International Journal of Pure and Applied Mathematics
Volume 21 No. 2 2005, 255-258

SPLIT VECTOR BUNDLES ON A SMOOTH CURVE: DEGENERATIONS AND SHORT EXACT SEQUENCES

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Abstract: Let X be a smooth curve of genus $g > 0$. Here we study the degeneration of flat families of vector bundles on X, all of them being isomorphic to direct sums of line bundles.

AMS Subject Classification: 14H60
Key Words: split vector bundles, vector bundles on curves

1. Split Vector Bundles

Let X be a smooth and connected projective curve of genus g. A vector bundle E on X is said to be *split* if it is isomorphic to a direct sum of line bundles. By the Krull-Schmidt Unique Factorization Theorem ([1], Theorem 3) the indecomposable factors of E are uniquely determined, up to isomorphisms and up to a permutation. A *marking* of a split bundle is a choice of their indecomposable factor, up to a permutation, but not up to isomorphisms. Every split bundle has a marking. It has a unique marking if its indecomposable factors are pairwise non-isomorphic. Our starting point was the following question.

Question 1. Let E, F be split vector bundles on X with the same rank. When there is a flat family $\{F_\lambda\}_{\lambda \in \Delta}$ of split vector bundles on X parametrized by an integral curve Δ and $o \in \Delta$ such that $F_o \cong E$ and for every $\lambda \in \Delta \backslash \{o\}$

Received: April 22, 2005

there is a marking of F_λ and of F such that the degree of the factors of F_λ and of F are the same? When there is a flat family as above, except we only require that (up to a permutation) the degrees of the irreducible factors of E and of E_o are the same, i.e. E and E_o have the same Harder-Narasimhan polygon?

When $g = 0$ everything is well-known: if and only if the Harder-Narasimhan filtration of F is higher (or equal) to the one of F and in this case we may take $F_\lambda \cong F$ for all $\lambda \in \Delta\backslash\{o\}$ ([2] or [5], Proposition 2). If $g > 0$ this condition is a necessary condition by the upper semicontinuity of the Harder-Narasimhan polygon ([4], Theorem 3).

Let E be a vector bundle on X. Let $\mu(E) := \deg(E)/\mathrm{rank}(E)$ denote the slope of E. Let $\mu_+(E)$ (resp. $\mu_-(E)$) denote the slope of the first (resp. last) subquotient of the Harder-Narasimhan filtration of E. Thus E is semistable if and only if $\mu_+(E) = \mu_-(E)$. If E is split, then $\mu_+(E)$ (resp. $\mu_-(E)$) is the maximal (resp. minimal) degree of one of the irreducible factors of E. Assume that E is split; we will say that E is *numerically rigid* if $\mu_+(E) \leq \mu_-(E) + 1$. Here it is our first result.

Theorem 1. *Let X be a smooth curve of genus $g \geq 1$ and E any split vector bundle on X. Then there is a flat family $\{F_\lambda\}_{\lambda\in\Delta}$ of split vector bundles on X parametrized by an integral curve Δ and $o \in \Delta$ such that $F_o \cong E$, $\mu_+(F_\lambda)-\mu_-(F_\lambda) \leq 2g-1$ for every $\lambda \in \Delta\backslash\{o\}$ and $F_\lambda \cong F_\mu$ for all $\lambda, \mu \in \Delta\backslash\{o\}$.*

Remark 1. Notice that when $g = 1$ the vector bundle F_λ, $\lambda \neq o$, appearing in the statement of Theorem 1 is numerically rigid. Hence we get that on an elliptic curve every split vector bundle is the flat limit of a family of isomorphic and numerically rigid split vector bundles.

Let E, F be split vector bundles on X with the same rank. We will say that E and F are *numerically equivalent* if there are markings of E and F, say $E \cong \oplus_{i=1}^r M_i$ and $F \cong \oplus_{i=1}^r L_i$ such that $\deg(M_i) = \deg(L_i)$ for all i.

Then we consider the following question.

Question 2. For which split bundles S, E, Q on X there is an exact sequence

$$0 \to S \to E \to Q \to 0 \tag{1}$$

on X? For which numerical type there are split vector bundles S, E, Q fitting in an exact sequence (1)?

