International Journal of Pure and Applied Mathematics
Volume 21 No. 3 2005, 335-350

ON CERTAIN RELATIVE COHOMOLOGICAL INVARIANTS

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Abstract: In this work we study general properties of the invariant end $E(G, \mathcal{S}, M)$, where G is a group, $\mathcal{S}$ is a family of subgroups with infinite index in G and M is a $\mathbb{Z}_2G$-module. When $M = \mathbb{Z}_2G \otimes_{\mathbb{Z}_2S} \overline{\mathbb{Z}_2S}$ and $\mathcal{S} = \{S\}$, we denote $E(G, \{S\}, \mathbb{Z}_2G \otimes_{\mathbb{Z}_2S} \overline{\mathbb{Z}_2S})$ by $\tilde{E}(G, S)$. Using the theory of cohomology of groups, we study $\tilde{E}(G, S)$ and its relations with the ends $e(G)$ (defined by Hopf and Specker), $e(G, S)$ (due to Houghton and Scott) and $\tilde{e}(G, S)$ (defined by Kropholler and Roller). We also obtain some results about duality of groups.

AMS Subject Classification: 20J06, 55M05, 20E06
Key Words: cohomology of groups, ends of pairs of groups, duality, Eilenberg-Maclane spaces

Received: May 2, 2005

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1. Introduction

Houghton, [9], and Scott, [15], gave a definition of the number of ends $e(G,S)$, of a pair of groups (G,S), where S is a subgroup of G. The number $e(G,S)$ is an algebraic invariant for pairs of groups which generalizes the classical end of groups $e(G)$ defined by Hopf, [8], and Specker, [17]. Lately Kropholler and Roller [12] defined another invariant end for pairs of groups (G,S), denoted by $\tilde{e}(G,S)$, which is closely related to the end $e(G,S)$. Andrade and Fanti, [1], defined a generalized invariant end $E(G,\mathcal{S},M)$, where G is a group, $\mathcal{S}$ is a family of subgroups with infinite index in G and M is a $\mathbb{Z}_2G$-module, and studied the invariant $E(G,\mathcal{S},M)$ in the case $M=\mathbb{Z}_2(G/S)$ and $\mathcal{S}=\{S\}$, the family with only one subgroup, denoting $E(G,\{S\},\mathbb{Z}_2(G/S))$ by $E(G,S)$. They studied the relations of $E(G,S)$ with the invariants $e(G,S)$ and $\tilde{e}(G,S)$ and obtained some results in groups satisfying certain finiteness conditions. Considering $M=\mathbb{Z}_2$ and denoting $E(G,\mathcal{S},M)$ by $E'(G,\mathcal{S})$, Andrade et al, [3], studied the invariant $E'(G,\mathcal{S})$, obtaining some results in splitting of groups and duality pairs.

In this paper, we study general properties of the invariant $E(G,\mathcal{S},M)$, which were not studied by Andrade and Fanti, [1], we calculate this invariant when the group pair $(G,\mathcal{S})$ satisfies certain properties of duality and, in particular, we study this invariant when $\mathcal{S}=\{S\}$ and $M=\mathbb{Z}_2G\otimes_{\mathbb{Z}_2S}\overline{\mathbb{Z}_2S}$ (where $\overline{\mathbb{Z}_2S}=\mathrm{Coind}^G_{\{1\}}\mathbb{Z}_2=\mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2S,\mathbb{Z}_2)$. Denoting $E(G,\{S\},\mathbb{Z}_2G\otimes_{\mathbb{Z}_2S}\overline{\mathbb{Z}_2S})$ by $\tilde{E}(G,S)$, we show that the invariant $\tilde{E}(G,S)$ is closely related with the invariant $\tilde{e}(G,S)$ and we obtain particular results when the group G and the subgroup S satisfy certain finiteness conditions.

We assume that the reader is familiar with the theory of relative cohomology of groups (Bieri and Eckmann, [5], Andrade et al, [2]) and duality (for groups and group pairs – Brown, [6], Bieri and Eckmann, [5]).

We should have in mind the following result.

Result 1.1. (see Bieri and Eckmann, [5], Proposition 1.1) *Let $(G,\mathcal{S})$ be a group pair, where $\mathcal{S}=\{S_i,\ i\in I\}$, and let M be a $\mathbb{Z}_2G$-module M. Denote $\prod_{i\in I} H^k(S_i;M)$ by $H^k(\mathcal{S};M)$. Then we have the following long exact sequence*

$$0\to H^0(G;M)\to H^0(\mathcal{S};M)\xrightarrow{\delta} H^1(G,\mathcal{S};M)\xrightarrow{J} H^1(G;M) \xrightarrow{\mathrm{res}^G_{\mathcal{S}}} H^1(\mathcal{S};M)\to\cdots,$$

which is natural in the module M and in the group pair $(G,\mathcal{S})$.

Next we recall the definition of the invariant $E(G,\mathcal{S},M)$.

Result 1.2. *Let* $(G, \mathcal{S})$ *be a group pair with* $\mathcal{S} = \{S_i, i \in I\}$ *a family of subgroups of* G *(not necessarily distinct) and* M *a* $\mathbb{Z}_2G$*-module. We assume that* $[G : S_i] = \infty$ *for all* $i \in I$. *Then* $E(G, \mathcal{S}, M) := 1 + \dim \ker \mathrm{res}^G_{\mathcal{S}}$, *where* $\mathrm{res}^G_{\mathcal{S}} : H^1(G; M) \to \prod_{i \in I} H^1(S_i; M)$ *is the map of the exact sequence (1.1).*

As we have stated in [1], $E(G, \mathcal{S}, M)$ is an algebraic invariant of the category $\mathcal{C}$ whose objects are pairs $((G, \mathcal{S}), M)$, where $(G, \mathcal{S})$ is a group pair with $[G : S] = \infty$, for all $S \in \mathcal{S}$, and M is a $\mathbb{Z}_2G$-module and whose morphisms are maps $\psi : ((G, \mathcal{S} = \{S_i \mid i \in I\}), M) \to ((L, \mathcal{R} = \{R_j \mid j \in J\}, N)$ consisting of a homomorphism $\alpha : G \to L$, a map $\pi : I \to J$ such that $\alpha(S_i) \subset R_{\pi(i)}$ and a homomorphism $\phi : N \to M$ such that $\phi(\alpha(g).n) = g.\phi(n)$.

We begin with certain general properties of the invariant $E(G, \mathcal{S}, M)$.

