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STABLE COHERENT SYSTEMS OF TYPE $(n, d, n+1)$ ON SMOOTH CURVES AND MAPS TO PROJECTIVE SPACES

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Abstract: Let X be a smooth and connected projective curve and (E, V) a spanned coherent system on X of type $(n, d, n+1)$ such that E has no trivial factor. Here we prove that the coherent system (E, V) is α-stable for all $\alpha \gg 0$. Furthermore, (E, V) is α-stable for all $\alpha \geq 0$ (resp. $\alpha > 0$) if and only if E is stable (resp. semistable).

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1. Spanned Coherent Systems of Type $(n, d, n+1)$

Let X be a smooth and connected projective curve. A coherent system (E, V) of type (n, d, k) on X is the datum of a rank n vector bundle E on X and a linear subspace $V \subseteq H^0(X, E)$ such that $\dim(V) = k$. For any $\alpha \in \mathbb{R}$, let $\mu_\alpha(E, V) := d/n + \alpha k/n = \mu(E) + \alpha k/n$ denote the α-slope of (E, V). Using μ_α it is easy to define the notions of α-stability and α-semistability of a coherent system (see [1]) and to study the moduli spaces of α-stable coherent systems of fixed type on X ([1] and [6]). For many existence or non-existence theorems for α-stable coherent systems, see [1], [4], [5], [2]). We will say that the coherent system (E, V) is *spanned* if V spans E. Here we will only consider the case

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$k = n+1$. We will use the classical " dual span construction " given in [1], 5.4, and used for the same purpose in [2]. We briefly recall it, because it gives an easy way to construct all such spanned coherent systems. Fix a spanned coherent system (E,V) on X of type (d,n,k). Set $L := \det(E) \in \mathrm{Pic}^d(X)$. Since V spans E the $(n+1)$-dimensional linear space $\bigwedge^n(V) \cong V^*$ spans L. Hence we have an exact sequence on X:

$$0 \to M_{L,V^*} \to V^* \otimes \mathcal{O}_X \to L \to 0 \tag{1}$$

with M_{L,V^*} a rank n vector bundle on X. Dualizing (1) we see that V^{**} spans ${M_{L,V^*}}^*$. We have $(E,V) \cong ({M_{L,V^*}}^*, V^{**})$, up to the identification of a finite-dimensional vector space with its double dual. The surjection $V^* \otimes \mathcal{O}_X \to L$ induces a morphism $f : X \to \mathbf{P}^n$. Since $h^0(X, {\mathcal{O}_X}^{\oplus n}) = n$ and $V \subseteq H^0(X,E)$, E is not trivial. Hence f is finite, i.e. $f(X)$ is a curve. The curve $f(X) \subset \mathbf{P}^n$ is nondegenerate (i.e. $f(X)$ spans $\mathbf{P}^n$) if and only if E has no trivial factor (see e.g. Remark 2). Consider the following twist of the Euler's sequence of $T\mathbf{P}^n$:

$$0 \to \mathcal{O}_{\mathbf{P}^n}(-1) \to {\mathcal{O}_{\mathbf{P}^n}}^{\oplus(n+1)} \to T\mathbf{P}^n(-1) \to 0 \tag{2}$$

By (2) the morphism f induces a coherent system (E,V) of type $(n,d,n+1)$ on X, where $E := f^*(T\mathbf{P}^n(-1))$, V is an $(n+1)$-dimensional linear subspace of $H^0(X,E)$ spanning E and $d := \deg(f)\cdot\deg(f(X))$. The last remark shows that to construct spanned coherent systems of type $(n,d,n+1)$ on X it is sufficient to take any nondegenerate integral curve $C \subset \mathbf{P}^n$ and any finite morphism $f : X \to C$ with the only restriction that $\deg(f)\cdot\deg(C) = d$. In this note we prove the following result.

Theorem 1. *Let X be a smooth and connected projective curve and (E,V) a spanned coherent system on X of type $(n,d,n+1)$ such that E has no trivial factor. The coherent system (E,V) is α-stable for all $\alpha \gg 0$. It is α-stable for all $\alpha \geq 0$ (resp. $\alpha > 0$) if and only if E is stable (resp. semistable).*

There are several papers in which there are constructions of smooth curves $X \subset \mathbf{P}^n$ such that $T\mathbf{P}^n(-1)|X$ is stable.

Remark 1. To get the result for small $\alpha > 0$ it is sufficient to make assumptions (or prove results) concerning the stability degrees of E introduced by H. Lange in [6] (see the proof of Theorem 1 given below). Notice that L. Brambila-Paz was able to prove (even dropping the spanned assumptions) much more for general X of genus $g \geq 2$ and for some special X in a certain range for g,d,n: she proved α-stability for all $\alpha > 0$ (essentially, proving stability or semistability of the kernel bundle) for ALL coherent systems on X of type $(n,d,n+1)$.

We work over an algebraically closed field $\mathbf{K}$.

