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PARADIGMS FOR DECISION-MAKING UNDER INCREASING LEVELS OF UNCERTAINTY

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Abstract: This paper outlines a paradigm for decision-making under various levels of uncertainty. The basic components of the general model are a finite state space and a finite set of possible actions (decisions). Each (state, action) pair induces a probability distribution over the space of possible outcomes. Associated with each (state, action, outcome) triple is a benefit function, which is modeled as a fuzzy set. Initially we consider the case in which the distribution over the state space is known. Our decision is based on the expected (fuzzy) benefit. Further uncertainty regarding the state space is modeled by postulating a family of probability distributions on the state space. In this scenario expected benefit is defined in terms of a Choquet integral. We also consider an approach based on maximum entropy.

Next we consider the case in which uncertainty is not expressible as a belief function. In this context we extend the concept of a Choquet integral.

Finally we define a preference relation on the distributions over the outcome space. We invoke the concept of fuzzy subsethood to make comparisons. We

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also use the concepts of positive responsiveness, weak dominance and weak independence to establish our preferences.

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1. Introduction

This paper deals with decision-making under increasing levels of uncertainty. Essentially, we try to answer the question, "How does a decision maker pick the best act?" In our model an act is defined as a function mapping a state space into a distribution or some family of distributions over the set of possible outcomes. Furthermore, we typically do not know precisely the current state, which adds another level of uncertainty. We investigate several approaches to this problem. The complexity of the approach depends on the specific level of uncertainty.

The first case deals with the case in which the distribution function over the state space is known and the distribution over the set of outcomes for each (state, act) pair is also known. We introduce a benefit function to construct the decision mechanism. Since the benefit for each outcome may not be precisely known, the benefit function is fuzzy-valued.

Next we consider the case in which we do not have complete knowledge of the distribution over the state space, but the uncertainty over this distribution is expressible as a belief function. We then propose two methods for decision-making, which involve maximizing the extended Choquet integral or maximizing the entropy function.

We then deal with the case in which uncertainty is not expressible as a belief function. Under mild conditions and under the assumption that we want to build flexibility into the process of picking the correct distribution over the state space, we build a utility function to generate our decision.

Finally, we assume that the value of an act is a family of distributions (as opposed to a single distribution) over the set of outcomes. We then define a maximizing set and a subsethood function together with two mild conditions – the weak dominance principle and weak independence – to construct the utility function to use in the decision-making process.

2. Formulation of the Problem

In Anscombe et al [1], the set of outcomes and the set of consequences are distinct sets. The set of consequences (or the set of lotteries) is a set of distributions with finite support over the set of outcomes. We now formally define the components of our decision-making model.

Let S be the finite set of states.

Let X be the set of outcomes.

Let C be the set of consequences.

Let F be the finite set of acts (decisions).

C denotes the set of distributions with finite support defined over X, and every element of F is a function mapping S onto C. The decision problem can now be stated as follows: Select the "best act" from set F, or equivalently, rank the acts in F. A classical decision theory approach is to construct a utility function u: F $\to$ R (the real numbers), and exercise a preference for act f over act g if u(f) $\geq$ u(g) (f, g $\in$ F).

Let $P_{s,f}$ denote the discrete probability function over outcome space X associated with state s $\in$ S, by act f $\in$ F. That is,

$$f(s) = P_{s,f}(\cdot).$$

We now try to define ways to select the best element (i.e., act) in set F.

3. The Benefit Function

For every act f, state s, and outcome x, we assume that there is some benefit associated with the triplet (f, s, x). We denote this benefit by B_f (s, x). Since we may not be certain about the benefit, B_f (s, x) will be assumed to be a fuzzy subset of some set W. Typically W denotes a set of dollar amounts, so B_f (s, x) is a fuzzy dollar amount. B_f (s, x)($\cdot$) is the membership function associated with (f, s, x). That is, B_f (s, x)(w) gives the membership of w $\in$ W in fuzzy set B_f (s, x).