2. Properties of the Invariant $E(G, \mathcal{S}, M)$

Consider the map $\mathrm{res}^G_{\mathcal{S}} : H^1(G; M) \to \prod_{i \in I} H^1(S_i; M)$ of the exact sequence (1.1) and the map $(\mathrm{res}_i)_{i \in I} : H^1(G; M) \to \prod_{i \in I} H^1(S_i; M)$ defined by $(\mathrm{res}_i)_{i \in I}\ ([f]) = (\mathrm{res}_i[f])_{i \in I}$, where $\mathrm{res}_i = \mathrm{res}^G_{S_i} : H^1(G; M) \to H^1(S_i; M)$ is the restriction map, for all $i \in I$. It is easy to see that $\mathrm{res}^G_{\mathcal{S}} = (\mathrm{res}_i)_{i \in I}$ and $\ker \mathrm{res}^G_{\mathcal{S}} = \bigcap_{i \in I} \ker \mathrm{res}_i$. Hence, we have the following result.

Proposition 1. *If* $(G, \mathcal{S})$ *is a group pair and* M *is a* $\mathbb{Z}_2G$*-module then the maps* $\mathrm{res}^G_{\mathcal{S}}$ *and* $(\mathrm{res}_i)_{i \in I}$ *coincide. Consequently, if* $[G : S_i] = \infty$, *for all* $i \in I$, *then* $E(G, \mathcal{S}, M) = 1 + \dim \bigcap_{i \in I} \ker\ \mathrm{res}_i$.

Remark 1. By Lemma 1.4 in Andrade and Fanti, [1], we have $\ker\ \mathrm{res}_i \simeq \frac{\mathrm{Der}(G, S_i, M)}{P(G, S_i, M)}$. Thus the above result provides an interpretation of $E(G, \mathcal{S},\ M)$ in terms of derivations.

Proposition 2. *Let* $(G, \mathcal{S})$ *be a group pair with* $\mathcal{S} = \{S_i, i \in I\}$ *and* $[G : S_i] = \infty, \forall i \in I$.

(1) If $\mathcal{S}' = \{S_{i_k}, i_k \in I' \subset I\}$ *is a subfamily of* $\mathcal{S}$ *then* $E(G, \mathcal{S}, M) \leq E(G, \mathcal{S}', M)$ *for all* $\mathbb{Z}_2G$*-modules* M. *In particular* $E(G, \mathcal{S}, M) \leq E(G, S_i, M)$ *for all* $i \in I$.

(2) If $\mathcal{S}' = \{H_i \leq S_i, i \in I\}$ *then* $E(G, \mathcal{S}, M) \leq E(G, \mathcal{S}', M)$ *for all* $\mathbb{Z}_2G$*-module* M.

(3) If M and N are $\mathbb{Z}_2G$-modules and $\phi : N \to M$ is a $\mathbb{Z}_2G$-homomorphism that induces a monomorphism $\phi^* : H^1(G;N) \to H^1(G;M)$ then $E(G,\mathcal{S},N) \leq E(G,\mathcal{S},M)$.

(4) If $\mathcal{S} = \bigcup_{l\in L} \mathcal{S}_l = \{S_i, i \in I\}$, where $\mathcal{S}_l = \{S_{i_l} = R_l, i_l \in I_l\}$ and, $\forall i \in I$, $[G : S_i] = \infty$, then $E(G,\mathcal{S},M) = E(G,\mathcal{S}',M)$, where $\mathcal{S}' = \{R_l, l \in L\}$ and M is any $\mathbb{Z}_2G$-module. Thus, to calculate $E(G,\mathcal{S},M)$ it is enough to consider the distinct subgroups of the family $\mathcal{S}$.

(5) If $\mathcal{S}' = \{L_i, i \in I\}$, with $L_i = \bigcap_{j\in I} S_j$ for all $i \in I$, then $E(G,\mathcal{S},M) \leq E(G,\mathcal{S}',M) = E(G,\bigcap_{i\in I} S_i, M)$.

Proof. The items (1), (2), (4) and (5) follow directly from Proposition 1.

We will prove (3). We have that the $\mathbb{Z}_2G$-homomorphism $\phi : N \to M$ induces the commutative diagram, for all $i \in I$:

$$\begin{array}{ccc} H^1(G;N) & \xrightarrow{\phi^*} & H^1(G;M) \\ \downarrow \text{res}_i^N & & \downarrow \text{res}_i^M \\ H^1(S_i;N) & \xrightarrow{\phi_i^*} & H^1(S_i;M) \end{array}$$

Let $i \in I$ and $\alpha \in \ker \text{res}_i^N$. Then $\text{res}_i^M \circ \phi^*(\alpha) = \phi_i^* \circ \text{res}_i^N(\alpha) = \phi_i^*(0) = 0$. Hence $\phi^*(\ker \text{res}_i^N) \subset \ker \text{res}_i^M$. Thus we have

$$\bigcap_{i\in I} \ker \text{res}_i^N \simeq \phi^*(\bigcap_{i\in I} \ker \text{res}_i^N) = \bigcap_{i\in I} \phi^*(\ker \text{res}_i^N) \subset \bigcap_{i\in I} \ker \text{res}_i^M.$$

Therefore, by Proposition 1 we have the desired inequality. □

Example 1. Consider the group $G =< s > * < t >\simeq \mathbb{Z} * \mathbb{Z}$ and the family of subgroups $\mathcal{S} = \{S_1, S_2\}$, where $S_1 =< sts^{-1}t^{-1} >\simeq \mathbb{Z}$ and $S_2 = [G,G]$ is the commutator subgroup of G. By Proposition 2, we have $E(G,\mathcal{S},\mathbb{Z}_2(G/S_1)) \leq E(G,S_1,\mathbb{Z}_2(G/S_1))$ and $E(G,\mathcal{S},\mathbb{Z}_2(G/S_2)) \leq E(G,S_2,\mathbb{Z}_2(G/S_2))$. However $E(G,S_1,\mathbb{Z}_2(G/S_1)) = E(G,S_1)$ and $E(G,S_2,\mathbb{Z}_2(G/S_2)) = E(G,S_2)$. By Andrade and Fanti, [1]((2.11)(a) and Theorem 3.1), and Scott, [15], we have $E(G,S_1) = 1$ and $E(G,S_2) = e(G,S_2) = 1$. Hence $E(G,\mathcal{S},\mathbb{Z}_2(G/S_1) = E(G,\mathcal{S},\mathbb{Z}_2(G/S_2)) = 1$.

3. The Invariant $E(G, \mathcal{S}, M)$ and Duality

If $(G, \mathcal{S})$ is a D^n-pair then it is known that $\mathcal{S}$ is finite and, in this case, we will write $\mathcal{S} = \{S_i, i = 1, 2, \dots, r\}$.

Proposition 3. *Let $(G, \mathcal{S})$ be a D^n-pair with dualizing module C and $\mathcal{S} = \{S_i, i = 1, 2, \dots, r\}$. Then:*

(1) $H^1(G, \mathcal{S}; \bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^G M_i) = \bigoplus_{i=1}^{r} M_i^{S_i}$, where M_i is a $\mathbb{Z}_2 S_i$-module for all $i \in I$. In particular, if $M_i = \mathbb{Z}_2$ we have $H^1(G, \mathcal{S}; \bigoplus_{i=1}^{r} \mathbb{Z}_2(G/S_i)) = \bigoplus_{i=1}^{r} \mathbb{Z}_2$.

