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NUMERICAL ANALYSIS OF STABILIZED METHOD FOR TRANSIENT VISCOELASTIC FLOWS

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Abstract: We study a new approximation scheme of transient viscoelastic fluid flow obeying an Oldroyd-B type constitutive equation. The stress, velocity and pressure are approximated by, respectively P_1-continuous, P_2-continuous, P_1-continuous. The new stabilized formulation bases on the choice of a modified Euler method connected to the streamline upwinding Petrov-Galerkin(SUPG) method (cf. Bensaada et al [5]). In this method, the standard Euler-SUPG formulation is modified in order to introduce some upwinding and to stabilize the tensoriel transport term of the Oldroyd derivative. The continuous problem is supposed to admit a sufficiently smooth and sufficiently small solution. We show that the approximate problem has a solution and we give an error bound.

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1. Introduction

Accurate numerical simulations of time-dependent viscoelastic flows are impor-

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tant to the understanding of many phenomena in non-Newtonian fluid mechanics, particularly those associated with flow instabilities. For many years, the numerical simulation of viscoealatic fluid flows have been marked by the loss of convergence of the iterative techniques when the Weissenberg number increases. Last research are motivated by a clearer understanding of this numerical phenomena. It is observed that, the cause of the failure of the numerical simulation is mainly due to the hyperbolic nature of differential viscoelastic constitutive models, which does not contain a dissipative(stabilizing) term for the stress (see Keunings [18]). To palliate these numerical deficiencies, the stabilized methods like upwind techniques and SUPG method are used in the non-newtonian fluid flows, like viscoelastics fluids, to overcome to the difficulties in analysis and computation of this fluids. Successfully, Marchal and Crochet [21] have used the streamline-upwind(SU) method, and no instability have been observed in numerical solution for reasonable Weissenberg number. After, others 'stabilization' methods are used, let us mention for example the works of Fortin et al [12, 14, 15] and Esselaoui et al [10, 11, 22]. Almost all the authors treat only steady viscoelastic flows. The numerical analysis of the steady case of the viscoelastics fluids flows is abundant. Although the list is not exhaustive, one may see for example the works of Esselaoui et al [10], Liakos et al [19], Luo et al [20] and Sandri [26]. Few works are available in the transient case, either for numerical analysis point of view (see Baranger et al [2], Bensaada et al [6], Ervin et al [9]) or for concrete simulation(see for example Fortin et al [13], Saramito [27], Saramito et al [28], Smith [29]).

For the analysis of the time-dependent problem, Baranger et al [2] studied a discontinuous Galerkin(DG) approximation to inertialess flow in $\mathbb{R}^2$, using a technique similar to those used for the steady-state problem (see Baranger et al [3]). With the Hood-Taylor finite element(FE) pair used to approximate the velocity and pressure and a discontinuous linear approximation for the stress, they showed, under the assumption $\Delta t \leq C_1 h^{\frac{3}{2}}$, that the discrete H^1 and L^2 errors for the velocity and stress, respectively, were bounded by $C(\Delta t + h^{\frac{3}{2}})$. Ervin et al [9], showed, for ν denoting the SUPG coefficient, and assuming Hood-Taylor pair approximation for velocity and pressure, and continuous FE approximation for the viscoelastic stress, under the assumption $\max(\Delta t, \nu) \leq C_1 h$, they obtain that the discrete H^1 and L^2 errors for the velocity and stress, respectively, were bounded by $C(\Delta t + \nu + h^2)$.

The aim of this paper is to analyze the modified Euler-SUPG approximation to the time-dependent equations in $\mathbb{R}^2$. For the fully discrete analysis, the approach used in (cf. Bensaada et al [5]) is extended from transient convection linear problem to non-Newtonian flow. The Hood-Taylor pair approximation for velocity and pressure, and continuous FE approximation for the viscoelastic stress are assumed. A extension of the modified SUPG method linked to a variant of implicit Euler method (see, e.g. Bensaada et al [5]) is used for the tensorial transport term of the constitutive equation. For δ and δ_0 denoting the stabilization coefficients, the discrete H^1 and L^2 errors for the velocity and stress, respectively, were bounded by $C(\Delta t + \delta + h\sqrt{\delta_0} + \frac{h^2}{\sqrt{\delta_0}} + h^2)$ are showed under the assumption $(\Delta t + \delta) \leq C_1 h$ and $\frac{h}{\sqrt{\delta_0}} \leq C_2$. The good choice for the parameters δ and δ_0 leads to the optimal error bound:

$$\mathcal{O}(\Delta t + h^{\frac{3}{2}}) \text{ with the discretization constraint } \Delta t \leq c_3 h.$$

Section 2 describes a model of Oldroyd-B and some function spaces. Section 3 contains a description of the mathematical notation and several Lemmas used in the analysis. The numerical method and the discrete approximation are then presented and analyzed in Section 4 and 5 respectively.

2. Presentation of the Problem

We consider a fluid flowing in a bounded, connected open set Ω in $\mathbb{R}^2$ with lipschitzian boundary Γ. The vector $\mathbf{n}$ represents the outward unit normal to Γ.

In order to describe the flow we use the pressure p(scalar), the velocity vector u and the (total) stress tensor σ_{tot}. The rate of strain tensor, $D(u)$, is given by $D(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$, where ∇u represent the velocity gradient tensor.

An Oldroyd's model of differential type with a single relaxation time is obtained by setting:

$$\sigma_{tot} = -pI + \sigma + \sigma_N$$

where σ is the viscoelastic part of the extra stress tensor and $\sigma_N = 2(1-\alpha)D(u)$ its Newtonian part. The $(1-\alpha)$ represents that part of the total viscosity which is considered Newtonian. The parameter α represents a third dimensionless number, $0 < \alpha \leq 1$ which may be considered as the fraction of the viscosity ($\alpha = 1$ for Maxwell's Model and this case is excluded here). Throughout this paper we assume that $\alpha < 1$.

For $a \in [-1,1]$ we define an objective derivative of tensor σ by:

$$\frac{\partial_a \sigma}{\partial t} = \frac{\partial \sigma}{\partial t} + (u.\nabla)\sigma + g_a(\sigma, \nabla u)\,,$$

where $g_a(\sigma,\nabla u) = \frac{1-a}{2}(\sigma\nabla u + \nabla u^\top \sigma) - \frac{1+a}{2}(\nabla u \sigma + \sigma \nabla u^\top)$.

The viscoelastic stress tensor σ satisfies Oldroyd's constitutive equation

$$\sigma + \lambda \frac{\partial_a \sigma}{\partial t} - 2\alpha D(u) \ = \ 0,$$

which is coupled with the equation of motion and incompressibility condition:

$$\begin{cases} \mathrm{Re}\left[\dfrac{\partial u}{\partial t} + (u.\nabla)u\right] - \nabla.\sigma_{tot} = f\,, \\ \nabla.u = 0, \end{cases}$$

where λ and Re represents respectively the Weissenberg and the Reynolds numbers. The right hand side f is a given density of forces. In the following we neglect the inertia term(slow flow), this inertia term increases the technical difficulties of the proof but not introduce any new idea. We also need some boundary conditions. A possible one for u is $u = u_d$ given on Γ. Regarding σ and the hyperbolic character of constitutive equation for u fixed, we have to impose $\sigma = \sigma_d$ on the inflow boundary $\Gamma^- = \{x \in \Gamma;\ u.\mathbf{n}(x) \ < \ 0\}$. In order to make some theoretical analysis of the problem we make the assumption that $u_d = 0$; this implies that $\Gamma^- = \emptyset$ and there is no boundary condition for σ. Finally we impose initial condition on u and σ. We then obtain the following Oldroyd-B problem of viscoelastic transient flow:

$$\begin{cases} \lambda\left[\dfrac{\partial \sigma}{\partial t} + (u.\nabla)\sigma + \beta(\sigma,\nabla u)\right] + \sigma - 2\alpha D(u) = 0, \ \text{in } \Omega\times]0,T[; \\ \mathrm{Re}\ \dfrac{\partial u}{\partial t} - \nabla.\sigma - 2(1-\alpha)\nabla.D(u) + \nabla p = f, \ \text{in } \Omega\times]0,T[; \\ \nabla.u = 0, \ \text{in } \Omega\times]0,T[; \\ u = 0, \ \text{on } \Gamma\times]0,T[; \\ u = u_0, \ \sigma = \sigma_0, \ \text{in } \Omega fort = 0, \end{cases} \qquad (O)$$

$\beta(\sigma,\nabla u)$ denote $g_a(\sigma,\nabla u)$ for $a = 1$. The viscoelastic extra-stress tensor σ, the velocity u and the pressure belong to their respective spaces S, X and Q given by

$$S = \{\tau = (\tau_{ij})_{1\le i,j\le 2} : \tau_{ij} = \tau_{ji};\ \tau_{ij} \in L^2(\Omega);\ i,j = 1,2\}\,,$$

$$X = (H_0^1(\Omega))^2\,, \quad Q = \{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\}\,.$$

The divergence-free space is given by $V = \{v \in X : (q, \nabla.v) = 0, \forall q \in Q\}$.

The norm and scalar product in $L^2(\Omega)$ of functions, vectors and tensors are denoted respectively by $|.|$ and $(.,.)$; ($|.|_\Gamma$ and $(.,.)_\Gamma$ in $L^2(\Gamma)$); $\langle f, v\rangle$ will denote the duality between $f \in (H^{-1}(\Omega))^2$ and $v \in X$.

Let Y is a Banach space. The spaces $C^m([0,T];Y)$ and $L^p(0,T;Y)$ $(1 \leq p \leq \infty)$, equipped respectively with the norms

$$\|v\|_{C^m([0,T];Y)} = \max_{0\leq l\leq m} \sup_{0\leq t\leq T} \left\| \frac{d^l v(t)}{dt^l} \right\|_Y, \quad \|v\|_{L^p(0,T;Y)} = \left(\int_0^T \|v(t)\|_Y^p dt \right)^{1/p}$$

for $1 \leq p < \infty$ and $\|v\|_{L^\infty(0,T;Y)} = \sup_{t\in(0,T)} \|v(t)\|_Y$ for $p = \infty$.