If R is a (crisp) distribution over the finite outcome set

$$X = \{x_1, x_2, \ldots\ldots\ldots, x_n\},$$

we define the (fuzzy) expected benefit (in terms of its membership function) as

$$E_R[B_f(s,\cdot)](w) = Inf\{B_f(s,x_1)(w_{k_1}),\ B_f(s,x_2)(w_{k_2}), \ldots B_f(s,x_n)(w_{k_n})\},$$

where the infimum is taken over all n-tuples (k_1, k_2, ……..., k_n) such that

$$\sum_{i=1}^{n} R(x_i) w_{k_i} = w.$$

For notational purposes we write

$$E_R[B_f(s,\,\cdot)] = \sum_{x \in X} P_R(x) B_f(s,x).$$

However, the reader is cautioned that since B_f (s, x) is fuzzy, the above expression does not pertain to a simple algebraic sum. Instead think of $E_R[B_f(s,x_1)](\cdot)$ as a membership function over W. That is,

$$E_R[B_f(s,\,\cdot)] : W \to [0,\,1].$$

Now suppose that function B_f (s, x) denotes a single value in W. That is for each index i,

$$B_f(s,x_i)(w_k) = \begin{cases} 1, & \text{if } k = k_i^*\,, \\ 0, & \text{if } k \neq k_i^*\,. \end{cases}$$

Then,

$$Inf\{B_f(s,x_1)(w_{k_1}),\ B_f(s,x_2)(w_{k_2}), \ldots B_f(s,x_n)(w_{k_n})\} = \ 1,$$

when $k_i = k_i^*$ ∀ i = 1, 2, …...…, n, and

$$Inf\{B_f(s,x_1)(w_{k_1}),\ B_f(s,x_2)(w_{k_2}), \ldots B_f(s,x_n)(w_{k_n})\} = 0$$

if $k_i \neq k_i^*$ for at least one index i. Consequently,

$$E_R[B_f(s,\cdot)](w) = \begin{cases} 1, & \text{when } w = w*\,, \\ 0, & \text{when } w \neq w*\,, \end{cases} \quad , \quad \text{where} \ \ w* = \sum_{i=1}^{n} R\,(x_i)\, w_{k_i^*}$$

That is,

$$E_R[B_f(s,\,\cdot)] = w*,$$

and we recover the usual numerical expected value.

Now if we select act f, and we are in state s, the expected (fuzzy) benefit is denoted by

$$E_{P_{s,f}}[B_f(s,\,\cdot)] = \sum_{x} P_{s,f}(x) B_f(s,x).$$

Furthermore, if we know precisely the probability function Q(·) over state space S, then the expected value of act f is

$$V(f) = E_Q\left[E_{P_{s,f}}[B_f(s,\cdot)]\right] = \sum_s Q(s)\, E_{P_{s,f}}[B_f(s,\cdot)].$$

Since $E_{P_{s,f}}[B_f(s,\, \cdot)]$ is a fuzzy subset of W, V(f) is also a fuzzy subset of W whose member-ship function is

$V(f)(w) = \sum_s Q(s)\left\{E_{P_{s,f}}[B_f(s,\cdot)](w)\right\}$ w ∈ W.Having established that the (expected) value of an act is a fuzzy subset of W, we can develop a mechanism for selecting the best act (best decision). We want to select f* ∈ F where f* is the act which in some sense maximizes the fuzzy value V(f). One possibility is to defuzzify V(f) for all f ∈ F, and chose the act (or acts) which yield the highest real value. Another possibility is to compare the fuzzy V(f)'s via Jain's maximizing set. (R. Jain, 1976.)

4. Unknown State Space Distribution – The Belief Function Approach

However, it could easily happen that we do not have full knowledge of distribution Q over S. Suppose, for instance, that we know that Q ∈ $\Im$, where $\Im$ denotes a family of probability distributions over S. We define

$$\pi(A) = \underset{Q\in\Im}{Inf}\ Q(A),$$

where A ⊆ S. We are particularly interested in the case in which $\pi(\cdot)$ is a belief function. That is,

$$(1)\ \pi\ (\ \emptyset) = 0,\ (2)\ \pi\ (\mathrm{S}) = 1,$$

and

$$(3)\ \pi\left(\bigcup_j A_j\right) \geq \sum_{\substack{J\,\in\,\Im\\ J\,\neq\,\phi}} (-1)^{|J|+1}\pi\left(\bigcap_{j\in J} A_j\right).$$