(2) $H^1(G, \mathcal{S}; \mathrm{Ind}_S^G M) = M^S$, where S is a D^{n-1}- subgroup of G with dualizing module $\mathrm{Res}_S^G C$ and M is a $\mathbb{Z}_2 S$-module.

Proof. (1) Let $(G, \mathcal{S})$ be a D^n-pair with dualizing module C. Then S_i is a D^{n-1}-group, for all i. Thus, using duality, Shapiro's Lemma and some properties of cohomology of groups, we have $H^1(G, \mathcal{S}; \bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^G M_i) \simeq \bigoplus_{i=1}^{r} H_{n-1}(G; \mathrm{Ind}_{S_i}^G (C \otimes M_i)) \simeq \bigoplus_{i=1}^{r} H_{n-1}(S_i; (C \otimes M_i)) \simeq \bigoplus_{i=1}^{r} H^0(S_i; M_i) = \bigoplus_{i=1}^{r} M_i^{S_i}$.

(2) It is similar to (1). □

Corollary 1. *Let $(G, \mathcal{S})$ be a group pair. If $(G, \mathcal{S})$ is realised topologically by an Eilenberg-MacLane pair (X, Y), where X is a compact n-manifold with boundary $Y = \partial X$, then the number of components of ∂X is* $\dim H^1(G, \mathcal{S}; \bigoplus_{S \in \mathcal{S}} \mathbb{Z}_2(G/S)) = \dim\, H^1(G, \mathcal{S}; \mathbb{Z}_2(G/\mathcal{S}))$.

Proof. We have that $(G, \mathcal{S})$ is a PD^n-pair with $\mathcal{S} = \{S_i, i = 1, \cdots, r\}$ and $Y = \bigcup_{i=1}^{r} K(S_i, 1)$ (Bieri and Eckmann, [5]). Hence

$$\dim\, H^1(G, \mathcal{S}; \bigoplus_{S \in \mathcal{S}} \mathbb{Z}_2(G/\mathcal{S})) = r\,,$$

which is the number of components of Y. □

Example 2. Let X be the torus T^2 from which r open discs has been removed and $Y = \partial X$. Then (X, Y) is an Eilenberg-MacLane pair realising the group pair $(G, \mathcal{S}) = (\pi_1(X), \{S_i = \pi_1(Y_i), i = 1, \dots, r\})$ (see Bieri and

Eckmann, [5]). Hence dim $H^1(G,\mathcal{S};\bigoplus_{i=1}^{r} Z_2(G/S_i)) = r$.

We will present now a result about D^n-pairs for the invariant $E(G,\mathcal{S},M)$ which generalizes Proposition 2.10 in Andrade and Fanti, [1].

Theorem 1. *Let $(G,\mathcal{S})$ be a D^n-pair, $\mathcal{S} = \{S_i, i = 1,\cdots,r\}$. If $[G:S] = \infty$ for all $S \in \mathcal{S}$ then:*

(1) $E(G,\mathcal{S},\bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^{G} M_i) = 1$, where M_i is a $\mathbb{Z}_2 S_i$-module for $i = 1,\ldots,r$ with dim *$M_i^{S_i} < \infty$. In particular, $E(G,\mathcal{S},\bigoplus_{i=1}^{r} \mathbb{Z}_2(G/S_i)) = 1$.*

(2) $S_i \neq S_j$ for $i \neq j$ and S_i is not normal in G, for $i = 1,\ldots,r$.

Proof. (1) Consider the exact sequence (1.1) for the pair $(G,\mathcal{S})$ and $M = \bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^{G} M_i$: $0 \to H^0(G;\bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^{G} M_i) \to \bigoplus_{j=1}^{r} H^0(S_j;\bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^{G} M_i) \xrightarrow{\delta} H^1(G,\mathcal{S};\bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^{G} M_i)$ $\cdots$. We have that

$$H^0(G;\bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^{G} M_i) = \bigoplus_{i=1}^{r} (\mathrm{Ind}_{S_i}^{G} M_i)^G = 0$$

and hence δ is injective. Now $H^1(G,\mathcal{S};\bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^{G} M_i) \simeq \bigoplus_{i=1}^{r} M_i^{S_i}$ and so, dim $H^1(G,\mathcal{S};\bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^{G} M_i) < \infty$. Otherwise

$$\bigoplus_{j=1}^{r} H^0(S_j;\bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^{G} M_i) = \bigoplus_{j=1}^{r} [H^0(S_j;\mathrm{Ind}_{S_j}^{G} M_j) \oplus \bigoplus_{i\neq j=1}^{r} H^0(S_j;\mathrm{Ind}_{S_i}^{G} M_i)]$$
$$= \bigoplus_{j=1}^{r} [(\mathrm{Ind}_{S_j}^{G} M_j)^{S_j} \oplus \bigoplus_{i\neq j=1}^{r} (\mathrm{Ind}_{S_i}^{G} M_i)^{S_j}].$$

Since $M_j^{S_j} \hookrightarrow (\mathrm{Ind}_{S_j}^{G} M_j)^{S_j}$ we have

$$\bigoplus_{j=1}^{r} M_j^{S_j} \hookrightarrow \bigoplus_{j=1}^{r} (\mathrm{Ind}_{S_j}^{G} M_j)^{S_j}$$
$$\hookrightarrow \bigoplus_{j=1}^{r} [(\mathrm{Ind}_{S_j}^{G} M_j)^{S_j} \oplus \bigoplus_{i\neq j=1}^{r} (\mathrm{Ind}_{S_i}^{G} M_i)^{S_j}] \overset{\delta}{\hookrightarrow} \bigoplus_{j=1}^{r} M_j^{S_j}.$$

Using the fact that $\dim M_j^{S_j}$ is finite, for $j = 1, \ldots, r$, we conclude that $(\mathrm{Ind}_{S_j}^G M_j)^{S_j} = M_j^{S_j}$ and $(\mathrm{Ind}_{S_i}^G M_i)^{S_j} = 0$, for all $i \neq j$. Thus

$$\dim \bigoplus_{j=1}^{r} H^0(S_j; \bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^G M_i) = \dim \bigoplus_{j=1}^{r} M_j^{S_j} < \infty.$$

Therefore, by Lemma 2.1 in Andrade et al, [3], we have

$$E(G, \mathcal{S}, \bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^G M_i) = 1.$$

(2) If $M_i = \mathbb{Z}_2$ for $i = 1, \ldots, r$ then $\mathbb{Z}_2(G/S_j)^{S_j} = (\mathrm{Ind}_{S_j}^G \mathbb{Z}_2)^{S_j} = \mathbb{Z}_2$ e $\mathbb{Z}_2(G/S_i)^{S_j} = (\mathrm{Ind}_{S_i}^G \mathbb{Z}_2)^{S_j} = 0$ for $i \neq j$. It follows that $S_i \neq S_j$ for $i \neq j$ and S_i is not normal in G, otherwise we would have $\mathbb{Z}_2(G/S_i)^{S_i} = \mathbb{Z}_2(G/S_i)$ which is an infinite sum of copies of $\mathbb{Z}_2$. □

