International Journal of Pure and Applied Mathematics

Volume 21 No. 2 2005, 259-265

SHORT EXACT SEQUENCES OF POLYSTABLE AND SPLIT VECTOR BUNDLES ON ELLIPTIC AND BIELLIPTIC CURVES

E. Ballico

Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let X be either an elliptic curve or a bielliptic curve. Here we prove the existence (for suitable numerical invariants) of exact sequences

$$0 \to S \to E \to Q \to 0$$

of vector bundles on X such that S is a direct sum of line bundles, while E and Q are polystable. If X is elliptic, we also consider the case $E \cong \mathcal{O}_X^{\oplus e}$ (i.e. the pull-back of the universal exact sequence on a suitable Grassmannian).

AMS Subject Classification: 14H60
Key Words: vector bundles on curves, stable vector bundles, bielliptic curve, elliptic curve, polystable vector bundle

1. Introduction

Let X be a smooth curve of genus g. Fix integer a, x, b, y with $x > 0$, $y > 0$ and $a/x < b/y$. If $g \geq 2$ there exists an exact sequence of vector bundles on X:

$$0 \to S \to E \to Q \to 0 \quad (1)$$

such that $\deg(S) = a$, $\text{rank}(S) = x$, $\deg(Q) = b$ and $\text{rank}(Q) = y$. Similarly, if

Received: April 22, 2005

$g = 1$, there is such an exact sequence in which all the 3 vector bundles S, E and Q are polystable and no two of their irreducible factors are isomorphic ([3], Theorem 1 and Remark 2.6). Here we consider a similar problem in which S is isomorphic to a direct sum of line bundles when either $g = 1$ or $g \geq 2$ and X is bielliptic. If X is elliptic, we also consider the case $E \cong \mathcal{O}_X^{\oplus e}$ (i.e. the pull-back of the universal exact sequence on a suitable Grassmannian) (see Proposition 4).

We work over an algebraically closed field $\mathbb{K}$. We will first prove the following result.

Theorem 1. *Fix integers $e > s > 0$, $v > 0$, d, b, $s_j > 0$, $1 \leq j \leq v$, c_j, $1 \leq j \leq v$, such that $e = s + \sum_{j=1}^{v} s_j$, $\lceil d/s \rceil \leq 1 + b/e$ and $b/e + 1 \leq \min\{c_1/s_1, \ldots, c_v/s_v\}$. Let X be a smooth elliptic curve. Fix $M_t \in \mathrm{Pic}(X)$, $1 \leq t \leq s$, such that $\sum_{t=1}^{s} \deg(M_t) = d$ and either $\deg(M_t) = \lfloor d/s \rfloor$ or $\deg(M_t) = \lceil d/s \rceil$. Set $S := \oplus_{t=1}^{s} M_t$: Assume $M_t \ncong M_l$ for all $t \neq l$. Then there exists an exact sequence of vector bundles on X:*

$$0 \to S \to E \to Q \to 0\,, \tag{2}$$

in which $Q = \oplus_{j=1}^{v} Q_j$, $\deg(E) = b$, $\mathrm{rank}(E) = r$, $\deg(Q_j) = c_j$, $\mathrm{rank}(Q_j) = s_j$ for all j, E is polystable, each Q_j is polystable and no two of the indecomposable factors of $E \oplus Q$ are isomorphic. If $\lceil d/s \rceil < 1 + b/e$, this is true even if some of the line bundles $M_1, \ldots, M_s$ are isomorphic. If $\lceil d/s \rceil < 1 + b/e$ and $b/e + 1 < \min\{c_1/s_1, \ldots, c_v/s_v\}$, then we may also take as Q_j an arbitrary polystable vector bundle with $\deg(Q_j) = c_j$, $\mathrm{rank}(Q_j) = s_j$ (simultaneously for all vector bundles $M_1, \ldots, M_s, Q_1, \ldots, Q_v$).

Notice that in the statement of Theorem 1 we allow the case $s_j = 1$ for all j. In this case both S and Q are a direct sum of line bundles.

From Theorem 1 we easily obtain the following result.

