pair (U, t) is strongly proximinal if and only if $P_U(t) = P_{U_t^+}(t) \cap P_{U_t^-}(t)$.*

Proof. Let $u \in P_U(t)$. Since $u - t \in Z_- = -Z_+$ and $u \in U$, it follows that $u \in U \cap (t - Z_+)$. We have

$$\|t - u\| = \min_{u' \in U} \|t - u'\| \leq \min_{u' \in U_t^+} \|t - u'\|.$$

Since $u \in U_t^+$, we conclude that the equality $\|t - u\| = \min_{u' \in U_t^+} \|t - u'\|$ holds. Thus $u \in P_{U_t^+}(t)$. A similar argument shows that $u \in P_{U_t^-}(t)$. Assume now that $u \in P_{U_t^+}(t) \cap P_{U_t^-}(t)$. Then

$$\|u - t\| = d(t, U_t^+) = d(t, U_t^-).$$

Combining the equality $U = U_t^+ \cup U_t^-$ with (8.1), we get $\|u - t\| = \min_{u' \in U} \|u' - t\|$ and so $u \in P_U(t)$.

Conversely, assume that $P_U(t) = P_{U_t^+}(t) \cap P_{U_t^-}(t)$. Then

$$\begin{aligned} P_U(t) = P_{U_t^+}(t) \cap P_{U_t^-}(t) &= \{u \in U_t^+ : t - r\mathbf{1} \leq_{K_0} u\} \cap \{u \in U_t^- : u \leq_{K_0} t + r\mathbf{1}\} \\ &= \{u \in U_t^+ \cap U_t^- : t - r\mathbf{1} \leq_{K_0} u \leq_{K_0} t + r\mathbf{1}\}. \end{aligned}$$

Applying part (3) of Proposition 8.1, we conclude that

$$\begin{aligned} P_U(t) &= \{u \in U \cap (t - Z_0) : t - r\mathbf{1} \leq_{K_0} u \leq_{K_0} t + r\mathbf{1}\} \\ &= U \cap (t - Z_0) \cap B(t, r). \end{aligned}$$

Since by definition we have $P_U(t) = U \cap B(t,r)$, it follows that $P_U(t) \subset t - Z_0$, and hence the pair (U,t) is strongly proximinal. □

Let (U,t) be a proximinal pair. We are now interested in describing conditions which guarantee that $\tilde{v} := t - \tilde{u}$, where $\tilde{u}$ is a best approximation of t by U, belongs to Z_0. First, we give the following definition.

Definition 8.2. We say that the set $U \subset X$ enjoys the property $(*)$, if for each $u \in U$ there exists an element $q \in \text{int} K_0$ such that $u + \delta q \in U$ for all δ with the small enough $|\delta|$.

Proposition 8.3. *Assume that (U,t) is a proximinal pair such that the set U enjoys the property $(*)$. Let $\tilde{u} \in P_U(t)$. Then $\tilde{v} := t - \tilde{u} \in Z_0$.*

Proof. Let $\tilde{v} \notin Z_0$. Then $\tilde{v} \notin (Z_+ \cap Z_-)$. Assume for the sake of definiteness that $\tilde{v} \in Z_+$, that is, $\|\tilde{v}\| = p(\tilde{v}) > p(-\tilde{v})$. Since the property (*) holds and $\tilde{u} \in U$, it follows that there exists $q \in \text{int}\, K_0$ such that $\tilde{u} + \delta q \in U$ for all small enough $\delta > 0$. Then:

$$p(\tilde{v}) > p(\tilde{v} - \delta q) \geq p(-\tilde{v} + \delta q) = p(-(\tilde{v} - \delta q)).$$

Hence, $\|\tilde{v} - \delta q\| = p(\tilde{v} - \delta q) < p(\tilde{v}) = \|\tilde{v}\|$. Let $\bar{u} = \tilde{u} + \delta q$. Due to property $(*)$, we conclude that $\bar{u} \in U$ for all small enough $\delta > 0$. Since $\tilde{v} - \delta q = t - \tilde{u} - \delta q = t - \bar{u}$, we have

$$\min_{u \in U} \|t - u\| \leq \|t - \bar{u}\| = \|\tilde{v} - \delta q\| < \|\tilde{v}\| = \|t - \tilde{u}\|.$$

This is a contradiction because $\tilde{u} \in P_U(t)$. □

Example 8.1. Let $U' \subseteq X$ be a locally compact set and $q \in \text{int} K_0$. Consider

$$U = U' + \{\lambda q : \lambda \in \mathbb{R}\} = \{u' + \lambda q : u' \in U,\ \lambda \in \mathbb{R}\}.$$

Clearly U is a locally compact set and U enjoys the property $(*)$. Then for each $t \in X$, the pair (U,t) is strongly proximinal.

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