Corollary 2. *Let $(G, \mathcal{S})$ be a group pair, $\mathcal{S} = \{S_i, i \in I\}$ with $[G : S_i] = \infty$, for all $i \in I$. If there exists $S \in \mathcal{S}$ such that (G, S) is a D^n-pair, then $E(G, \mathcal{S}, \mathrm{Ind}_S^G M) = 1$ for any $\mathbb{Z}_2 S$-module M with $\dim M^S < \infty$.*

Example 3. (i) Let F be a closed surface and let X be equal to F minus r open disks D_i, $i = 1, \ldots, r$ (with r finite, and $r \geq 2$ if $F = S^2$). If $Y = \partial X = \bigcup_{i=1}^{r} Y_i$, where $Y_i = \partial D_i$ and $\mathcal{S} = \{S_i = \pi_1(Y_i), i = 1, \ldots, r\}$ then, by Theorem 6.3 in Bieri and Eckmann, [5], we have that $(G, \mathcal{S})$ is a PD^2-pair. Hence, by Theorem 1(1), we obtain $E(G, \mathcal{S}, \bigoplus_{i=1}^{r} \mathrm{Ind}_{S_i}^G M_i) = 1$, where $M_i's$ are $\mathbb{Z}_2 S_i$- modules with $\dim M_i^{S_i} < \infty$, $i = 1, \ldots, r$. Moreover, by Theorem 1(2), the subgroup S_i is not normal in G and $S_i \neq S_j$, for all $i \neq j = 1, \ldots, r$.

(ii) If in (i), F is an orientable surface of genus $k \geq 1$ and $r = 1$, then $G = \Pi_1(X)$ is a free group with generators $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$, S_1 is a cyclic group generated by $\prod_{i=1}^{k} [a_i, b_i]$ (where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$) and (G, S_1) is a PD^2-pair (see Bieri and Eckmann, [5], p. 304]). Now if $\mathcal{S}$ is any family of subgroups of G such that $S_1 \in \mathcal{S}$ then, for any $\mathbb{Z}_2 S_1$-module M, we have, by Corolary 2, $E(G, \mathcal{S}, \mathrm{Ind}_{S_1}^G M) = 1$.

Proposition 4. *If $(G, \mathcal{S})$ is a D^n-pair with $[G : S] = \infty$, for all $S \in \mathcal{S}$, and T is a subgroup of G with homological dimension $hdT \leq n - 2$, then $E(G, \mathcal{S}, \mathrm{Ind}_T^G M) = 1$, for a $\mathbb{Z}_2 T$-module M with $\dim M^T < \infty$.*

Proof. We have that $[G:T]=\infty$, otherwise, since G has no $\mathbb{Z}_2$-torsion, then $hdT = n-1$ (by Theorem 5.13 in Bieri, [4]). Hence $H^0(G;\mathrm{Ind}_T^G M)=0$. Moreover, using Shapiro's Lemma and the hypothesis $hdT \leq n-2$, we have $H^1(G,\mathcal{S};\mathrm{Ind}_T^G M) = H_{n-1}(G;C\otimes \mathrm{Ind}_T^G M) = H_{n-1}(T;C\otimes M)=0$. Consequently, considering the exact long sequence (1.1) for the module $\mathrm{Ind}_T^G M$, we conclude that $H^0(\mathcal{S};\mathrm{Ind}_T^G M)=0$ and so, by Lemma 2.1 in Andrade et al, [3], we get the result. □

4. The Invariant $\tilde{E}(G,S)$

In this section we study some properties of the invariant $\tilde{E}(G,S) := E(G,\{S\},\mathbb{Z}_2G\otimes_{\mathbb{Z}_2S}\overline{\mathbb{Z}_2S})$. In particular we analyse its relations with the invariant $E(G,S)$ (see Andrade and Fanti, [1]). We will make use of some properties of the invariants ends $e(G)$ and $e(G,S)$ (see Scott, [15], Scott and Wall, [16]).

Let G be a group. Consider $\mathcal{P}G$ the power set of G and $\mathcal{F}G$ the set of the finite subsets of G. It is easy to see that $\mathcal{P}G \simeq Coind_{\{1\}}^G\mathbb{Z}_2$ (denoted by $\overline{\mathbb{Z}_2G}$) and $\mathcal{F}G \simeq \mathrm{Ind}_{\{1\}}^G\mathbb{Z}_2 \simeq \mathbb{Z}_2G$. Let $\mathcal{F}_SG := \{B\subset G \mid B\subset F.S$ for some finite subset F of G $\}$, where S is a subgroup of G. Clearly $\mathcal{F}_SG$ is a $\mathbb{Z}_2G$-submodule of $\mathcal{P}G$. Consider the $\mathbb{Z}_2G$-module $\mathrm{Ind}_S^G\overline{\mathbb{Z}_2S} = \mathbb{Z}_2G\otimes_{\mathbb{Z}_2S}\overline{\mathbb{Z}_2S}$ with the natural G-action of the induced module $(g.(g_1\otimes m) = gg_1\otimes m)$. We have that the modules $\mathbb{Z}_2G\otimes_{\mathbb{Z}_2S}\overline{\mathbb{Z}_2S}$ and $\mathcal{F}_SG$ are $\mathbb{Z}_2G$-isomorphic.

Proposition 5. *If (G,S) is a group pair with $[G:S]=\infty$ then $E(G,S)\leq \tilde{E}(G,S)$.*

Proof. Consider the exact sequence of $\mathbb{Z}_2S$-modules

$$0\to\mathbb{Z}_2=\{\emptyset,S\}\to\mathcal{P}S\to Q=\frac{\mathcal{P}S}{\{\emptyset,S\}}\to 0.$$

Since $\mathbb{Z}_2G$ is $\mathbb{Z}_2S$-free, we obtain the following exact sequence

$$0\to\mathbb{Z}_2G\otimes_{\mathbb{Z}_2S}\mathbb{Z}_2\xrightarrow{\phi}\mathbb{Z}_2G\otimes_{\mathbb{Z}_2S}\mathcal{P}S\xrightarrow{\pi}\mathbb{Z}_2G\otimes_{\mathbb{Z}_2S}Q\to 0\,,$$

which induces the long exact sequence in cohomology

$$0\to H^0(G;\mathrm{Ind}_S^G\mathbb{Z}_2)\to H^0(G;\mathrm{Ind}_S^G\mathcal{P}S)\to H^0(G;\mathrm{Ind}_S^G Q)$$
$$\to H^1(G;\mathrm{Ind}_S^G\mathbb{Z}_2)\xrightarrow{\phi^*}\xrightarrow{\phi^*} H^1(G;\mathrm{Ind}_S^G\mathcal{P}S)\to H^1(G;\mathrm{Ind}_S^G Q)\to\cdots$$