Proposition 1. *Assume $\mathrm{char}(\mathbb{K}) \neq 2$. Fix integers $e > s > 0$ $v > 0$, d, b, $s_j > 0$, $1 \leq j \leq v$, c_j, $1 \leq j \leq v$, such that $e = s + \sum_{j=1}^{v} s_j$, $\lceil d/s \rceil \leq 2 + b/e$ and $b/e + 2 \leq \min\{c_1/s_1, \ldots, c_v/s_v\}$. Assume that all integers d, b and c_j, $1 \leq j \leq v$, are even. Let X be a smooth bielliptic curve. Then there exists an exact sequence of vector bundles on X:*

$$0 \to S \to E \to Q \to 0\,, \tag{3}$$

in which $S \cong_{i=1}^{s} M_i$ for some $M_i \in \mathrm{Pic}(X)$, all $\deg(M_i)$ are even,

$$\max\{\deg(M_1), \ldots, \deg(M_s)\} \leq \min\{\deg(M_1), \ldots, \deg(M_s)\} + 2,$$

$Q = \oplus_{j=1}^{v} Q_j$, $\deg(E) = b$, $\operatorname{rank}(E) = r$, $\deg(Q_j) = c_j$, $\operatorname{rank}(Q_j) = s_j$ for all j, E is polystable, each Q_j is polystable and no two of the indecomposable factors of $E \oplus Q$ are isomorphic. If $\lceil d/s \rceil < 1 + b/e$, this is true even if some of the line bundles $M_1, \dots, M_s$ are isomorphic.

Then we will work more and obtain the following result.

Theorem 2. *Assume* $\operatorname{char}(\mathbb{K}) \neq 2$. *Fix integers $e > s > 0$ a, b such that d is even, $\lceil d/s \rceil \leq 4 + b/e$ and $b/e + 4 \leq (b - d)/(e - s)$. Let X be a smooth bielliptic curve. Then there exists an exact sequence of vector bundles on X:*

$$0 \to S \to E \to Q \to 0\,, \tag{4}$$

in which $S \cong_{i=1}^{s} M_i$ for some $M_i \in \operatorname{Pic}(X)$, $\max\{\deg(M_1), \dots, \deg(M_s)\} \leq \min\{\deg(M_1), \dots, \deg(M_s)\} + 2$, each $\deg(M_i)$ is even,

$$\operatorname{rank}(E) = e, \quad \deg(E) = b,$$

and E and Q are stable.

2. The Proofs and Related Results

Proof of Theorem 1. Fix any polystable vector bundle $\tilde{Q}_j$, $1 \leq j \leq v$, such that $\deg(Q_j) = c_j$ and $\operatorname{rank}(Q_j) = s_j$. Set $\tilde{Q} := \oplus_{j=1}^{v} \tilde{Q}_j$. Take the general extension $\tilde{E}$ of $\tilde{Q}$ by S. To prove the theorem it is sufficient to show that $\tilde{E}$ is polystable and its indecomposable factors are pairwise non-isomorphic. Fix any polystable vector bundle $\bar{E}$ such that $\deg(\bar{E}) = b$ and $\operatorname{rank}(\bar{E}) = r$. By [2], Proposition 1, there is an inclusion $M \to \bar{E}$ with locally free quotient and a surjection $\bar{E} \to \tilde{Q}$. Hence we may copy the proof of [3], Theorem 1 (see [3], last 11 lines of page 541). □

Proposition 2. *Fix integers $k \geq 2$, $e > s > 0$, $a_1 \geq \cdots \geq a_s$, $b_1 \geq \cdots \geq b_e$, an elliptic curve X and $M \in \operatorname{Pic}^k(X)$. Set $S := \oplus_{i=1}^{s} M^{\otimes a_i}$ and $E := \oplus_{j=1}^{e} M^{\otimes b_j}$. S is isomorphic to a subbundle of E if and only if for the first integer x such that $a_x \leq b_x$ we have $a_i \leq b_{i+1}$ for all $x \leq i \leq s$.*