Existence results for problem (O) are proved in Guillopé et al [17]. For $(1-\alpha)$ sufficiently small, Ω a bounded and regular domain,

$f \in L^2(\mathbb{R}_+, H^{-1}(\Omega)), \dfrac{\partial f}{\partial t} \in L^2_{loc}(\mathbb{R}_+, H^{-1}(\Omega)), u_0 \in H^2(\Omega) \cap V, \sigma_0 \in H^2,$

their exists a $T \geq 0$; and a solution (u, σ, p) of problem (O) such that:

$u \in L^2\left(0,T;(H^3(\Omega))^2\right) \cap C^0\left([0,T]; X \cap (H^2(\Omega))^2\right),$

$\sigma \in L^2\left(0,T;(H^2(\Omega))^{2\times 2}\right) \cap C^0\left([0,T];(H^2(\Omega))^{2\times 2}\right),$

$p \in L^2(0,T;H^2(\Omega)),$

and $\dfrac{\partial u}{\partial t} \in L^2(0,T;X) \cap C^0\left([0,T];(L^2(\Omega))^2\right);$

$\dfrac{\partial \sigma}{\partial t} \in L^2\left(0,T;(H^{-1}(\Omega))^{2\times 2}\right) \cap L^\infty\left(0,T;(L^2(\Omega))^{2\times 2}\right).$

3. Mathematical Notation

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and let $\Im_h$ be a triangulation of Ω made of triangles. Thus, the computational domain is defined by

$$\Omega = \bigcup K, K \in \Im_h.$$

We assume that there exists constants ν_1, ν_2 such that:

$$\nu_1 h \leq h_K \leq \nu_2 \rho_K,$$

where h_K is the diameter of triangle K, ρ_K is the diameter of the greatest ball included in K, and $h = \max_{K\in\Im_h} h_K$. Let $P_k(K)$ denote the space of polynomials on K of degree no greater than k. Then we define the Hood Taylor finite element space as follows:

$$X_h = \{v \in X \bigcap C(\bar{\Omega})^2 : v|_K \in P_2(K), \forall K \in \Im_h\},$$

$$S_h = \{\sigma \in T \bigcap C(\bar{\Omega})^{2\times 2} : \ \sigma|_K \in P_1(K), \forall K \in \Im_h\},$$

$$Q_h = \{q \in Q \bigcap C(\bar{\Omega}) : \ q|_K \in P_1(K), \forall K \in \Im_h\},$$

$$V_h = \{v \in X_h : (q, \nabla.v) = 0, \forall q \in Q_h\},$$

where $C(\bar{\Omega})^2$ denotes a vector valued function with two components continuous on $\bar{\Omega}$. Analogous to the continuous spaces, the pair (X_h, Q_h) satisfies the following discrete inf-sup condition (see Girault et al [16]):

$$\inf_{q\in Q_h} \sup_{v\in X_h} \frac{(q, \nabla.v)}{|q|\|v\|_1} \geq \tilde{\gamma} > 0. \tag{1}$$

On the following, we summarize several properties of finite element spaces and Sobolev's spaces which we will use in our subsequent analysis.

For $(u(t), p(t), \sigma(t)) \in (H^{m+1}(\Omega))^2 \times (H^m(\Omega) \times L_0^2(\Omega)) \times (H^m(\Omega))^{2\times 2}$ it is known from Girault et al [16] that there exists $(\tilde{u}(t), \tilde{p}(t)) \in V_h \times Q_h$ such that,

$$|(u - \tilde{u})(t_n)| + h|(u - \tilde{u})(t_n)|_{1,2} \leq C_1 h^{m+1} \|u(t_n)\|_{m+1,2} \tag{2}$$

$$|(p - \tilde{p})(t_n)| \leq C_2 h^m \|p(t_n)\|_{m,2} \tag{3}$$

and there exists $\tilde{\sigma}(t)$ in S_h such that

$$|(\sigma - \tilde{\sigma})(t_n)| \ + \ h|(\sigma - \tilde{\sigma})(t_n)|_{1,2} \leq C_3 h^m \|\sigma(t_n)\|_{m,2}, \tag{4}$$

also $\tilde{\sigma}(.)$ satisfies the following useful inequality:

$$|(\sigma - \tilde{\sigma})(t_n)|_{0,4} \ + \ h|(\sigma - \tilde{\sigma})(t_n)|_{1,4} \leq C_3 h^{m+\frac{1}{2}} \|\sigma(t_n)\|_{m,2}.$$

Remarking that if we define $\tilde{u}(.)$ by the elliptic projection of $u(.)$ on V_h:

$$(D(u - \tilde{u})(.), Dv_h) = 0, \forall v_h \in V_h$$

and $\tilde{\sigma}(.)$ by orthogonal the projection of $\sigma(.)$ on S_h:

$$((\sigma - \tilde{\sigma})(.), \tau_h) = 0, \forall \tau_h \in S_h.$$

Then the error estimate (2)-(4) and the following properties are satisfied (see Raviart et al [24]):

$$\frac{d\tilde{u}}{dt} = \widetilde{\frac{du}{dt}} \tag{5}$$

$$\left|\frac{du}{dt}(s)-\widetilde{\frac{du}{dt}}(s)\right| \leq C_4 h \left\|\frac{du}{dt}(s)-\widetilde{\frac{du}{dt}}(s)\right\|_{1,2}$$
$$\leq C_5 h^{m+1}\left\|\frac{du}{dt}(s)\right\|_{m+1,2},\ if\ u\in C^1([0,T];(H^{m+1}(\Omega))^2). \quad (6)$$

and

$$\frac{d\tilde{\sigma}}{dt}(s)=\widetilde{\frac{d\sigma}{dt}}(s) \quad (7)$$

$$\left|\frac{d\sigma}{dt}(s)-\widetilde{\frac{d\sigma}{dt}}(s)\right| \leq C_6 h \left\|\frac{d\sigma}{dt}(s)-\widetilde{\frac{d\sigma}{dt}}(s)\right\|_{1,2}$$
$$\leq C_7 h^{m}\left\|\frac{d\sigma}{dt}(s)\right\|_{m,2},\ if\ \sigma\in C^1([0,T];(H^{m}(\Omega))^{2\times 2}). \quad (8)$$

In the sequel we shall use the following inverse inequalities (see Ciarlet [7]):

Lemma 1. *Let $l \geq 0$ be an integer and $W_h = \{v, v|_K \in P_l(K)\ \forall K \in \Im_h\}$.*
Let r and p be real with $1 \leq r, p \leq \infty$ and let $l \geq 0$ and $m \geq 0$ be integers such that $l \leq m$. Then there exists a constant $C = C(\nu_1, \nu_2, l, r, m, p, k)$ such that

$$\forall v \in W_h \bigcap W^{l,r}(\Omega) \bigcap W^{m,p}(\Omega),\ |v|_{m,p} \leq C h^{l-m-2max\{0,1/r-1/p\}} |v|_{l,r}.$$

We shall also use the following Sobolev's imbedding theorems :

Lemma 2. *. Let $m \geq 0$ be an integer. The following imbedding hold algebraically and topologically:*

$$W^{1,4} \subset L^\infty(\Omega);\ H^2(\Omega) \subset L^\infty(\Omega),$$
$$W^{m+1,2}(\Omega) \subset W^{m,q}(\Omega) \quad \forall q \in [1, \infty[,$$
$$\text{and } W^{m,p}(\Omega) \subset C^0(\bar{\Omega}), \forall 1 \leq p \leq \infty \text{ such that } mp > 2.$$

4. Numerical Method and Discrete Approximation of Problem (*O*)

Let N be an integer. We divide the interval $[0,T]$ into N intervals of equal length Δt. We denote $t_n = n\Delta t, 0 \leq n \leq N$. A conforming finite element method to discretize the momentum equation is used. For the constitutive equation, which include transient transport term, the Euler-Streamline Upwind Petrov-Galerkin

(SUPG) modified method introduced by Bensaada et al [5] is proposed. This method introduces an additional backward difference quotient for $\partial\sigma/\partial t$ in the streamline direction. Hence, for given $\sigma^n(.)$ we would like that $\sigma^{n+1}(.)$ satisfy the following equation:

$$\lambda[\frac{\sigma^{n+1}-\sigma^n}{\Delta t}+\delta(.,\Delta t)u^n.\nabla(\frac{\sigma^{n+1}-\sigma^n}{\Delta t})]+\lambda u^n.\nabla\sigma^{n+1}+\ \sigma^{n+1}-2\alpha D(u^{n+1})=-\lambda\beta(\sigma^n,\nabla \tag{9}$$

where $\delta(.,\Delta t)$ is a positive stabilization term, to be specified below. Moreover we use the SUPG-discretisation as it was done in Sandri [25].

For a given h and Δt, an approximation of the solution at each time step $n\Delta t, n=0,...,N$, is constructed in the following way: we start with $u_h^0=\tilde{u}_0$, the elliptic projection of u_0 onto V_h and $\sigma_h^0=\tilde{\sigma}_0$ the orthogonal projection of σ_0 onto S_h.

Given $u_h^0,...,u_h^n,\ \sigma_h^0,...,\sigma_h^n$ and $p_h^0,...,p_h^n$, we look for the solution of the following problem: find $(u_h^{n+1},\sigma_h^{n+1},p_h^{n+1})\in X_h\times S_h\times Q_h$, such that

$$(O_h^{n+1})\begin{cases}\lambda(d_{,t\delta}\sigma_h^{n+1},\tau_{u_h^n,\lambda})+(\sigma_h^{n+1},\tau_{u_h^n,\lambda})+B(\lambda u_h^n,\sigma_h^{n+1};\tau)\\ -2\alpha(D(u_h^{n+1}),\tau_{u_h^n,\lambda})=-\lambda(\beta(\sigma_h^n,\nabla u_h^n),\tau_{u_h^n,\lambda})\quad\forall\tau\in S_h,\\ \mathrm{Re}\,(d_{,t}u_h^{n+1},v)+(\sigma_h^{n+1},D(v))+2(1-\alpha))(D(u_h^{n+1}),D(v))\\ -(p_h^{n+1}I,D(v))=\langle f(t_{n+1},.),v\rangle\ \ \forall v\in X_h,\\ (\nabla.u_h^{n+1},q_h)=0\ \ \forall q_h\in Q_h,\end{cases}$$

where

$$d_{,t\delta}\sigma_h^{n+1}=\frac{\sigma_{hu_h^n,\delta}^{n+1}-\sigma_{hu_h^n,\delta}^{n}}{\Delta t}\ \text{and}\ d_{,t}u_h^{n+1}=\frac{u_h^{n+1}-u_h^n}{\Delta t}.$$

The operator B is defined on $X\times(H^1(\Omega))^4\times(H^1(\Omega))^4$ by:

$$B(\lambda u_h^n,\tau_{h1};\tau_{h2})=(\lambda(u_h^n.\nabla)\tau_{h1},\tau_{h2u_h^n,\lambda})+\frac{\lambda}{2}(\nabla.u_h^n\tau_{h1},\tau_{h2}),$$

$$\sigma_{hu_h^n,\delta}^i=\sigma_h^i+\delta(h,\Delta t)(u_h^n.\nabla)\sigma_h^i(i=n,n+1),$$

and

$$\tau_{u_h^n,\lambda}=\tau+\lambda\delta_0(h,\Delta t)(u_h^n.\nabla)\tau,\ \forall\tau\in S_h.$$

The positive stabilization terms $\delta(,)$ and $\delta_0(,)$ satisfied $\lim\limits_{(h,\Delta t)\longrightarrow 0}\delta(h,\Delta t)=0$ and $\lim\limits_{(h,\Delta t)\longrightarrow 0}\delta_0(h,\Delta t)=0$.