Now, since $[G:S]=\infty$, we have that

$$H^0(G;\mathrm{Ind}_S^G\mathbb{Z}_2)=H^0(G;\mathrm{Ind}_S^G\mathcal{P}S)=H^0(G;\mathrm{Ind}_S^G Q)=0$$

and so $\phi^* : H^1(G; \mathrm{Ind}_S^G \mathbb{Z}_2) \to H^1(G; \mathrm{Ind}_S^G \mathcal{P}S)$ is a monomorphism. Hence, by Proposition 2 (3), we have

$$E(G,S) = E(G,S,\mathrm{Ind}_S^G \mathbb{Z}_2) \leq E(G,S,\mathrm{Ind}_S^G \mathcal{P}S) = \tilde{E}(G,S)\,. \qquad \square$$

Example 4. If $G = < a > * < b > \simeq \mathbb{Z} * \mathbb{Z}$ and $S = < b >$, then $\tilde{E}(G,S) = \infty$. In fact, by Example 2.2 (c) of Andrade and Fanti, [1], we have $E(G,S) = \infty$. Thus the result follows from the above inequality.

The equality $E(G,S) = \tilde{E}(G,S)$ may not occur, as the next example shows.

Example 5. Let $G =< a,b : a^2 = b^2 = 1 > \simeq \mathbb{Z}_2 * \mathbb{Z}_2$ and $S =< a : a^2 = 1 > \simeq \mathbb{Z}_2$. Then $E(G,S) = 1 < \tilde{E}(G,S) = 2$. In fact, by Lemma 2.6 in Scott, [15], it follows that $e(G,S) = 1$. Hence, by Theorem 3.1 in Andrade and Fanti, [1], we also have $E(G,S) = 1$. Now, $e(G) = 2$ (see Theorem 5.12 in Scott and Wall, [16]) and so, follows from Theorem 5.1 (2) (of the next section) that $\tilde{E}(G,S) = 2$.

Remark 2. In spite of the inequalities $E(G,S) \leq e(G,S)$ and $E(G,S) \leq \tilde{E}(G,S)$ being true, there are no relations between $e(G,S)$ and $\tilde{E}(G,S)$. In Example 5 we have $e(G,S) = 1 < \tilde{E}(G,S) = 2$. On the other hand, if (G,S_1) is the pair of the Example 1, we can conclude, by the Proposition 8 (1) below, that $\tilde{E}(G,S_1) = 1$, since (G,S_1) is a PD^2-pair. However, $e(G,S_1) = \infty$ (by Scott, [15], Section 3). Thus $\tilde{E}(G,S_1) = 1 < e(G,S_1) = \infty$.

Proposition 6. *Let S and T be subgroups of G with $S \leq T \leq G$. If $[G:T] = \infty$ then:*
(1) $E(G,T,\mathcal{F}_S G) \leq \tilde{E}(G,S) \leq E(G,S,\mathcal{F}_T G)$;
(2) $E(G,T,\mathcal{F}_S G) \leq \tilde{E}(G,T) \leq E(G,S,\mathcal{F}_T G)$.

Proof. We will show (1). The statement (2) is proved by a similar way. The inequality $E(G,T,\mathcal{F}_S G) \leq \tilde{E}(G,S) = E(G,S,\mathcal{F}_S G)$ follows from Proposition 2 (2), taking $\mathcal{S} = \{T\}$, $\mathcal{S}' = \{S\}$ and $M = \mathcal{F}_S G$. To obtain $\tilde{E}(G,S) \leq E(G,S,\mathcal{F}_T G)$ we use the fact that the map $\phi^* : H^1(G;\mathcal{F}_S G) \to H^1(G;\mathcal{F}_T G)$ induced by the inclusion $\mathcal{F}_S G \subset \mathcal{F}_T G$ is a monomorphism (see Kropholler, [10], (4.2)(ii))). Hence, by Proposition 2 (3), for $\mathcal{S} = \{S\}$, $N = \mathrm{Ind}_S^G \overline{\mathbb{Z}_2 S} \simeq \mathcal{F}_S G$ and $M = \mathrm{Ind}_T^G \overline{\mathbb{Z}_2 T} \simeq \mathcal{F}_T G$, we have $\tilde{E}(G,S) = E(G,S,\mathcal{F}_S G) \leq E(G,S,\mathcal{F}_T G)$. $\square$

Corollary 3. *Let G be a group.*
(1) Let $S \leq T \leq G$ with $[G:T] = \infty$. If $[T:S] < \infty$ then $\tilde{E}(G,T) \leq \tilde{E}(G,S)$.
(2) If T is a finitely generated subgroup of G, $[G:T] = \infty$ and $e(T) = 1$ then $e(G) \leq \tilde{E}(G,T)$.

Proof. (1) If $[T : S] < \infty$ then $\mathcal{F}_S G = \mathcal{F}_T G$. Hence the inequality follows immediately from above proposition.

(2) Consider, again, the above proposition and take $S = \{1\}$. Since $\mathcal{F}_{\{1\}}G = \mathcal{F}G \simeq \mathbb{Z}_2 G$ we have $E(G, T, \mathbb{Z}_2 G) \leq \tilde{E}(G, T)$. Now, by Mackey's Formula (see Brown, [6], III.5.6), we obtain the $\mathbb{Z}_2 T$-isomorphism $\mathbb{Z}_2 G \simeq \mathrm{Ind}_{\{1\}}^G \mathbb{Z}_2 \simeq \bigoplus_{g \in E} \mathrm{Ind}_{\{1\}}^T g\mathbb{Z}_2$, where E is a set of representatives for the right cosets of T in G. Hence, since that $1 = e(T) = 1 + \dim H^1(T, \mathbb{Z}_2 T)$ and T is finitely generated, we have $H^1(T; \mathbb{Z}_2 G) \simeq \bigoplus_{g \in E} H^1(T; \mathrm{Ind}_{\{1\}}^T g\mathbb{Z}_2) \simeq \bigoplus_{g \in E} H^1(T; \mathbb{Z}_2 T) = 0$. Thus, $\ker \mathrm{res}_T^G = H^1(G; \mathbb{Z}_2 G)$, and so $E(G, T, \mathbb{Z}_2 G) = 1 + \dim H^1(G; \mathbb{Z}_2 G) = e(G)$. □

Proposition 7. *Let S, T be subgroups of G with $S \leq T \leq G$. If $[G : S] = \infty$ and $[G : T] < \infty$ then $\tilde{E}(G, S) \leq \tilde{E}(T, S)$.*

Proof. Since $\mathcal{F}_S G \simeq \mathrm{Ind}_S^G \overline{\mathbb{Z}_2 S}$ and $\mathcal{F}_S T \simeq \mathrm{Ind}_S^T \overline{\mathbb{Z}_2 S}$, the result follows from the Corollary 1.9 in Andrade and Fanti, [1], considering $M = \overline{\mathbb{Z}_2 S}$. □

To finish this section we notice that, although $\tilde{E}(G, S) \geq E(G, S)$, the properties about duality for the invariant $E(G, S)$ (see Andrade and Fanti, [1], Proposition 2.10 and Proposition 2.12) are preserved by $\tilde{E}(G, S)$.