Proof. Since $k \geq 2$, M is spanned by its global sections and $h^0(X, M) = k \geq 2$. Hence a general linear subspace $V \subseteq H^0(X, M)$ such that $\dim(V) = 2$ induces a degree k morphism $f : X \to \mathbf{P}^1$ such that $f^*(\mathcal{O}_{\mathbf{P}^1}(1)) \cong M$. Hence the "if" part follows from [6], Proposition 3. The "only if" part may be proved copying verbatim the proof of [6], Proposition 3. □

Proposition 3. *Fix integers $k \geq 2$, $e > s > 0$, $a_1 \geq \cdots \geq a_s$, $b_1 \geq \cdots \geq b_e$, $c_1 \geq \cdots \geq c_{e-s}$, a a smooth and connected projective curve X and $M \in \mathrm{Pic}^k(X)$. Assume that M is spanned by its global sections. Set $S := \oplus_{i=1}^{s} M^{\otimes a_i}$, $E := \oplus_{j=1}^{e} M^{\otimes b_j}$ and $Q := \oplus_{h=1}^{e-s} M^{\otimes c_h}$. There is an exact sequence*

$$0 \to S \to E \to Q \to 0\,, \tag{5}$$

if there is a partition of $\{1, \ldots, e\}$ into two disjoint subsets $H = \{h_1 < \cdots < h_{e-s}$ and $K := \{k_1 < \cdots < k_s\}$ such that, if $(x_1, \ldots, x_e) \in \mathbb{Z}^e$ is defined by $x_{h_i} = c_i$ and $x_{k_i} = a_i$, the following conditions holds:

(i) $x_i \geq b_i$ *for all* $i \in H$;

(ii) $x_i \leq b_i$ *for all* $i \in K$;

(iii) $\sum_{i=1}^{e} x_i = \sum_{i=1}^{e} b_i$;

(iv) $\sum_{i=1}^{j} x_i \geq \sum_{i=1}^{e} b_i$ *for all* $j \in \{1, \ldots, e-1\}$.

Proof. Since M is spanned by its global sections and $k > 0$, we have $h^0(X, M) = k \geq 2$. Hence a general linear subspace $V \subseteq H^0(X, M)$ such that $\dim(V) = 2$ induces a degree k morphism $f : X \to \mathbf{P}^1$ such that $f^*(\mathcal{O}_{\mathbf{P}^1}(1)) \cong M$. Apply [5], Theorem 2.1. □

Proposition 4. *Let C be an elliptic curve and E a rank r indecomposable (resp. semistable, resp. polystable) vector bundle on C such that $d := \deg(E) > r$. Hence $h^0(C, E) = d$, $H^1(C, E) = 0$, E is spanned and the vector bundle $\mathrm{Ker}(e_E)$ is indecomposable (resp. semistable, resp. polystable).*

Proof. By [1], Lemma 15, only the last assertion must be proved. Set $F := \mathrm{Ker}(e_E)$. Hence $\mathrm{rank}(F) = d - r$, $\deg(F) = -d$, $h^0(C, F) = 0$ and hence $h^1(C, F^*) = 0$, F^* is spanned and $E^* \cong \mathrm{Ker}(e_{F^*}$. If $E \cong A \oplus B$, then $F \cong e_A \oplus e_B$, $\mu(A) = \mu(B)$. If $F \cong M \oplus N$, then $E^* \cong e_{M^*} \oplus e_{N^*}$. This gives the first assertion. By Atiyah's classification of semistable vector bundles the first assertion and the numerical characterers of F imply the second and third assertions, too. □

To prove Proposition 1 and Theorem 2 we will use the following well-known lemma.