Since the pair (X_h, Q_h) satisfies the inf-sup condition (1), the problem (O_h^{n+1}) is also equivalent to the following: Find $(\sigma_h^{n+1}, u_h^{n+1}) \in S_h \times V_h$ such that,

$$(\widehat{O_h}^{n+1}) \begin{cases} \lambda(d_{,t\delta}\sigma_h^{n+1}, \tau_{u_h^n,\lambda}) + (\sigma_h^{n+1}, \tau_{u_h^n,\lambda}) + B(\lambda u_h^n, \sigma_h^{n+1}; \tau) \\ -2\alpha(D(u_h^{n+1}), \tau_{u_h^n,\lambda}) = -\lambda(\beta(\sigma_h^n, \nabla u_h^n), \tau_{u_h^n,\lambda}) \quad \forall \tau \in S_h, \\ \mathrm{Re}\,(d_{,t}u_h^{n+1}, v) + (\sigma_h^{n+1}, D(v)) + 2(1-\alpha))(D(u_h^{n+1}), D(v)) \\ = \langle f(t_{n+1}, .), v\rangle \quad \forall v \in V_h. \end{cases}$$

In order to show that the scheme $(\widehat{O_h}^{n+1})$ defines uniquely $(u_h^{n+1}, \sigma_h^{n+1})$, we multiply the second equation by 2α and add the equation obtained to the first equation. We get

$$\begin{aligned} &\lambda(d_{,t\delta}\sigma_h^{n+1}, \tau_{u_h^n,\lambda}) + 2\alpha\mathrm{Re}\,(d_{,t}u_h^{n+1}, v_h) + B(\lambda u_h^n, \sigma_h^{n+1}; \tau_h) \\ &+A(u_h^n; (\sigma_h^{n+1}, u_h^{n+1}), (\tau, v)) = -\lambda(\beta(\sigma_h^n, \nabla u_h^n), \tau_{u_h^n,\lambda}) + 2\alpha\langle f(t_{n+1}), v_h\rangle, \\ &\forall(\tau_h, v_h) \in S_h \times V_h. \end{aligned}$$

where, the bilinear form $A(u_h^n; ., .)$ on $(H^1(\Omega)^4 \times X) \times (H^1(\Omega)^4 \times X)$ is defined by $A(u_h^n; (\sigma, u), (\tau, v)) = (\sigma, \tau_{u_h^n,\lambda}) + 2\alpha(\sigma, D(v)) - 2\alpha(D(v), \tau_{u_h^n,\lambda}) + 4\alpha(1-\alpha)(D(u), D(v))$.

From this, for $0 < \alpha < 1$ we have the following existence result for problem (O_h^{n+1}):

Proposition 1. . *Let $h_0 = \min\{h_{max}, (1-\alpha)/2\alpha\}$. If $\delta(h, \Delta t) = \lambda\delta_0(h, \Delta t)$ then for all $0 \le n \le N-1$, problem (O_h^{n+1}) admits a solution $(\sigma_h^{n+1}, u_h^{n+1}, p_h^{n+1}) \in S_h \times X_h \times Q_h$.*

Proof. When there is no ambiguity, we omit in $\delta(h, \Delta t)$ (resp. $\lambda\delta_0(h, \Delta t)$) the h and Δt dependence and we denote simply δ (resp.δ_0). We observe that

$$B(\lambda u_h^n, \tau; \tau) = \delta_0 |\lambda u_h^n.\nabla\tau|^2$$

and

$$A(u_h^n; (\tau, v), (\tau, v)) \ge |\tau|^2 + 4\alpha(1-\alpha)|D(v)|^2 - \delta_0|\tau||\lambda(u_h^n.\nabla)\tau| - 2\alpha\delta_0|D(v)||\lambda u_h^n.\nabla\tau|;$$

then, if we have $\delta = \lambda\delta_0$, $\dfrac{\lambda}{\Delta t}(\tau_{u_h^n,\delta}, \tau_{u_h^n,\lambda}) + \dfrac{2\alpha\mathrm{Re}}{\Delta t}(v, v) + A(u_h^n; (\tau, v), (\tau, v)) + B(\lambda u_h^n, \tau; \tau)$

$$\begin{aligned} &\ge |\tau|^2 + 4\alpha(1-\alpha)|D(v)|^2 + \delta_0|\lambda u_h^n.\nabla\tau|^2 + \frac{2\alpha\mathrm{Re}}{\Delta t}|v|^2 + \frac{\lambda}{\Delta t}|\tau_{u_h^n,\lambda}|^2 \\ &\quad -\delta_0|\tau||\lambda u_h^n.\nabla\tau| - 2\alpha\delta_0|D(v)||\lambda u_h^n.\nabla\tau| \ . \end{aligned}$$

Here, we suppose that for $h \leq h_0$, we have : $2(1-\alpha-\alpha\delta_0) \geq 1-\alpha$ then: $\dfrac{\lambda}{\Delta t}(\tau_{u_h^n,\delta}, \tau_{u_h^n,\lambda}) + \dfrac{2\alpha\mathrm{Re}}{\Delta t}(v,v) + A(u_h^n;(\tau,v),(\tau,v)) + B(\lambda u_h^n,\tau;\tau) \geq \dfrac{1}{2}\|(\tau,v)\|^2_{h,u_h^n}$, where $\|(\tau,v)\|_{h,u_h^n}$ is the norm defined by

$$\|(\tau,v)\|_{h,u_h^n} = \{|\tau|^2 + (2\sqrt{\alpha(1-\alpha)}|D(v)|)^2 + \frac{\delta_0}{2}|\lambda u_h^n.\nabla\tau|^2\}^{\frac{1}{2}}.$$

Hence there exists a unique solution $(\sigma_h^{n+1}, u_h^{n+1})$ of problem $(\widehat{O_h}^{n+1})$ and from the hypothesis (1) there exists $p_h^{n+1} \in Q_h$ such that $(\sigma_h^{n+1}, u_h^{n+1}, p_h^{n+1})$ is a solution of problem (O_h^{n+1}).

Remark 1. . In Bensaada et al [4], we studied a variant scheme (O_h^{n+1}) where $\beta(\sigma_h^n, \nabla u_h^n)$ is replaced by $\beta(\sigma_h^{n+1}, \nabla u_h^{n+1})$ and we showed existence and unicity by a fixed point method.

Remark 2. . For $\alpha = 1$ (Maxwell's model) the discrete problem requires an additional inf-sup condition linking S_h and X_h (see Fortin et al [15]). Here, this condition as not satisfied, and thus the $\alpha = 1$ case is excluded. Then assumption $\alpha < 1$ is essential in proving the proposition result.

Remark 3. . The existence result of the proposition 1 can be done without the condition $\delta = \lambda\delta_0$.

5. Main Result and Error Bounds

In this section we denote by C the generic positive constant independent of both the discretisation parameters and the solution, which can take different values at different places. Constants $C_i (i = 0, 1, ...)$ have particular meanings in this paper, their definition are given. We suppose here that

$$f \in L^2(0,T;(L^2(\Omega))^2),\ u_0 \in (H^3(\Omega))^2 \bigcap V \ and\ \sigma_0 \in (H^2(\Omega))^4 \bigcap S.$$

The corresponding solution (σ, u, p) will be assumed to satisfy the following regularity hypotheses:

$$\begin{array}{l}\sigma \in C^1([0,T];(H^2(\Omega))^{2\times 2}) \bigcap C^2([0,T];(L^2(\Omega))^{2\times 2}),\\ u \in C^1([0,T];(H^3(\Omega))^2) \bigcap C^2([0,T];(L^2(\Omega)^2),\\ and,\ p \in L^2(0,T;H^2(\Omega) \bigcap L_0^2(\Omega)) \bigcap C^0([0,T];H^2(\Omega)).\end{array} \tag{10}$$

With these assumption, we show the following result:

Theorem 1. . *There exists M_0 and h_0 such that if problem (O) admits a solution (σ, u, p) satisfying (10) and*
$\max\{\|\sigma\|_{C^1([0,T];(H^2(\Omega))^{2\times 2})}, \|u\|_{C^1([0,T];(H^3(\Omega))^2)}, \|p\|_{C^0([0,T];H^2(\Omega))}, \|\sigma\|_{C^2([0,T];(L^2(\Omega))^{2\times 2})},$
$\|u\|_{C^2([0,T];(L^2(\Omega))^2)}\} \leq M_0,$
then for $(\Delta t + \delta)h^{-1}$ and $\dfrac{h}{\sqrt{\delta_0}}$ bounded, there exists a constant C_0 independent of h and Δt such that,

$$\max_{0\leq n\leq N} |(\sigma_h^n - \sigma(t_n))_{u_h^{n-1},\lambda}| + \left(\sum_{n=0}^{N} \Delta t |\sigma_h^n - \sigma(t_n)|)^2\right)^{\frac{1}{2}} \leq C_0(\Delta t + \delta + h\sqrt{\delta_0} + \frac{h^2}{\sqrt{\delta_0}} + h^2) \text{ and } \max_{0\leq n\leq N} |u_h^n - u(t_n)| + \Bigg($$

where $\quad N \in N^* : N\Delta t = T,\ u_h^{-1} = u_h^0$
and

$$(\sigma_h^i, u_h^i)_{1\leq i\leq N} \quad \textit{are solutions of} \quad (O_h^i)_{1\leq i\leq N}.$$

Remark 4. . The good choice for the parameters of the numerical method are $\delta = \mathcal{O}(\Delta t)$ and $\delta_0 = \mathcal{O}(h)$: this leads to the optimal error bound $\mathcal{O}(\Delta t + h^{\frac{3}{2}})$ with the discretisation constraint $\Delta t \leq Ch$. In Baranger et al [2] with the Euler-Discontinuous Galerkin scheme, the error bound is equivalent, but with a stronger discretization constraint $\Delta t \leq Ch^{3/2}$. One can notice that the relationship of consistency sociable Δt and h is weaker than supposed in this work for the discontinuous stress approximation.