Proposition 8. *Let $(G : S)$ be a group pair with $[G : S] = \infty$.*
(1) If (G, S) is a D^n-pair then $\tilde{E}(G, S) = 1$.
(2) If G is a D^n-group with dualizing module C and S is a D^{n-1}-subgroup with dualizing module $\mathrm{Res}_S^G C$, then $\tilde{E}(G, S) \leq 2$.
(3) If G is a D^n-group $(n > 1)$ and hd *$S \leq n - 2$, then $\tilde{E}(G, S) = 1$.*

Proof. (1) Using Shapiro's Lemma, we have that $\overline{\mathbb{Z}_2 S}^S = H^0(S, \overline{\mathbb{Z}_2 S}) = H^0(\{1\}, \mathbb{Z}_2) = \mathbb{Z}_2$. Hence $\dim \overline{\mathbb{Z}_2 S}^S = 1 < \infty$. Since $\mathcal{F}_S G = \mathrm{Ind}_S^G \overline{\mathbb{Z}_2 S}$, the result follows from Corollary 2 considering $\mathcal{S} = \{S\}$.

The proof of the conditions (2) and (3) are similar to the proof of Proposition 2.12 in Andrade and Fanti, [1], considering $\overline{\mathbb{Z}_2 S}$ in the place of the module $\mathbb{Z}_2$. □

Example 6. (i) If $X = M^2 - D^2$, where M^2 is any oriented closed surface other than S^2, then $(G = \pi_1(X), S = \pi_1(\partial X))$ is a PD^2-par (Bieri and Eckmann, [5]) and so $\tilde{E}(G, S) = 1$.

(ii) If $G = < a > * < b > \simeq \mathbb{Z} * \mathbb{Z}$, $S = < aba^{-1}b^{-1} >$ and $T = < b >$, then $\tilde{E}(G, S) = 1$ and $\tilde{E}(G, T) = \infty$.

(iii) If $G = \mathbb{Z}^k$ and $S = \mathbb{Z}^{k-1}$, $k \geq 2$, then $\tilde{E}(G, S) = 2$. In fact, since S

is normal in G, we have $E(G,S) = e(G,S) = e(G/S) = 2$ (see Andrade and Fanti, [1], Theorem 3.1 and Scott, [15], Section 2). Hence, by Propositions 5 and 8(2), we have that $2 = E(G,S) \leq \tilde{E}(G,S) \leq 2$.

(iv) $\tilde{E}(\mathbb{Z}^k, \mathbb{Z}^r) = 1$ if $0 \leq r \leq k-2$.

(v) $\tilde{E}(< a, b; a^2b^2 = 1 >, S = < a >) \leq 2$.

(vi) Let $G = (\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}$, where $\theta : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z} \oplus \mathbb{Z})$ is given by

$$\theta(c)(a,b) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^c \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2c & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = (a, 2ca + b) .$$

The operation in G is defined by $((a,b),c)+((a_1,b_1),c_1) = ((a,b)+\theta(c)(a_1,b_1), c+c_1) = (a+a_1, b+b_1+2ca_1, c+c_1)$. We have that G is a PD^3-group, because G is the fundamental group of a asferical 3-manifold (see Raymond and Scott, [13]). If $S = \{((a,b),0); a,b \in \mathbb{Z}\}$ then S is a normal PD^2-subgroup of G isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Also, if $K = \{((0,b),c); b,c \in \mathbb{Z}\}$, we have that K is a normal Abelian subgroup of G. Similarly to the example (ii), we conclude that $\tilde{E}(G,S) = \tilde{E}(G,K) = 2$.

(vii) If G is as above and $S = \{((0,0),c); c \in \mathbb{Z}\}$, then S is not a normal subgroup of G. Since S is a PD^1-subgroup we have, by Proposition 8 (3), that $\tilde{E}(G,S) = 1$.

(viii) If $G = \mathbb{Z} \oplus (\mathbb{Z} * \mathbb{Z}) \oplus \mathbb{Z}_n$, with $\mathbb{Z}_n$ the cyclic group with n elements and $S = \mathbb{Z} * \mathbb{Z}$, then $\tilde{E}(G,S) = 2$. In fact, since $\mathbb{Z}$ is a PD^1-group and $S = \mathbb{Z} * \mathbb{Z}$ is a D^1-group (with dualizing module $C \simeq H^1(S; \mathbb{Z}_2 S)$) we have, by Theorem 9.10 in Bieri, [4], that $T = \mathbb{Z} \oplus (\mathbb{Z} * \mathbb{Z})$ is a D^2-group with dualizing module $\mathbb{Z}_2 \otimes C \simeq C$. Hence, by Proposition 7 and Proposition 8 (2), it follows that $\tilde{E}(G,S) \leq \tilde{E}(T,S) \leq 2$. Now, since S is normal in G, we have that $E(G,S) = e(G,S) = e(G/S) = e(\mathbb{Z} \oplus \mathbb{Z}_n) = e(\mathbb{Z}) = 2$ (see Andrade and Fanti, [1], Theorem (3.1) and Scott, [15], Section 2). Thus, by Proposition 5, we get $\tilde{E}(G,S) = 2$.

5. $\tilde{E}(G,S)$ and the Invariant end $\tilde{e}(G,S)$

In this section we study some relations between the invariant $\tilde{E}(G,S)$ and $\tilde{e}(G,S)$. When $[G:S] = \infty$ we have $\tilde{e}(G,S) = 1 + \dim H^1(G; \mathcal{F}_S G)$ (see Kropholler and Roller, [12], Lemma 1.2). Firstly we show the following result.

Proposition 9. *Let (G,S) be a group pair with $[G:S] = \infty$. We have*

(1) $\tilde{E}(G,S) \leq \tilde{e}(G,S)$;

(2) if $\mathrm{res}_S^G : H^1(G; \mathcal{F}_S G) \to H^1(S; \mathcal{F}_S G)$ is trivial then $\tilde{E}(G,S) = \tilde{e}(G,S)$, and when $\tilde{E}(G,S) = \tilde{e}(G,S) < \infty$, the converse is also true.

Proof. (1) Since $\ker \mathrm{res}^G_S \subset H^1(G;\mathcal{F}_S G)$, it is clear that $\tilde{E}(G,S) = 1 + \ker \mathrm{res}^G_S \leq 1 + \dim\, H^1(G;\mathcal{F}_S G)$.