Remark 2. Let X be a smooth and connected projective curve and $f : X \to \mathbf{P}^n$ a finite morphism. A spanned vector bundle on X has a trivial factor if and only if it has $\mathcal{O}_X$ as a quotient sheaf. Fix a homogeneous system $z_0, \dots, z_n$ on $\mathbf{P}^n$. Look at the Euler's sequence (2). We get that any surjection $f^*(T\mathbf{P}^n(-1)) \to \mathcal{O}_X$ corresponds to $n+1$ constants $a_i \in \mathbb{K}$, $0 \le i \le n$, not all zero, such that $f(X)$ is contained in the hyperplane $\{\sum_{i=0}^n a_i z_i = 0\}$. Hence $f^*(T\mathbf{P}^n(-1))$ has a trivial factor if and only if $f(X)$ is contained in a hyperplane of $\mathbf{P}^n$.

Lemma 1. *Let (E, V) be a spanned coherent system of type $(n, d, n+1)$ on X. Assume that E has no trivial factor. Then for any proper linear subspace $W \subsetneqq V$ the evaluation map $e_{E,W} : W \otimes \mathcal{O}_X \to E$ is injective as a map of sheaves.*

Proof. Set $x := \dim(W)$. Hence $1 \le x \le n$. Assume $e_{E,W}$ not injective as a map of sheaves. Let F denote the saturation of $\mathrm{Im}(e_{E,W})$ in E. Hence F is subbundle of E of rank at most $x-1$. The quotient bundle E/F has rank at least $n-x+1$ and it is spanned by V/W. Since $\dim(V/W) \le \mathrm{rank}(E/F)$, E/F is trivial. Since E is spanned, this implies that E has a trivial factor, contradiction. $\square$

Proof of Theorem 1. There is $\beta \in \mathbb{R}$ (depending only on E) such that $\mu(F) \le \mu(E) + \beta$ for all proper subbundles F of E; a sharp lower bound for β is given by the instability degrees of E introduced in [6]. Fix $\alpha \in \mathbb{R}$, $\alpha > 0$, for which we would like to check if (E, V) is α-stable. Fix an integer x such that $1 \le x \le n-1$ and a coherent subsystem (F, W) such that $\mathrm{rank}(F) = x$. Increasing if necessary $\mu_\alpha(F, W)$ we will also assume that F is saturated in E and that $W = V \cap H^0(X, E)$. First assume $\dim(W) \le x-1$. Hence $\mu_\alpha(F, W) \le \mu(F) + \alpha(x-1)/x \le \mu(E) + \beta + \alpha(x-1)/x$. Hence $\mu_\alpha(F, W) < \mu(E) + \alpha(n+1)/n = \mu_\alpha(F, V)$ for all α such that $\alpha(x+n)/xn > \beta$. Now assume $\dim(W) \ge x$. By Lemma 1 we have $\dim(W) = x$ and F is the saturation of $\mathrm{Im}(e_{E,W})$ in E. We have $\mu_\alpha(F, W) = \alpha + \mu(F) \le \alpha + \mu(E) + \beta$. Hence $\mu_\alpha(F, W) < \mu_\alpha(E, V)$ if $\alpha > n|\beta|$. The last assertion comes from the proof, because we may take $\beta = 0$ if E is semistable.

Remark 3. For any linear subspace $M \subsetneqq V^*$, $M \ne \{0\}$ the surjection $V^* \otimes \mathcal{O}_X \to L := \det(E)$ induces a non-zero evaluation map map $u_{M,V^*} : M \otimes \mathcal{O}_X \to L$. Since $u_{M,V^*} \ne 0$, there is a unique zero-dimensional subscheme Z_M of X such that $\mathrm{Im}(u_{M,V^*}) = \mathcal{I}_{Z_M} \otimes L$. We will say that Z_M is the dependecy locus of M. When f is an embedding, Z_M is just the scheme-theoretic intersection of $f(X)$ with the linear subspace $\mathbf{P}(M)$ defined by M. Identify V^{**} with V. With this identivation any linear subspace $M \subset V^*$ corresponds to the linear subspace $W_M := (V^*/M)^* \subseteq V$ and any linear subspace $W \subseteq W$ corresponds to the linear subspace $M_W = (V/W)^* \subseteq V^*$. We have $\dim(M) + \dim(W_M) = n+1$ and $\dim(W) + \dim(M_W)$. Set ${}_WZ := Z_{W_M}$. Use the notation of the proof of Theorem 1. Take (F, W) as in the case $\dim(W) = x$. In this case $\deg(F) = \deg({}_WZ)$. Hence, starting from an integral nondegenerate curve $C \subset \mathbf{P}^n$ with or without " very multisecant " linear subspaces of dimension $x - 1 \leq n - 2$ and taking any finite morphism $f : X \to C$ for which the part of the checking of α-stability coming from these subbundles is perfectly controlled.

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