If $X \cong \mathbf{P}^1$, then there is a complete answer to Question 2, but we cannot expect a complete answer for higher genus curves. As a byproduct of the proof of Theorem 1 we will get the following result.

Theorem 2. *Let X be a smooth curve of genus $g \geq 1$ and S, Q split vector bundles on X such that $\mu_+(S) \leq \mu_-(Q) - 4g$. Fix any split vector bundle $\tilde{E}$ on X such that* $\mathrm{rank}(\tilde{E}) = \mathrm{rank}(S) + \mathrm{rank}(Q)$, $\deg(\tilde{E}) = \deg(S) + \deg(Q)$, $\mu_-(\tilde{E}) \geq \mu_+(S) + 2g$ *and* $\mu_+(\tilde{E}) \leq \mu_-(Q) - 2g$. *Then there exists an exact sequence (1) on X in which E is a split vector bundle numerically equivalent to $\tilde{E}$.*

We work over an algebraically closed field $\mathbb{K}$.

To prove all the stated results we prove the following statement.

Proposition 1. *Let X be a smooth curve of genus $g \geq 1$. Fix $L, M \in \mathrm{Pic}(X)$ such that $\deg(L) - \deg(M) \geq 2g$. Then for a general $P \in X$ the vector bundle $M \oplus L$ is the flat limit of vector bundles on X isomorphic to $M(P) \oplus L(-P)$.*

Proof. First assume $t := \deg(L) - \deg(M) \geq 2g+1$. Let $j : L(-P) \to L$ be the obvious inclusion with $L|\{P\} \cong \mathbb{K}$. Since $\deg(L \otimes (M(P)^*)) = t - 1 \geq 2g$, the line bundle $L \otimes (M(P)^*)$ is spanned and hence there is an inclusion $i : M(P) \to L$ whose fiber at P is surjective. Hence the map $u : M(P) \oplus L(-P)$ is surjective. Since $\mathrm{Ker}(u) \cong M$, we have an exact sequence

$$0 \to M \to M(P) \oplus L(-P) \to L \to 0\,. \tag{2}$$

Call $\wp$ the extension (2). For every $\lambda \in \mathbb{K}\backslash\{0\}$ the extension $\lambda\wp$ has middle term isomorphic to $M(P) \oplus L(-P)$. Obviously, $M \oplus L$ is the middle term of the zero-extension. Hence we get the required flat family with $\mathbb{K}$ as parameter space and with 0 as special point. Now assume $t = 2g$. Set $R := L \otimes M^*$. To apply the previous proof need to check that $R(-P)$ is spanned at P for general $P \in X$. Since $h^1(X, R) = 0$ for degree reasons, it is sufficient to show that $h^1(X, R(-2P)) = 0$ for a general $P \in X$ (use Riemann-Roch). Hence by Serre duality it is sufficient to remark that $R \ncong \omega_X(2P)$ for a general $P \in X$. □

Proof of Theorem 1. Apply several times Proposition 1. □

Proof of Theorem 2. Set $\mathbb{E} := S \oplus Q$. Apply several times the proof of Proposition 1; in the set-up of Theorem 1 the vector bundle $\mathbb{E}$ would be called E, while we take F_λ, $\lambda \neq 0$, as the middle term of the exact sequence (1). □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] M. Atiyah, On the Krull-Schmidt Theorem with application to sheaves, *Bull. Soc. Math. France*, **84** (1956), 307–317; Reprinted in: *Michael Atiyah, Collected Works*, Volume **1**, Clarendon Press, Oxford (1988), 81-93.

[2] M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices III, *Ann. of Math.*, **70** (1959), 167-205.

[3] E. Schlesinger, Extensions of vector bundles on $\mathbb{P}^1$, *Comm. Algebra*, **28**, No. 12 (2000), 5883-5889.

[4] S.S. Shatz, The decomposition and specialization of algebraic families of vector bundles, *Compositio Math.*, **35**, No. 2 (1977), 163-187.

[5] S.S. Shatz, On subbundles of vector bundles over $\mathbf{P}^1$, *J. Pure Appl. Alg.*, **10**, No. 3 (1977), 315-322.