Proof of the theorem. Let us introduce the following notations. Let $u^n(.) = u(., t_n)$, $\sigma^n = \sigma(., t_n)$ and $p^n = p(., t_n)$ represent the solution of problem (O), (u_h^n, σ_h^n) denote the solution of $(\widehat{O_h}^{n+1})$.
Define, $\quad \eta_h^n = u^n - \tilde{u}_h^n,\ e_h^n = u_h^n - \tilde{u}_h^n,$
$\xi_h^n = \sigma^n - \tilde{\sigma}_h^n$ and $\varepsilon_h^n = \sigma_h^n - \tilde{\sigma}_h^n$.
By definition we have from the formulation of the problem $(\widehat{O_h}^{n+1})$ that, for all $(\tau_h, v_h) \in S_h \times V_h$

$$\begin{aligned}
&\lambda(d_{,t\delta}\varepsilon_h^{n+1}, \tau_{u_h^n,\lambda}) + 2\alpha \mathrm{Re}\,(d_{,t}e_h^{n+1}, v_h) + B(\lambda u_h^n, \varepsilon_h^{n+1}; \tau_h) \\
&+A(u_h^n; (\varepsilon_h^{n+1}, e_h^{n+1}), (\tau_h, v_h)) = -\lambda(\beta(\sigma_h^n, \nabla u_h^n), \tau_{u_h^n,\lambda}) + 2\alpha\langle f(t_{n+1}), v_h\rangle \\
&+\lambda(d_{,t\delta}\tilde{\sigma}_h^{n+1}, \tau_{u_h^n,\lambda}) + 2\alpha \mathrm{Re}\,(d_{,t}\tilde{u}_h^{n+1}, v_h) + B(\lambda u_h^n, \tilde{\sigma}_h^{n+1}; \tau_h) \\
&+A(u_h^n; (\tilde{\sigma}_h^{n+1}, \tilde{u}_h^{n+1}), (\tau_h, v_h)).
\end{aligned} \tag{11}$$

Since (σ, u, p) is the exact solution of (O), it satisfies the following consistency equation,

$$\begin{aligned}
&2\alpha \mathrm{Re}\left(\frac{du}{dt}(t), v_h\right) + \lambda\left(\frac{d\sigma}{dt}(t), \tau_{u_h^n,\lambda}\right) + B(\lambda u(t), \lambda u_h^n, \sigma(t); \tau_h)\\
&+A(u_h^n; (\sigma(t), u(t)), (\tau_h, v_h)) = 2\alpha(p(t), \nabla.v_h) + 2\alpha\langle f(t), v_h\rangle\\
&\qquad -\lambda(\beta(\sigma(t), \nabla u(t)), \tau_{u_h^n,\lambda})\ ;\ \forall(\tau_h, v_h) \in S_h \times V_h
\end{aligned}$$

Subtracting at $t = t_{n+1}$, the previous equation to the equation (11), we obtain that:

$$\begin{aligned}
&\lambda\left(d_{,t\delta}\varepsilon_h^{n+1}, \tau_{u_h^n,\lambda}\right) + 2\alpha \mathrm{Re}\ \left(d_{,t}e_h^{n+1}, v_h\right)\\
&+ B\left(\lambda u_h^n, \varepsilon_h^{n+1}; \tau_h\right) + A\left(u_h^n; (\varepsilon_h^{n+1}, e_h^{n+1}), (\tau_h, v_h)\right)\\
&= \lambda\left(\beta(\sigma^{n+1}, \nabla u^{n+1}) - \beta(\sigma_h^n, \nabla u_h^n), \tau_{u_h^n,\lambda}\right) + 2\alpha(p^{n+1}, \nabla.v_h)\\
&+ \lambda\left(\frac{d\sigma^{n+1}}{dt} - d_{,t\delta}\sigma^{n+1}, \tau_{u_h^n,\lambda}\right) + 2\alpha \mathrm{Re}\left(\frac{du^{n+1}}{dt} - d_{,t}u^{n+1}, v_h\right)\\
&+ \lambda\left(d_{,t\delta}\xi_h^{n+1}, \tau_{u_h^n,\lambda}\right) + 2\alpha \mathrm{Re}\ \left(d_{,t}\eta_h^{n+1}, v_h\right)\\
&+ B\left(\lambda(u^{n+1} - u_h^n), \lambda u_h^n, \sigma^{n+1}; \tau_h\right) + B\left(\lambda u_h^n, \xi_h^{n+1}; \tau_h\right)\\
&+ A\left(u_h^n; (\xi_h^{n+1}, \eta_h^{n+1}), (\tau_h, v_h)\right),\ \forall(\tau_h, v_h) \in S_h \times V_h.
\end{aligned} \tag{12}$$

Taking $v = e_h^{n+1}$ and $\tau = \varepsilon_h^{n+1}$ into equation (12) and using the identity $(a - b, a) = \dfrac{1}{2}(|a|^2 - |b|^2 + |a-b|^2)$, we obtain,

$$\begin{aligned}
&\frac{\alpha \mathrm{Re}}{\Delta t}\left\{|e_h^{n+1}|^2 - |e_h^n|^2 + |e_h^{n+1} - e_h^n|^2\right\}\\
&+ \frac{\lambda}{2\Delta t}\left\{|\varepsilon_{hu_h^n,\lambda}^{n+1}|^2 - |\varepsilon_{hu_h^n,\lambda}^{n}|^2 + |\varepsilon_{hu_h^n,\lambda}^{n+1} - \varepsilon_{hu_h^n,\lambda}^{n}|^2\right\}\\
&+ 2\alpha(1-\alpha)|D(e_h^{n+1})|^2 + (1/2)|\varepsilon_h^{n+1}|^2 + (\delta_0/4)|\lambda u_h^n.\nabla\varepsilon_h^{n+1}|^2\\
&\leq (second\ member\ of\ (12)).
\end{aligned} \tag{13}$$

We multiply inequality (13) by Δt and sum it for $n=0$ to $n=m-1 (m \leq N)$, and we use $e_h^0 = \varepsilon_h^0 = 0$. Then we get:

$$\begin{aligned}
&\alpha \mathrm{Re}\left[|e_h^m|^2 + \sum_{n=0}^{m-1} |e_h^{n+1} - e_h^n|^2\right] + (1/2)\sum_{n=0}^{m-1} \Delta t |\varepsilon_h^{n+1}|^2 \\
&+\frac{\lambda}{2}\left[|\varepsilon_{hu_h^n,\lambda}^m|^2 + \sum_{n=0}^{m-1} |\varepsilon_{hu_h^n,\lambda}^{n+1} - \varepsilon_{hu_h^n,\lambda}^n|^2\right] + 2\alpha(1-\alpha)\sum_{n=0}^{m-1} \Delta t |D(e_h^{n+1})|^2 \\
&+(\delta_0/4)\sum_{n=0}^{m-1} \Delta t |\lambda u_h^n . \nabla \varepsilon_h^{n+1}|^2 \\
&\leq \sum_{n=0}^{m-1} \Delta t \left(\lambda[\beta(\sigma^{n+1}, \nabla u^{n+1}) - \beta(\sigma_h^n, \nabla u_h^n)], \varepsilon_{hu_h^n,\lambda}^{n+1}\right) \\
&+\sum_{n=0}^{m-1} \Delta t \left[\lambda\left(\frac{d\sigma^{n+1}}{dt} - d_{,t\delta}\sigma^{n+1}, \varepsilon_{u_h^n,\lambda}^{n+1}\right) + 2\alpha \mathrm{Re}\left(\frac{du^{n+1}}{dt} - d_{,t}u^{n+1}, e_h^{n+1}\right)\right. \\
&\qquad\qquad \left. + 2\alpha(p^{n+1}, \nabla . e_h^{n+1})\right] \\
&+\sum_{n=0}^{m-1} \Delta t \left[\lambda(d_{,t\delta}\xi_h^{n+1}, \varepsilon_{u_h^n,\lambda}^{n+1}) + 2\alpha \mathrm{Re}\,(d_{,t}\eta_h^{n+1}, e_h^{n+1})\right] \\
&+\sum_{n=0}^{m-1} \Delta t \left[B(\lambda(u^{n+1} - u_h^n), \lambda u_h^n, \sigma^{n+1}; \tau) + B(\lambda u_h^n, \xi_h^{n+1}; \varepsilon_h^{n+1})\right] \\
&+\sum_{n=0}^{m-1} \Delta t A\left(u_h^n; (\xi_h^{n+1}, \eta_h^{n+1}), (\varepsilon_h^{n+1}, e_h^{n+1})\right).
\end{aligned} \tag{14}$$

The proof of the theorem takes three steps:
1. The induction hypotheses are defined.
2. Assuming the induction hypotheses, each term of the right-hand side of (14) is estimated.
3. Conclusion of the proof.

Step1. Induction hypotheses. For $C > 0$, we define the ball $B_{h,\Delta t}^m$ $(0 \leq m \leq N)$, by

$$\begin{aligned}
&B_{h,\Delta t}^m = \Big\{(\tau_i, v_i)_{i=0,..,m} \in (S_h \times V_h)^{m+1} : \\
&\max_{0\leq i\leq m}\{|(\tau_i - \sigma(t_i))_{v_{i-1},\lambda}|^2 + |v_i - u(t_i)|^2\}^{\frac{1}{2}} \leq C_0(\Delta t + \delta + h\sqrt{\delta_0} + \frac{h^2}{\sqrt{\delta_0} + h^2)} \\
&\textit{and} \\
&[\sum_{n=0}^{m} \Delta t\{|\tau_i - \sigma(t_i)|^2 + |D(v_i - u(t_i))|^2\}]^{\frac{1}{2}} \leq C_0(\Delta t + \delta + h\sqrt{\delta_0} + \frac{h^2}{\sqrt{\delta_0}} + h^2)\Big\}.
\end{aligned}$$

Our aim is to prove that we can choose M_0, h_0, C_0 such that, for all m satisfying $m\Delta t \leq T$ we have:

(R): for $M \leq M_0,\ h \leq h_0$
if $(\sigma_h^n, u_h^n)_{0\leq n\leq m-1} \in B_{h,\Delta t}^{m-1}$ for a $C_0 = C(M_0, h_0, C)$,
then $(\sigma_h^n, u_h^n)_{0\leq n\leq m} \in B_{h,\Delta t}^{m}$ for the same C_0.

Next, we set :

$$M = \max\big\{\|u\|_{C^1([0,T];(H^3(\Omega))^2)}, \|\sigma\|_{C^1([0,T];(H^2(\Omega))^4)}, \|p\|_{C^0([0,T];H^2(\Omega))}, \\ \|u\|_{C^2([0,T];(L^2(\Omega))^2)}, \|\sigma\|_{C^2([0,T];(L^2(\Omega))^4)}\big\};$$

and we assume that $\max\{(\Delta t+\delta)h^{-1}, \dfrac{h}{\sqrt{\delta_0}}\} \leq C$.

Lemma 3. *There exists $h_0 > 0$ such that for $h \leq h_0$, we have $B_{h,\Delta t}^0 \neq \emptyset$.*

Proof. From the equations properties (2) and (4) we have,

$$|(\sigma_h^0 - \sigma(0))_{u_h^0,\lambda}| + |u_h^0 - u(0)| \leq Mh^2\{C_3(1+\lambda M\delta_0 h^{-1}) + C_1\}$$

and

$$[\Delta t\{|\sigma_h^0 - \sigma(0)|^2 + |D(u_h^0 - u(0)|^2\}]^{\frac{1}{2}} \leq \sqrt{2\Delta t}(C_1 + C_3)Mh^2.$$

In order to ensure that $(\sigma_h^0, u_h^0) = (\tilde{\sigma}_0, \tilde{u}_0) \in B_{h,\Delta t}^0$, it is sufficient to impose for $h < h_0$,
$Mh^{\frac{1}{2}}\{C_3(1+\lambda M\sqrt{\delta_0}) + C_1\} \leq C_0$ and for $h \leq h_2$, $(C_1+C_3)\sqrt{2\Delta t}M\sqrt{\delta_0} \leq C_0$.
Hence, choosing $h_0 = min\{h_1, h_2\}$ the result of the current Lemma follow.