Lemma 1. *Assume $\mathrm{char}(\mathbb{K}) \neq 2$. Let X, Y be smooth and connected projective curves, $f : X \to Y$ a double covering and E a stable (resp. semistable) vector bundle. Assume that f is not ètale, i.e. assume $p_a(X) \geq 2p_a(Y) - 1$. Then $f^*(E)$ is stable (resp. semistable).*

Proof. We will check only the stability part, since the proof of the semistable case is similar and the semistability assertion is true in characteristic zero for arbitrary finite coverings. Assume E stable and that $f^*(E)$ is semistable, but not stable. Let $\sigma : X \to X$ be the order two automorphism associated to f. Take a proper sybsheaf A of $f^*(E)$ such that $\mu(A) \geq \mu(f^*(E)) = 2\mu(E)$ and with maximal slope. Since $f^*(E)$ is semistable, we have $\mu(A) = \mu(f^*(E))$. Since $\mu(A)$ is maximal, A is saturated in $f^*(E)$, i.e. $f^*(E)/A$ is locally free. Since $f^*(E)$ is σ-invariant, σ acts on $f^*(E)$. Taking rank(A) minimal among all subbundles of $f^*(E)$ with maximal slope we may allso assume that A is stable.

(a) Here we assume $A = \sigma^*(A)$ as subsheaves of $f^*(E)$. Fix a ramification point Q of f. Since $f^*(E)$ comes from Y, σ acts as the identity over its fiber $f^*(E)|\{Q\}$. Since A is saturated in $f^*(F)$ at Q and $A = \sigma^*(A)$, σ acts as the identity on the fiber $A|\{Q\}$. Since this is true for all ramification points of f, descent theory gives the existence of a subbundle D of E such that $A \cong f^*(D)$. We have $\deg(D) = \deg(A)/2$ ([3], Remark 3.1). Hence $\mu(D) = \mu(A)/2 = \mu(E)$, contradicting the stability of E.

(b) Here we assume $A \neq \sigma^*(A)$. Hence the image $A + \sigma^*(A)$ of the map $u : A \oplus \sigma^*(A) \to f^*(E)$ has rank at least $1+\text{rank}(A)$. Let B be the saturation of $A+\sigma^*(A)$ in $f^*(E)$. Notice that $U+\sigma^*(U) \subseteq H^0(X, A+\sigma^*(A)) \cap f^*(W))$. Since $A \oplus \sigma^*(A)$ is semistable and $A+\sigma^*(A)$ is an image of it, we have $\mu(A+\sigma^*(A)) \geq \mu(A)$. By the maximality of $\mu(A)$, we have $B = A + \sigma^*(A)$. Since $A + \sigma^*(A)$ is saturated and σ-invariant, the proof of part (a) gives a contradiction, unless $A + \sigma^*(A) = f^*(E)$. Since $A \oplus \sigma^*(A)$ is semistable, we also obtained $\mu(A) = \mu(f^*(E)) = \mu(E)$. First assume $A \cap \sigma^*(A) = \{0\}$. Hence $f^*(E) = A \oplus \sigma^*(A)$. Notice that the projection $f^*(E) = A \oplus \sigma^*(A)$ is not σ-invariant and hence it does not come from an endomorphism of f, contradicting [3], Proposition 3.5; here we use that f has some ramification point. Now assume $D := A \cap \sigma^*(A) \neq \{0\}$. Since $\mu(A) = \mu(f^*(E))$ and $A + \sigma^*(A) = f^*(E)$, we get $\mu(D) = \mu(A)$, contradicting the minimality of rank(A) and the assumption $A \neq \sigma^*(A)$. □

Proof of Proposition 1. Just apply Theorem 1, Proposition 1 and the definition of polystability. □

Proof of Theorem 2. Let $f : X \to Y$, Y an elliptic curve, be the bieliptic double covering. Let $\sigma : X \to X$ be the involution associated to f. Since d is even, b is even if and only if $b - d$ is even.