Let u(·) be a function from S into **R**. The Choquet integral of function u(·) with respect to belief function $\pi(\cdot)$ is defined to be

$$c\int_S u(s)d\pi(s) = \int_0^\infty \pi\left\{s \mid u(s) > t\right\} dt + \int_{-\infty}^0 \left(\pi\left\{s \mid u(s) > t\right\} - 1\right) dt.$$

We denote this integral by $E_\pi[u(\cdot)]$. It is known that if $\pi(\cdot)$ is a belief function, then

$$E_\pi[u(\cdot)] = Inf\,\{E_Q[u(\cdot)] \mid Q \in \Im\}\,.$$

Thus, the question arises, "When is π a belief function?" Suppose that a family $\Im$ of distributions over state space S has the property that their values are known on some subset of states forming a partition of S. That is,

$$\text{Q}(\text{A}_i) = \alpha_i \text{ for Q} \in \Im\,,$$

where A_1 , A_2, ..., A_n , ... partition S, $\alpha_i \geq 0$, $\forall$ i = 1, 2, 3, ..., and

$$\sum_i \alpha_i = 1.$$

Then if

$$\pi(A) = \underset{Q \in \Im}{Inf}\; Q(A),$$

as above, $\pi(\cdot)$ is a belief function. From now on we are assuming that the uncertainty of the distribution over the state space can be expressed as a belief function.

Now suppose that function u($\cdot$) maps S into the fuzzy subsets of W. That is,

$$\text{u(s)} : \text{W} \rightarrow [0, 1]\ \forall \text{ s} \in \text{S}.$$

Then it is natural to extend the Choquet integral to function u($\cdot$) by

$$cE \int_S u(s) d\pi(s) = \underset{Q \in \Im}{Inf}\, E_Q(u)\,.$$

Recall that E_Q (u) has been defined and is a fuzzy subset of W. Thus, the above Choquet integral is a well-defined fuzzy subset of W.

Returning to our decision problem, and assuming that π is a belief function, our goal is to select act f* which in some sense (i.e. defuzzification or Jain's approach) maximizes the extended Choquet integral

$$cE \int_S E_{P_{s,f}}[B_f(s,\cdot)] d\pi(s) = \underset{Q \in J}{Inf}\, E_Q\left[E_{P_{s,f}}[B_f(s,\cdot)]\right].$$

Recall that π is the belief function generated by Q $\in$ J.

An alternate approach to picking the best f ∈ F is to pick the f associated with the "optimal" distribution Q*. The problem now is to determine which distribution over S is the optimal distribution. Since the distributions in $\Im$ are defined over the finite set S, each Q ∈ $\Im$ is in essence probability function $p_Q(\cdot)$ given by

$$p_Q(s) = Q(\{s\}).$$

The idea is to pick Q* so that its probability function $p_{Q*}(\cdot)$ maximizes the entropy

$$H(p_Q) = -\sum_{s \in S} p_Q(s) Log_e(p_Q(s)).$$

An alternate approach is to select Q* with the goal of maximizing the expected belief function. At this point it is appropriate to recall an algorithm for constructing p_{Q*} in terms of belief function π . In the following algorithm S denotes the state space (as always), while S $_i$ denotes a (not necessarily proper) subset of S.

Algorithm.

S ← S $_i$;

While S $_i$ $\neq$ $\emptyset$ Do

b $_i$ ← $\underset{\substack{K \subset S_i \\ K \neq \phi}}{Max} \left[\frac{\pi(K \cup S_i) - \pi(\bar{S}_i)}{|K|}\right]$; Note: $\bar{S}_i = S - S_i$ (complement of S $_i$) .

Let K $_i$ be the largest subset of S $_i$ such that $\pi(K \cup S_i) - \pi(\bar{S}_i) = b_i K_i$;

S $_{i+1}$ ← S $_i$ − K $_i$;

End While.