Step2. Estimate the terms of the second member of (14). This step is divided into some Lemmas result. In the following we denote by $\mathcal{TC} = \left(\Delta t + \delta + h\sqrt{\delta_0} + \dfrac{h^2}{\sqrt{\delta_0}} + h^2\right)$.

Lemma 4. *(β-term). Assume that the hypotheses of theorem holds. Then* *there exist a constant C_i ($i = 9, 10$) independent of h and Δt, such that*

$$\lambda \sum_{n=0}^{m-1} \Delta t(\beta(\sigma(t_{n+1}), \nabla u(t_{n+1})) - \beta(\sigma_h^n, \nabla u_h^n), \varepsilon_{hu_h^n,\lambda}^{n+1})$$
$$\leq \lambda(1+\sqrt{\delta_0})\{C_9 \mathcal{TC}[M^2 + MC_0 + M^2 \Delta t h^{-1}] + \lambda C_{10}(C_0 \mathcal{TC})^2 h^{-1}\}\times$$
$$\left[\left(\sum_{n=0}^{m-1} \Delta t |\varepsilon_h^{n+1}|^2\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{m-1} \Delta t \delta_0 |\lambda u_h^n.\nabla \varepsilon_h^{n+1}|^2\right)^{\frac{1}{2}}\right].$$

Bounding the β terms on the right of (14) is more involved. We rewrite the difference as the sum of six terms and then bound each of the terms individually. We have:

$$\begin{aligned}
&\beta(\sigma(t_{n+1}), \nabla u(t_{n+1})) - \beta(\sigma_h^n, \nabla u_h^n) \\
&\quad = \beta(\sigma(t_{n+1}, \nabla(u(t_{n+1}) - u(t_n)) + \beta(\sigma(t_{n+1}) - \sigma_h^n, \nabla u(t_n)) \\
&\qquad + \ \beta(\sigma(t_{n+1}) - \sigma(t_n), \nabla(u(t_{n+1}) - u(t_n))) + \beta(\sigma(t_n) - \sigma_h^n, \nabla u(t_n)) \\
&\qquad\quad + \beta(\sigma(t_n), \nabla(u(t_n) - u_h^n)) + \beta(\sigma(t_n) - \sigma_h^n, \nabla(u(t_n) - u_h^n)).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lambda \sum_{n=0}^{m-1} \Delta t |(\beta(\sigma(t_{n+1}), \nabla u(t_{n+1})) - \beta(\sigma_h^n, \nabla u_h^n), \varepsilon_{hu_h^n,\lambda}^{n+1})| \\
&\leq \lambda \sum_{n=0}^{m-1} \Delta t |(\beta(\sigma(t_{n+1}), \nabla(u(t_{n+1}) - u(t_n))), \varepsilon_{hu_h^n,\lambda}^{n+1})| \\
&+ \lambda \sum_{n=0}^{m-1} \Delta t |(\beta(\sigma(t_{n+1}) - \sigma_h^n, \nabla u(t_n)), \varepsilon_{hu_h^n,\lambda}^{n+1})| \\
&+ \lambda \sum_{n=0}^{m-1} \Delta t |(\beta(\sigma(t_{n+1}) - \sigma(t_n), \nabla(u(t_{n+1}) - u(t_n))), \varepsilon_{hu_h^n,\lambda}^{n+1})| \qquad (15) \\
&+ \lambda \sum_{n=0}^{m-1} \Delta t |(\beta(\sigma(t_n) - \sigma_h^n, \nabla u(t_n)), \varepsilon_{hu_h^n,\lambda}^{n+1})| \\
&+ \lambda \sum_{n=0}^{m-1} \Delta t |(\beta(\sigma(t_n), \nabla(u(t_n) - u_h^n)), \varepsilon_{hu_h^n,\lambda}^{n+1})| \\
&+ \lambda \sum_{n=0}^{m-1} \Delta t |(\beta(\sigma(t_n) - \sigma_h^n, \nabla(u(t_n) - u_h^n)), \varepsilon_{hu_h^n,\lambda}^{n+1})|
\end{aligned}$$

For the three first term on the right of (15), we have by regularity of (σ, u) and the definition of $\beta(\tau, .)$ and $\beta(., \nabla v)$ that,

- $\lambda \sum_{n=0}^{m-1} \Delta t |(\beta(\sigma(t_{n+1}), \nabla(u(t_{n+1}) - u(t_n)), \varepsilon^{n+1}_{hu^n_h,\lambda})|$

Proof. $\leq 2\lambda \sum_{n=0}^{n=m-1} \Delta t |\sigma(t_{n+1})|_{0,\infty} |\nabla(u(t_{n+1}) - u(t_n))||\varepsilon^{n+1}_{hu^n_h,\lambda}|$

$$\leq 2\lambda \Delta t M^2 (1+\sqrt{\delta_0}) \left(\sum_{n=0}^{m-1} \Delta t\right)^{\frac{1}{2}} \left[\left(\sum_{n=0}^{m-1} \Delta t |\varepsilon^{n+1}_h|^2\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{m-1} \Delta t \delta_0 |\lambda u^n_h . \nabla \varepsilon^{n+1}_h|^2\right)^{\frac{1}{2}}\right],$$

- $\lambda \sum_{n=0}^{m-1} \Delta t \left|\left(\beta(\sigma(t_{n+1}) - \sigma^n_h, \nabla u(t_n)), \varepsilon^{n+1}_{hu^n_h,\lambda}\right)\right|$

$$\leq 2\lambda \sum_{n=0}^{m-1} \Delta t |\nabla(u(t_{n+1}))|_{0,\infty} |\sigma(t_{n+1}) - \sigma^n_h||\varepsilon^{n+1}_{hu^n_h,\lambda}|$$

$$\leq 2\lambda M (1 + \sqrt{\delta_0}) \left(\sum_{n=0}^{m-1} \Delta t |\sigma(t_{n+1}) - \sigma^n_h|\right)^{\frac{1}{2}} \times$$

$$[(\sum_{n=0}^{m-1} \Delta t |\varepsilon^{n+1}_h|^2)^{\frac{1}{2}} + (\sum_{n=0}^{m-1} \Delta t \delta_0 |\lambda u^n_h . \nabla \varepsilon^{n+1}_h|^2)^{\frac{1}{2}}],$$

- $\lambda \sum_{n=0}^{m-1} \Delta t |(\beta(\sigma(t_{n+1}) - \sigma(t_n), \nabla(u(t_{n+1}) - u(t_n))), \varepsilon^{n+1}_{hu^n_h,\lambda})|$

$$\leq 2\lambda \sum_{n=0}^{m-1} \Delta t |\sigma(t_{n+1}) - \sigma(t_n)||\nabla(u(t_{n+1}) - u(t_n))|\|\varepsilon^{n+1}_{hu^n_h,\lambda}\|_{0,\infty,\Omega}$$

$$\leq \lambda C h^{-1} \sum_{n=0}^{m-1} \Delta t \left[\Delta t^2 \|\sigma\|_{C([t_n,t_{n+1}];H^1)} \|u\|_{C([t_n,t_{n+1}];H^2)}\right] |\varepsilon^{n+1}_{hu^n_h,\lambda}|$$

$$\leq C \Delta t^2 h^{-1} M^2 (1+\sqrt{\delta_0}) \left(\sum_{n=0}^{m-1} \Delta t\right)^{\frac{1}{2}} \times \left[\left(\sum_{n=0}^{m-1} \Delta t |\varepsilon^{n+1}_h|^2\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{m-1} \Delta t \delta_0 |\lambda u^n_h . \nabla \varepsilon^{n+1}_h|^2\right)^{\frac{1}{2}}\right],$$

where $C = C(\nu_1, \nu_2)$ is a constant independent of $(h, \Delta t)$. In the same way we have,

$$\lambda \sum_{n=0}^{m-1} \Delta t \left|\left(\beta(\sigma(t_n) - \sigma^n_h, \nabla(u(t_n)), \varepsilon^{n+1}_{hu^n_h,\lambda}\right)\right|$$

$$\leq 2\lambda \sum_{n=0}^{m-1} \Delta t |\nabla(u(t_n))|_{0,\infty} |\sigma(t_n) - \sigma^n_h||\varepsilon^{n+1}_{hu^n_h,\lambda}|$$

$$\leq 2\lambda M(1+\sqrt{\delta_0})\left(\sum_{n=0}^{m-1}\Delta t|\sigma(t_n)-\sigma_h^n|^2\right)^{\frac{1}{2}}\times\left[\left(\sum_{n=0}^{m-1}\Delta t|\varepsilon_h^{n+1}|^2\right)^{\frac{1}{2}}+\left(\sum_{n=0}^{m-1}\Delta t\delta_0|\lambda u_h^n.\nabla\varepsilon_h^{n+1}|^2\right)^{\frac{1}{2}}\right],$$

and,

$$\lambda\sum_{n=0}^{m-1}\Delta t\left|\left(\beta(\sigma(t_n),\nabla(u(t_n)-u_h^n)),\varepsilon^{n+1}_{hu_h^n,\lambda}\right)\right|$$
$$\leq 2\lambda M(1+\sqrt{\delta_0})\left(\sum_{n=0}^{m-1}\Delta t|\nabla(u(t_n)-u_h^n)|^2\right)^{\frac{1}{2}}\times\left[\left(\sum_{n=0}^{m-1}\Delta t|\varepsilon_h^{n+1}|^2\right)^{\frac{1}{2}}+\left(\sum_{n=0}^{m-1}\Delta t\delta_0|\lambda u_h^n.\nabla\varepsilon_h^{n+1}|^2\right)^{\frac{1}{2}}\right].$$

For the last term on the right of (15), we have

$$\left|\left(\beta(\sigma(t_n)-\sigma_h^n,\nabla(u(t_n)-u_h^n)),\varepsilon^{n+1}_{hu_h^n,\lambda}\right)\right|$$
$$\leq 2|\sigma(t_n)-\sigma_h^n||\nabla(u(t_n)-u_h^n)|\|\varepsilon^{n+1}_{hu_h^n,\lambda}\|_{0,\infty}$$
$$\leq Ch^{-1}|\sigma(t_n)-\sigma_h^n||\nabla(u(t_n)-u_h^n)||\varepsilon^{n+1}_{hu_h^n,\lambda}|.$$