(i) Here we assume b odd. In this case we will prove Theorem 2 under the weaker assumptions "$\lceil d/s \rceil \leq 2 + b/e$ and $b/e + 2 \leq (b-d)/(e-s)$". By the case $v = 1$ of Proposition 1 there is an exact sequence of vector bundles on Y:

$$0 \to S_1 \to E_1 \to Q_1 \to 0 \tag{6}$$

such that $S_1 \cong_{i=1}^s N_i$, $N_i \in \mathrm{Pic}(Y)$, $\lfloor d/2s \rfloor \leq \deg(N_i)\lfloor d/2s \rfloor + 1$,

$$\sum_{i=1}^{s} \deg(N_i) = d/2,$$

E_1 and Q_1 polystable, $\mathrm{rank}(E_1) = e$ and $\deg(E_1) = (b-1)/2$. Set $S := f^*(S_1)$, $E' := f^*(E_1)$ and $Q' := f^*(Q_1)$. Hence there is an exact sequence on Y:

$$0 \to S \xrightarrow{\psi} E' \to Q' \to 0\,. \tag{7}$$

By Lemma 1 the vector bundles E' and Q' are polystable. Let E be the general vector bundle obtained from E' making a positive elementary transformation. By the generality of the elementary transformation, we may assume that one fixed proper subbundle of E' remains a subbundle (i.e. it is again saturated) in E. Hence we may assume that $\psi(S)$ is a subbundle of E. Thus we have an exact sequence

$$0 \to S \to E \to Q \to 0 \tag{8}$$

with Q obtained from Q' making a general positive elementary transformation. Hence to prove Theorem 2 when b is odd it is sufficient to prove that E and Q are stable. First assume that E is not stable and take a proper subsheaf A of E with maximal slope. Let P the support of the positive elementary transformation made to obtain E from E'. By the generality of the elementary transformation we may assume $P \neq \sigma(P)$. The vector bundle $\sigma^*(E)$ is obtained from E' making a positive elementary transformation supported by P. Let E'' be the vector bundle obtained from E' making both the positive elementary transformations. Hence $\deg(E'') = b + 1$, $\sigma^*(E'') = E''$ and σ acts on E''. Both E and $\sigma^*(E)$ are subsheaf of E'' and E' is their intersection inside E''. The subbundle $\sigma^*(A)$ destabilizes $\sigma^*(E)$ and it is a proper subbundle of $\sigma^*(E)$ with maximal slope. Since $E \cap \sigma^*(E) = E'$, we have $A \cap \sigma^*(A) \subset E'$. $E'' = f^*(E_2)$, where E_2 is obtained from E_1 making a general positive elementary transformation supported by $f(P)$. By [3], Corollary 2.4, E_2 is polystable. Hence E'' is polystable (Lemma 1). By [3], Lemma 3.2, E is semistable. Hence $\mu(A) = \mu(E)$ and A is semistable. Hence $\sigma^*(A)$ is semistable. Let B denote the saturation of $A + \sigma^*(A)$ in E''. Since both E and $A + \sigma^*(A)$ are σ-invariant, we have $\deg(B) - \deg(A + \sigma^*(A))$ even. Since A is semistable and $\mu(A) = \mu(E)$, we have $\mu(A+\sigma^*(A)) \geq \mu(E)$. Since $\mu(E'') = \mu(E)+1/e$ and E'' is polystable, we get $B = A + \sigma^*(A)$. Since $A + \sigma^*(A)$ is saturated in E'' and σ-invariant, the proof of Lemma 1 gives the existence of a subbundle D of E_2 such that $A + \sigma^*(A) = f^*(D)$. Taking as A a minimal subbundle of E with slope $\mu(E)$

we may assume A stable. Since $\mu(E') < \mu(E)$, $A \cap \sigma^*(A) \subseteq E'$, we have $A \cap \sigma^*(A) \neq A$, i.e. $A \neq A + \sigma^*(A)$. Since $\mu(A + \sigma^*(A)) = \mu(A)$ and A is stable, we get $A+\sigma^*(A) \cong A \oplus \sigma^*(A)$ and that $f^*(D) = A+\sigma^*(A)$ is polystable with two indecomposable factors, each of them exchanged by σ. This implies that D is stable. By Lemma 1 $A + \sigma^*(A)$ is stable, contradiction. Thus E is stable. The same proof shows that Q is stable.

(ii) Here we assume b even. By the first part (proved under the weaker assumption "$\lceil d/s \rceil \leq 2 + b/e$ and $b/e + 2 \leq (d-e)/(e-s)$") we may assume that the statement of the theorem is true for the integers $d, s, e, b' := b - 1$. $\square$

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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