We now define $p_{Q*}(s) = b_i$ if s ∈ K, and to pick the best act f* ∈ F, we seek to maximize

$E_{Q*}\left[E_{P_{s,f}}[B_f(s,\cdot)]\right]$ over all f ∈ F.

That is,

$$E_{Q*}\left[E_{P_{s,f*}}[B_{f*}(s,\cdot)]\right] = \underset{f \in F}{Max}\left\{E_{Q*}\left[E_{P_{s,f}}[B_f(s,\cdot)]\right]\right\}.$$

5. Greater Uncertainty Concerning the State Space

We now consider the case in which π is not a belief function, and there is even less certainty regarding the distribution over the state space. Suppose that we have a finite set of families of distributions over S from which to chose. We label them J_1, J_2, ..., J_t. The choice of J_i is determined by the i th expert. Once J_i is selected, an appropriate distribution within J_i will be chosen (perhaps to maximize the entropy as indicated above or perhaps on the basis of additional information).

Thus the choice of an appropriate distribution over S is done in two stages. In stage one, we select an appropriate family of distributions (dictated by an expert's opinion). In stage two, a particular distribution in the chosen family is selected according to some specific criteria. Also, we assume that the experts involved have varying credibility, and so we would prefer family J_i to family J_j if the i th expert has more credibility than the j th expert. On the other hand, we prefer J_1 to J_2 if $J_1 \supset J_2$ since this gives more flexibility for stage two. Thus we have a preference relation on the families of distributions under consideration.

$J_1 \geq J_2$ denotes the fact that J_1 is at least as preferred as J_2. J_1 ¿ J_2 denotes the fact that J_1 is (strictly) preferred to J_2. We use the notation $J_1 \sim J_2$ if we are indifferent as to the choice of J_1 or J_2. That is, $J_1 \sim J_2$ means that $J_1 \geq J_2$ and $J_2 \geq J_1$.

We assume that the following two properties hold:

$J_1 \supset J_2 \Rightarrow J_1$ ¿ J_2.

$J \sim J \cup J' \Rightarrow J \cup J'' \sim J \cup J' \cup J'' \; \forall J''$

The rationale for property 1 is that J is preferred because it contains more choices of distributions (i.e., is more flexible) than any proper subset of itself. The rationale for property 2 is that if adding set J' is of no value when we have set J, then adding set J' is of no value when we have any set larger than J.

Using the results from Kreps [5], we proceed to build a utility function on the set of possible families of distributions recommended by experts (with built in flexibility).

Recall that $\Im$ denotes the set of families of distributions over S that are under consideration. We define a function ξ on $\Im$ by

$\xi\,(J) = \bigcup_{J \geq J'} J'$.

Let

$$\hat{S} = \{J \in \Im \mid J = \xi(J)\}.$$

Let G denote all possible distributions over S. We define a function

$$U: G \times \hat{S} \rightarrow R$$

as follows.

Consider the system ($\hat{S}$, ¿, $\supset$) . Since

$\hat{s}_1 \supset \hat{s}_2$ and $\hat{s}_1 \neq \hat{s}_2 \Rightarrow \neg (\hat{s}_2 > \hat{s}_1)$, there exists a negative-valued function n($\cdot$) on $\hat{S}$ such that $\hat{s}_1 \geq \hat{s}_2$ if and only if

$$\sum_{\hat{s} \mid \hat{s} \supset \hat{s}_1} n(\hat{s}) \geq \sum_{\hat{s} \mid \hat{s} \supset \hat{s}_2} n(\hat{s}).$$

We now define

$$U(g, \hat{s}) = \begin{cases} n(\hat{s}) & if\, g \in \hat{s} \\ 0 & if\, g \notin \hat{s} \end{cases} .$$

Let

$$G(J) = \sum_{\hat{s} \in \hat{S}} \{ \underset{g \in J}{Max}\, U(g, \hat{s}) \}.$$

Then according to Kreps [5], G defines a preference relation on $\Im$. Since all of our sets are finite, Lemma 3 of Kreps [5] provides a way to construct the function n($\cdot$) . The above considerations offer a method of selecting an element in class $\Im$.