Consequently this term is bounded as

$$\lambda\sum_{n=0}^{m-1}\Delta t|(\beta(\sigma(t_n)-\sigma_h^n,\nabla(u(t_n)-u_h^n)),\varepsilon^{n+1}_{hu_h^n,\lambda})|$$
$$\leq \lambda Ch^{-1}(\sum_{n=0}^{m-1}\Delta t|\sigma(t_n)-\sigma_h^n|^2)^{\frac{1}{2}}(\sum_{n=0}^{m-1}\Delta t|\nabla(u(t_n)-u_h^n)|^2)^{\frac{1}{2}}\times$$
$$(1+\sqrt{\delta_0})\left[\left(\sum_{n=0}^{m-1}\Delta t|\varepsilon_h^{n+1}|^2\right)^{\frac{1}{2}}+\left(\sum_{n=0}^{m-1}\Delta t\delta_0|\lambda u_h^n.\nabla\varepsilon_h^{n+1}|^2\right)^{\frac{1}{2}}\right]$$

Returning to the inequality (15), remarking that $\sum_{n=0}^{m-1}\Delta t = T$ and putting everything back together, we can see that there exist a constant C_9 and C_{10}

independent of $(h, \Delta t)$ such that,

$$\lambda \sum_{n=0}^{m-1} \Delta t |(\beta(\sigma(t_{n+1}), \nabla u(t_{n+1})) - \beta(\sigma_h^n, \nabla u_h^n), \varepsilon_{hu_h^n,\lambda}^{n+1})|$$
$$\leq \lambda C_9 (1+\sqrt{\delta_0})[\Delta t M^2 + M(\sum_{n=0}^{m-1} \Delta t |\sigma(t_n) - \sigma_h^n|^2)^{\frac{1}{2}} + M^2 \Delta t^2 h^{-1}$$
$$+M(\sum_{n=0}^{m-1} \Delta t |\nabla(u(t_n) - u_h^n)|^2)^{\frac{1}{2}}][(\sum_{n=0}^{m-1} \Delta t |\varepsilon_h^{n+1}|^2)^{\frac{1}{2}} + (\sum_{n=0}^{m-1} \Delta t \delta_0 |\lambda u_h^n . \nabla \varepsilon_h^{n+1}|^2)^{\frac{1}{2}}]$$
$$+\lambda C_{10} h^{-1} (\sum_{n=0}^{m-1} \Delta t |\sigma(t_n) - \sigma_h^n|^2)^{\frac{1}{2}} (\sum_{n=0}^{m-1} \Delta t |\nabla(u(t_n) - u_h^n)|^2)^{\frac{1}{2}} \times$$
$$(1+\sqrt{\delta_0})[(\sum_{n=0}^{m-1} \Delta t |\varepsilon_h^{n+1}|^2)^{\frac{1}{2}} + (\sum_{n=0}^{m-1} \Delta t \delta_0 |\lambda u_h^n . \nabla \varepsilon_h^{n+1}|^2)^{\frac{1}{2}}].$$

The result of the Lemma follows by using the induction (**R**)-hypothesis.

Lemma 5. *(Truncation error). Assume that the hypotheses of theorem holds. Then there exists a constant C_i $(i = 11, 12, 13)$ independent of h and Δt, such that*

$$\lambda \sum_{n=0}^{m-1} \Delta t \left(\frac{d\sigma^{n+1}}{dt} - d_{,t\delta} \sigma^{n+1}, \varepsilon_{u_h^n,\lambda}^{n+1} \right) \leq \lambda C_{11} \left[M \Delta t + \delta(h^{-1}(\mathcal{TC}) + M) \right]$$
$$(1+\sqrt{\delta_0}) \times \left[\left(\sum_{n=0}^{m-1} \Delta t |\varepsilon_h^{n+1}|^2 \right)^{\frac{1}{2}} + \left(\sum_{n=0}^{m-1} \Delta t \delta_0 |\lambda u_h^n . \nabla \varepsilon_h^{n+1}|^2 \right)^{\frac{1}{2}} \right]. \tag{16}$$

$$2\alpha \mathrm{Re} \sum_{n=0}^{m-1} \Delta t \left(\frac{du^{n+1}}{dt} - d_{,t} u^{n+1}, e_h^{n+1} \right) \leq C_{12} \alpha \mathrm{Re}\, \Delta t M \max_{0 \leq n \leq m-1} |e_h^{n+1}| \tag{17}$$

$$2\alpha \sum_{n=0}^{m-1} \Delta t (p^{n+1}, \nabla . e_h^{n+1}) \leq \alpha C_{13} M h^2 \left(\sum_{n=0}^{m-1} \Delta t |D(e_h^{n+1})|^2 \right)^{\frac{1}{2}}. \tag{18}$$

Proof. From the definition included in section 4, we have

$$\Delta t \left(\frac{d\sigma^{n+1}}{dt} - d_{,t\delta} \sigma^{n+1}, \varepsilon_{hu_h^n,\lambda}^{n+1} \right) = \Delta t \left(\frac{d\sigma}{dt}(t_{n+1}) - \frac{\sigma_{u_h^n,\delta}(t_{n+1}) - \sigma_{u_h^n,\delta}(t_n)}{\Delta t}, \varepsilon_{hu_h^n,\lambda}^{n+1} \right)$$

$$= \Delta t \left(\frac{d\sigma}{dt}(t_{n+1}) - \frac{\sigma(t_{n+1}) - \sigma(t_n)}{\Delta t}, \varepsilon_{hu_h^n,\lambda}^{n+1} \right) + \delta \left(u_h^n . \nabla(\sigma(t_{n+1}) - \sigma(t_n)), \varepsilon_{hu_h^n,\lambda}^{n+1} \right).$$

The first term of the second member is equal to $\left(\int_{t_n}^{t_{n+1}} (s-t)\frac{d^2\sigma}{dt^2}(s)\,ds, \varepsilon^{n+1}_{hu^n_h,\lambda}\right)$ and for the second, we have

$$\left(\int_{t_n}^{t_{n+1}} u^n_h.\nabla(\sigma(t_{n+1})-\sigma(t_n)), \varepsilon^{n+1}_{hu^n_h,\lambda}\right) =$$
$$\left(\int_{t_n}^{t_{n+1}} (u^n_h-u^n).\nabla\frac{d\sigma}{dt}(s)\,ds, \varepsilon^{n+1}_{hu^n_h,\lambda}\right) + \left(\int_{t_n}^{t_{n+1}} u^n.\nabla\frac{d\sigma}{dt}(s)\,ds, \varepsilon^{n+1}_{hu^n_h,\lambda}\right).$$

Hence, we obtain that,

$$\begin{aligned}
&\Delta t\left(\frac{d\sigma^{n+1}}{dt} - d_{,t\delta}\sigma^{n+1}, \varepsilon^{n+1}_{hu^n_h,\lambda}\right)\\
&\le \Delta t^2 \max_{t_n\le s\le t_{n+1}} |\frac{d^2\sigma}{dt^2}(s)||\varepsilon^{n+1}_{hu^n_h,\lambda}|\\
&+\delta\Delta t \max_{t_n\le s\le t_{n+1}} |\frac{d\sigma}{dt}(s)|_{1,2}[|u^n_h-u^n|\|\varepsilon^{n+1}_{hu^n_h,\lambda}\|_{0,\infty} + \|u^n\|_{0,\infty}|\varepsilon^{n+1}_{hu^n_h,\lambda}|],
\end{aligned}$$

so, using the regularity assumption of σ, we obtain

$$\begin{aligned}
&\Delta t\left(\frac{d\sigma^{n+1}}{dt} - d_{,t\delta}\sigma^{n+1}, \varepsilon^{n+1}_{hu^n_h,\lambda}\right) \le \left|\left(\int_{t_n}^{t_{n+1}} (s-t)\frac{d^2\sigma}{dt^2}(s)\,ds, \varepsilon^{n+1}_{hu^n_h,\lambda}\right)\right|\\
&+\delta\Delta t \max_{t_n\le s\le t_{n+1}} \left|\frac{d\sigma}{dt}(s)\right|_{1,2}\left[|u^n_h-u^n|\|\varepsilon^{n+1}_{hu^n_h,\lambda}\|_{0,\infty} + \|u^n\|_{0,\infty}|\varepsilon^{n+1}_{hu^n_h,\lambda}|\right]\\
&\le \Delta t^2\|\sigma\|_{C^2(t_n,t_{n+1};L^2)}|\varepsilon^{n+1}_{hu^n_h,\lambda}| + C\delta\Delta t\|\sigma\|_{C^1(H^1)}(h^{-1}|u^n_h-u^n|+M)|\varepsilon^{n+1}_{hu^n_h,\lambda}|\\
&\le M\left(\Delta t^2 + C\delta\Delta t[h^{-1}|u^n_h-u^n|+M]\right)|\varepsilon^{n+1}_{hu^n_h,\lambda}|.
\end{aligned}$$

It follows that there exists a constant C_{11} independent of $(h,\Delta t)$ such that (16) hold.

For the second result of Lemma, we remark that $\Delta t\left(\frac{du^{n+1}}{dt} - d_{,t}u^{n+1}, e^{n+1}_h\right)$ is equal to $\left(\int_{t_n}^{t_{n+1}} (s-t_n)\frac{d^2u}{dt^2}(s)\,ds, e^{n+1}_h\right)$.

Hence, $u \in C^2([0,T];(L^2(\Omega))^2)$, (16) result follow, because we have,

$$2\alpha\Delta t\left(\frac{du^{n+1}}{dt} - d_{,t}u^{n+1}, e^{n+1}_h\right) \le 2\alpha\mathrm{Re}\,\Delta t^2\left\|\frac{d^2u}{dt^2}\right\|_{C([t_n,t_{+1}],L^2)}|e^{n+1}_h|.$$

On the other hand, since $(\tilde p^{n+1}, e^{n+1}_h) \in Q_h \times V_h$ we have,

$$(p^{n+1}, \nabla.e^{n+1}_h) = ((p-\tilde p)^{n+1}, \nabla.e^{n+1}_h),$$

then from (3) there exists a constant C independent of $(h, \Delta t)$ such that

$$2\alpha \sum_{n=0}^{m-1} \Delta t |(p(t_{n+1}), \nabla . e_h^{n+1})| \leq 2\alpha C M h^2 \left(\sum_{n=0}^{m-1} \Delta t \right)^{\frac{1}{2}} \left(\sum_{n=0}^{m-1} \Delta t |D(e_h^{n+1})|^2 \right)^{\frac{1}{2}}.$$