(2) We have $\mathrm{res}^G_S = 0 \Leftrightarrow \ker \mathrm{res}^G_S = H^1(G;\mathcal{F}_S G) \overset{(*)}{\Rightarrow} \dim \ker \mathrm{res}^G_S = \dim\, H^1(G;\mathcal{F}_S G) \Leftrightarrow \tilde{E}(G,S) = \tilde{e}(G,S)$. Now, if $\tilde{E}(G,S) = \tilde{e}(G,S) < \infty$, the converse implication of (*) is true. □

Corollary 4. *Let (G,S) be a group pair with $[G:S] = \infty$. If S is a PD^n-group, and $\{g \in G \mid cd(S^g \cap S) = n-1\} = \emptyset$, then $\tilde{E}(G,S) = \tilde{e}(G,S)$.*

Proof. By Lemma 5.3 in Kropholler and Roller, [11], we get $H^1(S;\mathcal{F}_S G)$ $= 0$. Therefore the result follows from the above proposition. □

As a consequence of the last result, we have the following application.

Example 7. Let G be the PD^3-group $\mathbb{Z} \oplus (\mathbb{Z} \rtimes \mathbb{Z}) = \Pi_1(S^1 \times KB)$, where KB is the Klein bottle. Consider $S = \{(a,0,c) \mid a,c \in \mathbb{Z}\}$. Then S is not a normal subgroup of G and $\tilde{E}(G,S) = \tilde{e}(G,S) = 2$.

In fact, since the addition in G is defined by $(a,b,c) + (x,y,z) = (a+x, b+(-1)^c y, c+z)$, it is easy to see that S is not a normal subgroup of G, and S is a PD^2-group because $S \simeq \mathbb{Z} \oplus \mathbb{Z}$. Now we have $\mathrm{cd}(S^g \cap S) = 2$, for all $g \in G$:

• If $g \in S$ it is clear that $S^g \cap S = S$ and the statement is true because S is a PD^2-group.

• If $g = (a,b,c) \in G - S$ (thus $b \neq 0$), consider $S_0 = \{(x,0,2k) \mid x,k \in Z\}$ and $S_1 = \{(x,0,2k+1) \mid x,k \in \mathbb{Z}\}$. Hence $S = S_0 \cup S_1$ and $S^g \cap S = (S_0 \cup S_1)^g \cap S = (S_0^g \cap S) \cup (S_1^g \cap S)$. Now $S_0^g \cap S = S_0$ and $S_1^g \cap S = \emptyset$. Then $S^g \cap S = S_0 \simeq \mathbb{Z} \oplus \mathbb{Z}$ and so $\mathrm{cd}(S^g \cap S) = \mathrm{cd}(S_0) = 2$

We have that G is a PD^3-group, S is a PD^2-subgroup and, by Corollary 4.3 in Kropholler and Roller, [12], we obtain $\tilde{e}(G,S) = 2$. By the way, since $\mathrm{cd}(S^g \cap S) = 2$, for all $g \in G$, then $\{g \in G \mid \mathrm{cd}(S^g \cap S) = 1\} = \emptyset$, and so, it follows from the former result, that $\tilde{E}(G,S) = \tilde{e}(G,S) = 2$.

Theorem 2. *Let (G,S) be a group pair with $[G:S] = \infty$.*

(1) If S is finitely generated and normal in G, then $\tilde{E}(G,S) = \tilde{e}(G,S) = e(G/S)$.

(2) If S is finite, then $\tilde{E}(G,S) = \tilde{e}(G,S) = e(G)$.

Proof. (1) If S is normal in G we have, by Mackey's Formula, the $\mathbb{Z}_2 S$-isomorphism $\mathrm{Ind}^G_S \mathcal{P}S \simeq \bigoplus_{g \in G/S} g\mathcal{P}S$, where E is a set of representatives for the right cosets of S in G. Hence, since S is finitely generated, it follows that $H^1(S;\mathcal{F}_S G) = H^1(S; \bigoplus_{g \in E} g\mathcal{P}S) \simeq \bigoplus_{g \in E} H^1(S; g\mathcal{P}S)$. Now, $g\mathcal{P}S$ is a $\mathbb{Z}_2 S$-

module with the S-action given by $s * (gA) = gsg^{-1}.gA = gsA$, for all $s \in S$, $A \in \mathcal{P}S$, and $\mathcal{P}S$ is a $\mathbb{Z}_2S$-module with the S-action given by the left translation. Consider the map $\gamma : g\mathcal{P}S \to \mathcal{P}S$ defined by $\gamma(gA) = A$. Clearly γ is a bijection. Moreover, γ is a $\mathbb{Z}_2S$-homomorphism, since $\gamma(s{*}gA) = \gamma(gsA) = sA = s\gamma(gA)$. Thus, for all $g \in E$, we have $H^1(S; g\mathcal{P}S) \simeq H^1(S;\mathcal{P}S) = 0$. Consequently, $H^1(S;\mathcal{F}_SG) = 0$ and so, by Proposition 9 above and Lemma 2.4 (iv) in Kropholler and Roller, [12], the result follows.

(2) If S is finite then $\mathcal{F}_SG = \mathcal{F}G \simeq \mathbb{Z}_2G \simeq \mathrm{Ind}^G_{\{1\}}\mathbb{Z}_2$. Using Mackey's Formula, for $T = \{1\}$, we obtain the $\mathbb{Z}_2S$-isomorphism

$$\mathrm{Ind}^G_{\{1\}}\mathbb{Z}_2 \simeq \bigoplus_{g\in E} \mathrm{Ind}^S_{\{1\}} g\mathbb{Z}_2.$$

Consequently, using the fact that S is finite and Shapiro's Lemma, we get

$$\begin{aligned} H^1(S;\mathcal{F}_SG) &\simeq H^1(S;\bigoplus_{g\in E}\mathrm{Ind}^S_{\{1\}}g\mathbb{Z}_2) \simeq H^1(S;\bigoplus_{g\in E} Coind^S_{\{1\}}g\mathbb{Z}_2) \\ &\simeq \bigoplus_{g\in E} H^1(S;Coind^S_{\{1\}}g\mathbb{Z}_2) \simeq \bigoplus_{g\in E} H^1(\{1\};g\mathbb{Z}_2) = 0. \end{aligned}$$

Therefore, by Proposition 5 and Lemma 2.4(i) in Kropholler and Roller, [12], we obtain the equalities $\tilde{E}(G,S) = \tilde{e}(G,S) = e(G)$ □

Corollary 5. *Let (G,S) be a group pair with $[G : S] = \infty$. If $e(G) = 2$ then $\tilde{E}(G,S) = \tilde{e}(G,S) = 2$.*

Proof. If $e(G) = 2$ then an arbitrary subgroup of G either is finite or it has finite index in G. In fact, if $e(G) = 2$, by [16], Theorem 5.12, G has an infinite ciclic subgroup H with $[G : H] < \infty$. Let S be a subgroup of G. Then $S \cap H = \{1\}$ or $[H : S \cap H] < \infty$. Since $[S : S \cap H] \le [G : H] < \infty$, then S is finite if $S \cap H = \{1\}$. Otherwise, if $S \cap H \neq \{1\}$, then $[G : S \cap H] \le [G : H][H : S \cap H] < \infty$ and so $[G : S] \le [G : S \cap H] < \infty$. Thus, since by hypothesis $[G : S] = \infty$, it follows that S is finite. Hence, by Theorem 2 (2), we get $\tilde{E}(G,S) = \tilde{e}(G,S) = e(G) = 2$. □

Remark 3. By using Remark 1 and Ribes, [14], we have $\tilde{E}(G,S) = 1 + \dim \frac{H^1(G,S;\mathcal{F}_SG)}{P(G,S.\mathcal{F}_SG)}$, and so, if $\tilde{E}(G,S) = \tilde{e}(G,S)$, we obtain an interpretation for $\tilde{e}(G,S)$ in terms of relative cohomology.