Again according to Kreps [5], the above result can be restated as follows: There exists a strictly increasing, real-valued function u($\cdot$) on $\hat{S}$ such that u ^{o} h is a utility function on $\Im$ where

$$h(J)(\hat{s}) = \underset{g \in J}{Max}\, U(g, \hat{s}).$$

In fact

$$\text{u(h(J)) = G(J).}$$

6. The Subsethood Approach

In this section we shift our point of view. Given a state s and an act f, we have a family of possible distributions over the set of outcomes X. We denote this family of distributions by $J_{s,f}$. The problem, of course, is still how to chose the best act f $\in$ F. We could define a preference relation on the distributions for state s as follows:

$P_{s,f_1} \geq P_{s,f_2}$ if and only if $E_{P_{s,f_1}}[B_{f_1}(s,\cdot)] \geq E_{P_{s,f_2}}[B_{f_2}(s,\cdot)]$, and $P_{s,f_1} > P_{s,f_2}$ if and only if $E_{P_{s,f_1}}[B_{f_1}(s,\cdot)] > E_{P_{s,f_2}}[B_{f_2}(s,\cdot)]$ .Of course the preference on the right is determined by comparing two fuzzy sets. We require that this comparison be transitive and complete (i.e., no ties). For simplicity we now assume that W, the set of possible benefits is finite. Let

$$W = \{b_1, b_2, \ldots\ldots, b_{Max}\},$$

where

$$b_i \leq b_{i+1} \ (1 \leq i \leq \max - 1).$$

Define fuzzy set M with membership M(·) , where

$M(b_i) = \frac{b_i}{b_{Max}}$, (1 ≤ i ≤ max – 1).M plays the role of the "maximizing set" (Jain [3]). The membership of b_i in fuzzy set M is the relative magnitude of b_i with respect to that of the maximal element of W. Note that b_{Max} has membership 1. We now define

$P_{s,f_1} \geq P_{s,f_2}$ if and only if $H\left(E_{P_{s,f_1}}[B_{f_1}(s,\cdot)],\, M\right) \geq H\left(E_{P_{s,f_2}}[B_{f_2}(s,\cdot)],\, M\right)$, and $P_{s,f_1} > P_{s,f_2}$ if and only if $H\left(E_{P_{s,f_1}}[B_{f_1}(s,\cdot)],\, M\right) > H\left(E_{P_{s,f_2}}[B_{f_2}(s,\cdot)],\, M\right)$, where H(·) denotes the subsethood function,

$$H(A,B) = \frac{\sum_i (A(b_i) \wedge B(b_i))}{\sum_i A(b_i)} = \frac{\sum_i Min\,\{A(b_i),\, B(b_i)\}}{\sum_i A(b_i)}.$$

That is, H(A,B) denotes the "fraction of A falling in B".

We now impose two mild conditions on our preference of P_{s,f_1} over P_{s,f_2} .

If $P_{s,f_1} > P_{s,f_2}$, then the preference attached to the set $\{ P_{s,f_1} , P_{s,f_2} \}$ falls between the preference attached to P_{s,f_1} and preference attached P_{s,f_2} .

We refer to property 1 as the *weak dominance* principle (WDP).

If $\{P_{s.g} \mid g \in A\} > \{P_{s.g} \mid g \in B\}$, and if

$$\{P_{s.g} \mid g \in A\} \cap \{P_{s.g} \mid g \in C\} = \{P_{s.g} \mid g \in B\} \cap P_{s.g} \mid g \in C\} = \emptyset,$$

then

$$\{P_{s.g} \mid g \in A \cup C\} \geq \{P_{s.g} \mid g \in B \cup C\}.$$

We refer to this property as *weak independence* (WIND).