Lemma 6. *(Interpolation error). Let (2) to (8) hold. Under the theorem and the induction hypotheses, there exists constants* C_i $(14 \leq i \leq 18)$ *independent of* $(h, \Delta t)$*, such that*

$$\lambda \sum_{n=0}^{m-1} \Delta t (d_{,t\delta} \xi_h^{n+1}, \varepsilon_{u_h^n,\lambda}^{n+1}) \leq \lambda C_{14} M \Delta t h^2 \max_{0 \leq n \leq m-1} |\varepsilon_{hu_h^n,\lambda}^{n+1}| + C_{15} \delta M h (\mathcal{TC}) \left(\sum_{n=0}^{m-1} \Delta t |\varepsilon_h^{n+1}|^2 \right)^{\frac{1}{2}}$$
$$+ C_{16} \{ \delta \lambda^{-1} M \frac{h^2}{\sqrt{\delta_0}} + M \delta \sqrt{\delta_0} (\mathcal{TC}) \} \left(\sum_{n=0}^{m-1} \Delta t \delta_0 |\lambda u_h^n . \nabla \varepsilon_h^{n+1}|^2 \right)^{\frac{1}{2}} \tag{19}$$

$$2\alpha \mathrm{Re} \sum_{n=0}^{m-1} \Delta t (d_{,t} \eta_h^{n+1}, e_h^{n+1}) \leq \alpha \mathrm{Re}\, C_{17} h^3 M \max_{0 \leq n \leq m-1} |e_h^{n+1}| \tag{20}$$

and,

$$\sum_{n=0}^{m-1} \Delta t A \left(u_h^n; (\xi_h^{n+1}, \eta_h^{n+1}), (\varepsilon_h^{n+1}, e_h^{n+1}) \right)$$
$$\leq C_{18} M h^2 \max(1, \sqrt{\delta_0}) \times$$
$$\left[\left(\sum_{n=0}^{m-1} \Delta t |D(e_h^{n+1})|^2 \right)^{\frac{1}{2}} + \left(\sum_{n=0}^{m-1} \Delta t |\varepsilon_h^{n+1}|^2 \right)^{\frac{1}{2}} + \left(\sum_{n=0}^{m-1} \Delta t \delta_0 |\lambda u_h^n . \nabla \varepsilon_h^{n+1}|^2 \right)^{\frac{1}{2}} \right]. \tag{21}$$

Proof. By definition we have in the first,

$$\Delta t (d_{,t\delta} \xi_h^{n+1}, \varepsilon_{u_h^n,\lambda}^{n+1}) = \left(\int_{t_n}^{t_{n+1}} \frac{d\xi_h}{dt}(t)\, dt, \varepsilon_{hu_h^n,\lambda}^{n+1} \right) + \delta \left(\int_{t_n}^{t_{n+1}} u_h^n . \nabla (\frac{d\xi_h}{dt})(t)\, dt), \varepsilon_{hu_h^n,\lambda}^{n+1} \right). \tag{22}$$

We estimate the first term in (22) by the use of (2),

$$\left| \left(\int_{t_n}^{t_{n+1}} \frac{d\xi_h}{dt}(t)\, dt, \varepsilon_{hu_h^n,\lambda}^{n+1} \right) \right|$$
$$\leq \Delta t \max_{t_n \leq t \leq t_{n+1}} \left| \int_{t_n}^{t_{n+1}} \frac{d\xi_h}{dt}(t) \right| \left| \varepsilon_{hu_h^n,\lambda}^{n+1} \right|$$

$$\leq C_7 M \Delta t h^2 |\varepsilon^{n+1}_{hu^n_h,\lambda}|.$$

For the second term in (22), we have

$$\left(\int_{t_n}^{t_{n+1}} u^n_h.\nabla\left(\frac{d\xi_h}{dt}\right)(t)\,dt, \varepsilon^{n+1}_{hu^n_h,\lambda}\right)$$

$$= -\left(\int_{t_n}^{t_{n+1}} \left(\frac{d\xi_h}{dt}\right)(t)\,dt, (u^n_h.\nabla)\varepsilon^{n+1}_h\right) + \left((\nabla.u^n_h)\int_{t_n}^{t_{n+1}} \left(\frac{d\xi_h}{dt}\right)(t)\,dt, \varepsilon^{n+1}_h\right)$$

$$+\delta_0\left(\int_{t_n}^{t_{n+1}} u^n_h.\nabla\left(\frac{d\xi_h}{dt}\right)(t)\,dt, \lambda(u^n_h.\nabla)\varepsilon^{n+1}_{hu^n_h,\lambda}\right),$$

thus

$$\delta\left|\left(\int_{t_n}^{t_{n+1}} u^n_h.\nabla\left(\frac{d\xi_h}{dt}\right)(t)\,dt, \varepsilon^{n+1}_{hu^n_h,\lambda}\right)\right|$$

$$\leq \delta\lambda^{-1}\Delta t \max_{t_n\leq t\leq t_{n+1}}\left|\left(\frac{d\xi_h}{dt}\right)(t)\right|\left|\lambda(u^n_h.\nabla)\varepsilon^{n+1}_h\right|$$

$$+\delta\Delta t \max_{t_n\leq s\leq t_{n+1}}\left|\left(\frac{d\xi_h}{dt}\right)(t)\right|\left|\nabla(u^n_h - u(t_n))\right|\left|\varepsilon^{n+1}_h\right|_{0,\infty}$$

$$+\Delta t\delta_0\delta|u^n_h|_{0,\infty}\max_{t_n\leq t\leq t_{n+1}}|\xi_h(t)|_{1,2}|\lambda(u^n_h.\nabla)\varepsilon^{n+1}_h|,$$

using (8) and Lemma 1, we can see that there exists a constant C independent of $(h,\Delta t)$ such that

$$\delta\left|\left(\int_{t_n}^{t_{n+1}} u^n_h.\nabla\left(\frac{d\xi_h}{dt}\right)(t)\,dt, \varepsilon^{n+1}_{hu^n_h,\lambda}\right)\right|$$

$$\leq \delta C C_7 M\delta\Delta t h|\nabla(u^n_h - u(t_n))||\varepsilon^{n+1}_h|$$

$$+\left\{\delta\lambda^{-1}C_7 M h^2 + C\frac{C_7}{C_6}Mh\delta_0\delta h^{-1}|u^n_h|\right\}\Delta t|\lambda(u^n_h.\nabla)\varepsilon^{n+1}_h|$$

$$\leq C_7 C M\delta\Delta t h|\nabla(u^n_h - u(t_n))||\varepsilon^{n+1}_h|$$

$$+\left\{C_7\delta\lambda^{-1}M\frac{h^2}{\sqrt{\delta_0}} + C\frac{C_7}{C_6}M\delta\sqrt{\delta_0}(M + \max_{0\leq n\leq m-1}|u^n_h - u(t_n)|)\right\}\times\Delta t\sqrt{\delta_0}|\lambda(u^n_h.\nabla)\varepsilon^{n+1}_h|.$$

Hence, by induction hypothesis (R), (19) follow for $C_{16} = \left(\sum_{n=0}^{n=m-1}\Delta t\right)^{\frac{1}{2}} \times \max\left(C_7, C\frac{C_7}{C_6}\right)$, $C_{15} = CC_7 \times \left(\sum_{n=0}^{n=m-1}\Delta t\right)^{\frac{1}{2}}$ and $C_{14} = C_7\left(\sum_{n=0}^{n=m-1}\Delta t\right)^{\frac{1}{2}}$.

In the same way we prove (20) by remarking that,

$$
\begin{aligned}
&2\alpha \mathrm{Re}\, \Delta t \left(d_{,t}\eta_h^{n+1}, e_h^{n+1}\right) \\
&\leq 2\alpha \mathrm{Re}\, \Delta t \max_{t_n \leq t \leq t_{n+1}} \left| \left(\frac{d\eta_h}{dt}\right)(t) \right| \left|e_h^{n+1}\right|.
\end{aligned}
$$

Then, using (6) we have,

$$
\begin{aligned}
&2\alpha \mathrm{Re}\, \Delta t \left(d_{,t}\eta_h^{n+1}, e_h^{n+1}\right) \\
&\leq 2\alpha \mathrm{Re}\, C_5 \Delta t h^3 \max_{t_n \leq t \leq t_{n+1}} \left\| \frac{du}{dt}(t) \right\|_{3,2} \left|e_h^{n+1}\right|, \\
&\leq 2\alpha \mathrm{Re}\, C_5 \Delta t h^3 M |e_h^{n+1}|.
\end{aligned}
$$

Hence, (19) follow for $C_{17} = 2C_5 \left(\sum_{n=0}^{n=m-1} \Delta t \right)$.

It remains to estimate the A-term, we easily see that there exists a constant $C = C(\alpha)$ such that $\forall (\tau_1, v_1); (\tau_1, v_1) \in S \times X$:

$$
A\left(u_h^n; (\tau_1, v_1), (\tau_2, v_2)\right) \leq C \left[|\tau_1|^2 + |D(v_1)|^2\right]^{\frac{1}{2}} \left[|\tau_{2u_h^n,\lambda}|^2 + |D(v_2)|^2\right]^{\frac{1}{2}},
$$

using again the Schwartz inequality and respectively (2) and (4), we obtain

$$
\begin{aligned}
&\left|A\left(u_h^n; (\xi_h^{n+1}, \eta_h^{n+1}), (\varepsilon_h^{n+1}, e_h^{n+1})\right)\right| \\
&\leq C(C_1 + C_3) M h^2 \max(1, \sqrt{\delta_0}) \left(\sum_{n=0}^{m-1} \Delta t \right)^{\frac{1}{2}} \times \\
&\quad \left[\left(\sum_{n=0}^{m-1} \Delta t |D(e_h^{n+1}\right) |^2)^{\frac{1}{2}} + \left(\sum_{n=0}^{m-1} \Delta t |\varepsilon_h^{n+1}|^2 \right)^{\frac{1}{2}} + \left(\sum_{n=0}^{m-1} \Delta t \delta_0 |\lambda u_h^n . \nabla \varepsilon_h^{n+1}|^2 \right)^{\frac{1}{2}} \right],
\end{aligned}
$$

then the last estimate of the Lemma is proved with $C_{18} = C(C_1+C_3) \left(\sum_{n=0}^{m-1} \Delta t \right)^{\frac{1}{2}}$.