Theorem 3. *Let (G,S) be a group pair with $[G : S] = \infty$. If S is a PD^n-subgroup with $n > 1$, malnormal in G (i.e., $S^g \cap S = \{1\}$, $\forall g \in G \setminus S$) then $\tilde{e}(G,S) = \tilde{E}(G,S) = \dim H^1(G,S;\mathcal{F}_SG)$.*

Proof. Since $[G : S] = \infty$ implies that $H^0(G; \mathcal{F}_S G) = 0$, we have, using (1.1) for $M = \mathcal{F}_S G$, the exact sequence

$$0 \to H^0(S; \mathcal{F}_S G) \to H^1(G, S; \mathcal{F}_S G) \xrightarrow{J} H^1(G; \mathcal{F}_S G) \xrightarrow{\text{res}_S^G} H^1(S; \mathcal{F}_S G) \to \cdots . \quad (1)$$

Let E be a set of representatives for the double cosets SgS, where $g \in G$ (with $1 \in E$). Using Mackey's Formula for $T = S$ and $M = \mathcal{P}S$, we get $\text{Ind}_S^G \mathcal{P}S = \bigoplus_{g \in E} \text{Ind}_{S \cap S^g}^S g\mathcal{P}S$ and so $H^0(S; \mathcal{F}_S G) \simeq H^0(S; \text{Ind}_S^G \mathcal{P}S) \simeq \bigoplus_{g \in E} H^0(S; \text{Ind}_{S \cap S^g}^S g\mathcal{P}S)$. Since S is malnormal in G, we have $S \cap S^g = \{1\}$, for all $g \in G \setminus S$. Hence, if $g \in E$ and $g \neq 1$, then $H^0(S; \text{Ind}_{S \cap S^g}^S g\mathcal{P}S) = H^0(S; \text{Ind}_{\{1\}}^S g\mathcal{P}S) = 0$. Now, for $g = 1 \in E$, we have $H^0(S; \text{Ind}_{S \cap S^g}^S g\mathcal{P}S) = H^0(S; \mathcal{P}S) = \mathbb{Z}_2$. Thus, $H^0(S; \mathcal{F}_S G) = \mathbb{Z}_2$ and therefore, using the sequence (1), we obtain the following short exact sequence:

$$0 \to \mathbb{Z}_2 \to H^1(G, S; \mathcal{F}_S G) \to \text{Im } J = \text{Ker } res_S^G \to 0.$$

Then, dim $H^1(G, S; \mathcal{F}_S G) = 1 + \dim \text{Ker res}_S^G = \tilde{E}(G, S)$. To obtain the equality $\tilde{E}(G, S) = \tilde{e}(G, S)$, observe that $\text{cd}(S^g \cap S) = n$, if $g \in S$, and $\text{cd}(S^g \cap S) = 0$, if $g \in G \setminus S$. Consequently, $\{g \in G \mid \text{cd}(S^g \cap S) = n - 1\} = \emptyset$, since $n > 1$. Now, the result follows from Corollary 4. ☐

6. Conclusions and Remarks

(1) The invariant $\tilde{e}(G, S)$ provides several results in the theory of duality and splitting of groups (see Kropholler and Roller, [12]) and, as we have seen, there are some connections between that invariant and $\tilde{E}(G, S)$. But, in general, they are different because, if the equality holds then we would have $e(G, S) \leq \tilde{E}(G, S)$ (Kropholler and Roller, [12], Lemma 2.5). However, by Remark 2, there are no relations between these invariants. Moreover, the Proposition 8 (1), is not true for $\tilde{e}(G, S)$ since, for the PD^2-pair ($G =< a > * < b >$, $S =< aba^{-1}b^{-1} >$) we have $\tilde{e}(G, S) = \infty$, because $\infty = e(G) \leq \tilde{e}(G, S)$ (by Kropholler and Roller, [12], Lemma 2.4) and so $\tilde{E}(G, S) = 1 \neq \tilde{e}(G, S) = \infty$.

(2) We saw in Theorem 2 (1) that, if S is normal and finitely generated in G then $\tilde{E}(G, S) = \tilde{e}(G, S)$. We do not know if the equality holds without the *finitely generated* hypothesis for S.

(3) We can also notice that, in all the previous examples, we have $\tilde{E}(G,S) = 1, 2$ or ∞. Thus, a interesting question is about how many values $\tilde{E}(G,S)$ can assume. For $\tilde{e}(G,S)$ the following result is true (see Kropholler and Roller, [12], Theorem 1.3): "*If* G *and* S *are finitely generated and* $[\mathrm{Comm}_G(S) : S] = \infty$, *where* $\mathrm{Comm}_G(S) = \{g \in G \mid (S : S \cap S^g) < \infty$ *and* $(S^g : S \cap S^g) < \infty\}$, *then* $\tilde{e}(G,S) = 1, 2$or ∞." We do not know if a similar result holds for $\tilde{E}(G,S)$. However, in agreement with Proposition 4.7 in Kropholler and Roller, [12], there exist pairs (G,S) for which $\tilde{e}(G,S)$ can take values other than $0, 1, 2,$ or ∞. In the proof of this result, the authors show that $H^1(S; \mathcal{F}_S G) = 0$. Hence, by Proposition 9(2), this result is also true for $\tilde{E}(G,S)$, i.e.: *If* $G = G_1 *_S G_2$ *is an amalgamated free product, where* $(G_1, \{S\})$ *is a* PD^3*-pair,* S *an Abelian free group with rank 2 and malnormal in* G_1*,* G_2 *is an Abelian free group with rank 2 and* S *has index* n *in* G_2*, then* $\tilde{E}(G,S) = \tilde{e}(G,S) = n$.

7. Acknowledgements

We want to thank Professor Adalberto Spezamiglio (UNESP, S.J. Rio Preto, Brazil) and Professor Gracinda M. S. Gomes Moreira da Cunha (CAUL, Lisbon, Portugal) by their help in the preparation of the manuscript.

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