Now for given state s ∈ S, let $\ell\,(J_{s,f}) = P_{s,f_\ell}$ denote the least preferred distribution over X if

$$H\left(E_{P_{s,\,f_\ell}}[B_{f_\ell}(s,\cdot)],\, M\right) = \underset{P_{s,\,f} \in J_{s,\,f}}{Inf} H\left(E_{P_{s,\,f}}[B_f(s,\cdot)],\, M\right).$$

Similarly, $g\,(J_{s,f}) = P_{s,f_g}$ denotes the most preferred distribution if

$$H\left(E_{P_{s,\,fg}}[B_{f_g}(s,\cdot)],\, M\right) = \underset{P_{s,\,f} \in J_{s,\,f}}{Sup} H\left(E_{P_{s,\,f}}[B_f(s,\cdot)],\, M\right).$$

For the moment we are assuming here that there are no ties induced on the ordering of $J_{f,s}$ according the preference relation defined by the subsethood function. That is, we have a complete linear ordering on the distributions in class $J_{f,s}$.

It then follows from Barbera et al [2] that if we have both WDF and WIND,

$$J_{f,s} \sim \ \{P_{f,s_g},\ P_{f,s_\ell}\}.$$

We now define

$$u(J_{s,f}) = \alpha\, P_{s,f_\ell} + (1-\alpha)\, P_{s,f_g}\ \ 0 \le \alpha \le\ 1.$$

The coefficient α is dependent on the decision maker. Alpha close to 1 indicates a tendency to look at the worst possible distribution, while alpha close to 0 indicates a tendency to look at the best possible distribution of $J_{f,s}$.

Now $u(J_{s,f})$ is a utility function defined on the class of distributions $J_{s,f}$, and its value is a specific distribution over X, the set of outcomes. If we are in state s, then it is natural to compare acts f $_1$ and f $_2$ by maximizing

$$H\left(E_{u(J_{s,f})}[B_f(s,\cdot)],\ M\right),$$

and if Q is a distribution on S, we pick act f to maximize

$$E_Q\{H(E_{u(J_{s,f})}[B_f(s,\cdot)],\ M)\}.$$

We now define *positive responsiveness* (PR) as follows: If

$$P_{s,f} > \{P_{s,g}\ |\, g \in A\}\,,$$

then

$$\{P_{s,g}\ |\ \ g \in A\} \cup \{P_{s,f}\} > \{P_{s,g}\ |\, g \in A\}\,,$$

and if

$$P_{s,g} > P_{s,f}\ \ \forall\, g \in A,$$

then

$$\{P_{s,g}\ |\ \ g \in A\} > \{P_{s,g}\ |\, g \in A\} \cup \{P_{s,f}\}\,.$$

Next we define independence (IND) as follows.

If $\{P_{s.g} \mid g \in A\} > \{P_{s.g} \mid g \in B\}$, and if

$$\{P_{s.g} \mid g \in A\} \cap \{P_{s.g} \mid g \in C\} = \{P_{s.g} \mid g \in B\} \cap P_{s.g} \mid g \in C\} = \emptyset,$$

then

$$\{P_{s,g} \mid g \in A \cup C\} > \{P_{s,g} \mid g \in B \cup C\}\,.$$

From Kannai et al [4], the following is known. If the total number of distributions in J $_{s,f}$ is at least 3, there does not exist a binary relation among the distributions of J $_{s,f}$ satisfying WNP and IND. Also, if the total number of distributions in J $_{s,f}$ is at least 6, then there does *not* exist an ordering on the class J $_{s,f}$ satisfying both PR and WIND. Because of these two results, the two properties that we postulate are WPD and WIND, rather than WPD and IND.

7. Dealing With Ties

The question now arises, "How can we handle ties?" Suppose, for instance, that according to the preference relation generated by the subsethood function, $\ell(J_{s,f_1}),\ \ell(J_{s,f_2}),\ \ldots,\ \ell(J_{s,f_k})$ are tied for the least preferred distributions, and $g(J_{s,h_1}),\ g(J_{s,h_2}),\ \ldots,\ g(J_{s,h_t})$ are tied for the most preferred distributions. We then might consider

$$u[J_{s,f}] = \alpha \tfrac{k}{k+t} \sum_{i=1}^{k} \ell(J_{s,f_i}) + (1-\alpha) \tfrac{t}{k+t} \sum_{j=1}^{t} g(J_{s,h_j}),\ 0 \le \alpha \le\ 1,$$

and proceed as before.

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