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ON THE k-GENERALIZED LUCAS NUMBERS

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Abstract: In this paper, we study the determinants of the matrices obtained by k sequences of the k-generalized Lucas numbers. Furthermore, we consider the relationship between the k-generalized Fibonacci numbers and the k-generalized Lucas numbers.

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The well-known Fibonacci numbers $\{F_n\}$ is defined by

$$F_0 = F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad \text{for } n > 1.$$

The Lucas numbers $\{L_n\}$ is defined by

$$L_0 = 2, L_1 = 1, \quad L_n = L_{n-1} + L_{n-2}, \quad \text{for } n > 1.$$

The closed formulas are: $F_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$, $L_n = \alpha^n - \beta^n$, and $L_n = 2F_n - F_{n-1}$, where $\alpha = \frac{1+\sqrt{5}}{2}$, and $\beta = \frac{1-\sqrt{5}}{2}$.

E. Karaduman [3] studied k sequences of the generalized order-k Fibonacci numbers as shown: for $1 \leq i \leq k$,

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$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i, \quad 1 \le i \le k, \tag{1}$$

with initial conditions: for $-(\mathrm{k}-1) \le n \le 0$,

$$g_n^i = \begin{cases} 1, & \text{if } n = -(i-1), \\ 0, & \text{otherwise}, \end{cases}$$

where g_n^i is the n-th term of i-th sequence.

Defining $k \times k$ square matrix G_n as follows:

$$G_n = \begin{pmatrix} g_n^1 & g_n^2 & g_n^3 & \cdots & g_n^k \\ g_{n-1}^1 & g_{n-1}^2 & g_{n-1}^3 & \cdots & g_{n-1}^k \\ g_{n-2}^1 & g_{n-2}^2 & g_{n-2}^3 & \cdots & g_{n-2}^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{n-k+1}^1 & g_{n-k+1}^2 & g_{n-k+1}^3 & \cdots & g_{n-k+1}^k \end{pmatrix}, \tag{2}$$

then one has $G_n = A^n G_0$, and $\det G_n = \begin{cases} 1, & \text{if } k \text{ is odd}, \\ (-1)^n, & \text{if } k \text{ is even}, \end{cases}$

where A and G_0 are $k \times k$ matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad G_0 = \begin{pmatrix} 1 & 1 & 2 & 2^2 & \cdots & 2^{k-3} & 2^{k-2} \\ 0 & 1 & 1 & 2 & \cdots & 2^{k-4} & 2^{k-3} \\ 0 & 0 & 1 & 1 & \cdots & 2^{k-5} & 2^{k-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We define k sequences of the k-generalized Lucas numbers as shown: for $1 \le i \le k$,

$$l_n^i = \sum_{j=1}^{k} l_{n-j}^i, \tag{3}$$

with boundary conditions

$$l_n^i = \begin{cases} b, & \text{if } n = -(i-2), \\ a, & \text{if } n = -(i-1), \quad \text{for } -(k-1) \le n \le 0, \\ 0, & \text{otherwise}, \end{cases}$$

where l_n^i is the n-th term of i-th sequence, and a, b are arbitrary constants.

Define $k \times k$ square matrix L_n as follows,

$$L_n = \begin{pmatrix} l_n^1 & l_n^2 & l_n^3 & \cdots & l_n^k \\ l_{n-1}^1 & l_{n-1}^2 & l_{n-1}^3 & \cdots & l_{n-1}^k \\ l_{n-2}^1 & l_{n-2}^2 & l_{n-2}^3 & \cdots & l_{n-2}^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n-k+1}^1 & l_{n-k+1}^2 & l_{n-k+1}^3 & \cdots & l_{n-k+1}^k \end{pmatrix}, \tag{4}$$

then

$$L_0 = \begin{pmatrix} a & b & a+b & 2(a+b) & \cdots & 2^{k-4}(a+b) & 2^{k-3}(a+b) \\ 0 & a & b & (a+b) & \cdots & 2^{k-5}(a+b) & 2^{k-4}(a+b) \\ 0 & 0 & a & b & \cdots & 2^{k-6}(a+b) & 2^{k-5}(a+b) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix},$$

and we have the following result.

Theorem 1. *$L_{n+1} = AL_n$, $L_n = A^n L_0$, and*

$$\det L_n = \begin{cases} a^k, & \text{if } k \text{ is odd ,} \\ (-1)^n a^k, & \text{if } k \text{ is even.} \end{cases}$$

Proof. By the recurence relations (3), we have $L_{n+1} = AL_n$, and $L_n = A^n L_0$, hence

$$\det L_n = (\det A)^n \det L_0 = ((-1)^{k+1})^n \cdot a^k = \begin{cases} a^k, & \text{if } k \text{ is odd ,} \\ (-1)^n a^k, & \text{if } k \text{ is even.} \end{cases} \qquad \square$$

Theorem 2. *Let B denote $k \times k$ square matrix such that,*

$$B=\begin{pmatrix} a & b-a & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & b-a & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & b-a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & b-a \\ 0 & 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix},$$

then $L_n = G_n B$, *and hence* $l_n^1 = ag_n^1, l_n^i = ag_n^i + (b-a)g_{n-1}^i$, *for* $2 \leq i \leq k$.

Proof. Since $G_n = A^n G_0$, and $L_n = A^n L_0$, hence $L_n = A^n L_0 = A^n G_0 (G_0^{-1} L_0) = G_n (G_0^{-1} L_0)$. By a routine computing, we have

$$G_0^{-1}=\begin{pmatrix} 1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ 0 & 1 & -1 & -1 & \cdots & -1 & -1 \\ 0 & 0 & 1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

and $G_0^{-1} L_0 = B$, hence $L_n = G_n B$. Therefore, $l_n^1 = ag_n^1, l_n^i = ag_n^i + (b-a)g_n^{i-1}$, for $2 \leq i \leq k$. Since $g_n^{i-1} = g_{n-1}^i$ for all $2 \leq i \leq k$ and $n \in Z$, we have $l_n^i = ag_n^i + (b-a)g_{n-1}^i$, for $2 \leq i \leq k$. □

Chen [1] has given the explicit formula for all elements in the n-th power of the companion matrix A, using this formula we can obtain explicit expressions for the k-generalized Lucas numbers.

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