Lemma 7. *(B-terms). Let (2) to (8) hold. Under the theorem and the induction hypotheses, there exists constants* C_i $(23 \leq i \leq 24)$ *independent of*

$(h, \Delta t)$, *such that*

$$\sum_{n=0}^{m-1} \Delta t \left[B\left(\lambda(u^{n+1}-u_h^n), \lambda u_h^n, \sigma^{n+1}; \tau\right) + B\left(\lambda u_h^n, \xi_h^{n+1}; \varepsilon_h^{n+1}\right)\right]$$
$$\leq \lambda C_{23}\left[\Delta t M^2 + M(\mathcal{TC})\right]\left(\sum_{n=0}^{n=m-1} \Delta t |\varepsilon_h^{n+1}|^2\right)^{\frac{1}{2}}$$
$$+\lambda C_{24}\left[\delta_0 M^2 \Delta t + M\frac{h^2}{\sqrt{\delta_0}} + M^2 h\sqrt{\delta_0} + \delta_0 M h(\sqrt{h}(1+h) + \frac{(\mathcal{TC})}{\sqrt{h}})\right] \times$$
$$\left(\sum_{n=0}^{n=m-1} \Delta t \delta_0 |\lambda u_h^n . \nabla \varepsilon_h^{n+1}|^2\right)^{\frac{1}{2}}. \tag{23}$$

Proof. In order to establish the estimate (23), we begin by rewriting the term, object of the Lemma, in a more convenient form:

$$\begin{aligned} &B\left(\lambda(u^{n+1}-u_h^n), \lambda u_h^n, \sigma^{n+1}; \tau\right) + B\left(\lambda u_h^n, \xi_h^{n+1}; \varepsilon_h^{n+1}\right) \\ &= B\left(\lambda(u^{n+1}-u^n), \lambda u_h^n, \sigma^{n+1}; \varepsilon_h^{n+1}\right) + B\left(\lambda u^n, \lambda u_h^n, \xi_h^{n+1}; \varepsilon_h^{n+1}\right) \\ &+B\left(\lambda(u^n-u_h^n), \lambda u_h^n, \xi_h^{n+1}; \varepsilon_h^{n+1}\right) + B\left(\lambda(u^n-u_h^n), \lambda u_h^n, \sigma^{n+1}; \varepsilon_h^{n+1}\right). \end{aligned}$$

since $\nabla.(u^{n+1}-u^n) = \nabla.u^n = 0$, we have

$$\begin{aligned} B\left(\lambda(u^n-u_h^n, \lambda u_h^n, \sigma^{n+1}; \varepsilon_h^{n+1}\right) &= \left(\lambda(u^n-u_h^n).\nabla\sigma, \varepsilon_h^{n+1}\right) \\ &+\delta_0\left(\lambda(u^n-u_h^n).\nabla\sigma^{n+1}, \lambda u_h^n.\nabla\varepsilon_h^{n+1}\right), \end{aligned}$$

and

$$\begin{aligned} B\left(\lambda u^n, \lambda u_h^n, \xi_h^{n+1}; \varepsilon_h^{n+1}\right) &= \left(\lambda u^n.\nabla\xi_h^{n+1}, \varepsilon_h^{n+1}\right) \\ &+\delta_0\left(\lambda u^n.\nabla\xi_h^{n+1}, \lambda u^n.\nabla\varepsilon_h^{n+1}\right), \end{aligned}$$

where we remark that,

$$\left(\lambda u^n.\nabla\xi_h^{n+1}, \varepsilon_h^{n+1}\right) = -\lambda\left(\xi_h^{n+1}, (u^n-u_h^n).\nabla\varepsilon_h^{n+1}\right) - \lambda\left(\xi_h^{n+1}, u_h^n.\nabla\varepsilon_h^{n+1}\right).$$

Thus,

$$\begin{aligned} &B\left(\lambda(u^{n+1}-u^n), \lambda u_h^n, \sigma^{n+1}; \varepsilon_h^{n+1}\right) + B\left(\lambda u^n, \lambda u_h^n, \xi_h^{n+1}; \varepsilon_h^{n+1}\right) \\ &\leq \lambda|\sigma|\|u^{n+1}-u^n\|_{0,\infty}|\varepsilon_h^{n+1}| + \lambda\delta_0\|u^{n+1}-u^n\|_{0,\infty}|\nabla\sigma^{n+1}||\lambda u_h^n.\nabla\varepsilon_h^{n+1}| \\ &+\lambda|\xi_h^{n+1}||u^n-u_h^n|\|\nabla\varepsilon_h^{n+1}\|_{0,\infty}+\lambda|\xi_h^{n+1}||\lambda(u_h^n.\nabla)\varepsilon_h^{n+1}|+\lambda\delta_0\|u_h^n\|_{0,\infty}|\nabla\xi_h^{n+1}||\lambda(u_h^n.\nabla)\varepsilon_h^{n+1}|. \end{aligned}$$

Owing to regularity assumption of the continuous solution we see that,

$$\|u^{n+1}-u^n\|_{0,\infty} = \left\| \int_{t_n}^{t_{n+1}} \frac{du}{dt}(s)\ ds \right\|_{0,\infty}$$
$$\leq \Delta t \|u\|_{C(t_n,t_{n+1};H^2)}.$$

On the other hand, using result of Lemma 1 and interpolation error (4), there exists a constant C independent of $(h,\Delta t)$ such that,

$$B\left(\lambda(u^{n+1}-u^n), \lambda u_h^n, \sigma^{n+1}; \varepsilon_h^{n+1}\right) + B\left(\lambda u^n, \lambda u_h^n, \xi_h^{n+1}; \varepsilon_h^{n+1}\right)$$
$$\leq \lambda C\left[\Delta t M^2 |\varepsilon_h^{n+1}| + \delta_0 M^2 \Delta t |\lambda(u_h^n.\nabla)\varepsilon_h^{n+1}| + C_3 M |u^n - u_h^n||\varepsilon_h^{n+1}| + C_3 M h^2 |\lambda u_h^n.\nabla \varepsilon_h^{n+1}|\right.$$
$$\left. + C_3 M^2 \delta_0 h |\lambda u_h^n.\nabla \varepsilon_h^{n+1}|\right].$$

Hence, according to the induction hypothesis, we have

$$\sum_{n=0}^{n=m-1} \Delta t B(\lambda(u^{n+1}-u^n), \lambda u_h^n, \sigma^{n+1}; \varepsilon_h^{n+1}) + B(\lambda u^n, \lambda u_h^n, \xi_h^{n+1}; \varepsilon_h^{n+1})$$
$$\leq \lambda C_{19}\left[\Delta t M^2 + M(\mathcal{TC})\right] \left(\sum_{n=0}^{n=m-1} \Delta t |\varepsilon_h^{n+1}|^2 \right)^{\frac{1}{2}}$$
$$+ \lambda C_{20} \left[\delta_0 M^2 \Delta t + M \frac{h^2}{\sqrt{\delta_0}} + M^2 h \sqrt{\delta_0}\right] \left(\sum_{n=0}^{n=m-1} \Delta t \delta_0 |\lambda u_h^n.\nabla \varepsilon_h^{n+1}|^2 \right)^{\frac{1}{2}}, \tag{24}$$

where C_{19}(resp.C_{20}) is a constant independent of $(h,\Delta t)$. A device similar to the one used to bound the first part of B-term yields,

$$B\left(\lambda(u^n-u_h^n), \lambda u_h^n, \xi_h^{n+1}; \varepsilon_h^{n+1}\right) + B\left(\lambda(u^n-u_h^n), \lambda u_h^n, \sigma^{n+1}; \varepsilon_h^{n+1}\right)$$
$$\leq \lambda C_{21} M(\mathcal{TC}) \left(\sum_{n=0}^{n=m-1} \Delta t |\varepsilon_h^{n+1}|^2 \right)^{\frac{1}{2}}$$
$$+ \lambda C_{22} \delta_0 M h \left[h^{\frac{1}{2}}(1+h) + h^{-\frac{1}{2}}(\mathcal{TC})\right] \left(\sum_{n=0}^{n=m-1} \Delta t \delta_0 |\lambda u_h^n.\nabla \varepsilon_h^{n+1}|^2 \right)^{\frac{1}{2}}, \tag{25}$$

where C_{21}(resp.C_{22}) is, also, a constant independent of $(h,\Delta t)$. Thus, the Lemma result follow from (24) with (25) with $C_{23} = C_{19} + C_{21}$ and $C_{24} = \max(C_{20}, C_{22})$.

Step 3. Conclusion of the proof. Before completing the proof of the theorem we are need to estimate $(e_h^n, \varepsilon^n_{hu_h^{n-1},\lambda})_{1\leq n\leq m-1}$. For this we establish the following result,

Lemma 8. *Assume that (2) and (4) hold. Under the regularity assumption of the continuous solution and the induction hypothesis, there exists constants* C_{25} *independent of* $(h, \Delta t)$*, such that,*

$$|e_h^n| \leq \mathcal{TC} + C_1 M h^3, \tag{26}$$

and

$$|\varepsilon^n_{hu_h^{n-1},\lambda}| \leq C_2 M h^2 + \lambda \delta_0 C_{25}(\mathcal{TC} + M h^3 + M^2 h). \tag{27}$$

Proof. By the definition we have,

$$|e_h^n| = |u_h^n - \tilde{u}(t_n)| \leq |u_h^n - u(t_n)| + |\eta_h^n|,$$

so from (2) and the induction hypothesis,

$$|e_h^n| \leq \mathcal{TC} + C_1 M h^3.$$

For the second result of the Lemma, we have in the first,

$$|\varepsilon^n_{hu_h^{n-1},\lambda}| = |\varepsilon_h^n + \lambda\delta_0(u_h^{n-1}.\nabla)\varepsilon_h^n|$$

$$\leq |(\sigma_h^n - \sigma(t_n)) + \lambda\delta_0(u_h^{n-1}.\nabla)(\sigma_h^n - \sigma(t_n))|$$

$$+|\xi_h^n + \lambda\delta_0(u_h^{n-1}.\nabla)\xi_h^n|$$

by the induction hypotheses **(R)**,

$$|(\sigma_h^n - \sigma(t_n))_{u_h^{n-1}}| \leq \mathcal{TC}.$$

On the other hand we have

$$|\xi_h^n + \lambda\delta_0(u_h^{n-1}.\nabla)\xi_h^n|$$

$$\leq |\xi_h^n| + \lambda\delta_0\{|(e_h^{n-1}.\nabla)\xi_h^n| + |\tilde{u}(t_{n-1}).\nabla\xi_h^n|\}.$$

Using the Lemma 1, there exists a constant C independent of $(h, \Delta t)$ such that,

$$|(e_h^{n-1}.\nabla)\xi_h^n| \leq |e_h^{n-1}|_{0,\infty}|\xi_h^n|_{1,2}$$

$$\leq C h^{-1}|\xi_h^n|_{1,2}|e_h^{n-1}|,$$

and from (4) and the above estimate $|e_h^{n-1}|$ we obtain,

$$|(e_h^{n-1}.\nabla)\xi_h^n| \leq CC_3Mh(\mathcal{TC} + C_1Mh^3).$$

Let us estimate the last part of this term, we have

$$|(\tilde{u}(t_{n-1}).\nabla(\sigma - \tilde{\sigma})(t_n)| \leq |\tilde{u}(t_{n-1})|_{0,4}|\xi_h^n|_{1,4}$$

from the Lemma 2, we have the Sobolev's imbedding $H^1(\Omega) \subset L^4(\Omega)$ and $H^2(\Omega) \subset W^{1,4}(\Omega)$ then there exists C constant independent of $(h, \Delta t)$ such that

$$\begin{aligned}|(\tilde{u}(t_{n-1}).\nabla\xi_h^n| &\leq C\|u^{n-1}\|_1 C_3 h\|\sigma^{n-1}\|_2 \\ &\leq CC_3M^2h.\end{aligned}$$

Therefore,

$$|\varepsilon^n_{hu_h^{n-1},\lambda}| \leq C_2Mh^2 + \lambda\delta_0C_{25}(\mathcal{TC} + Mh^3 + M^2h).$$

Combining the last inequality we get the Lemma result.

To finish the proof of the theorem, substituting results of Lemma 4 to Lemma

7 into second member of the inequality (14), we obtain

$$\alpha \text{Re}\left[|e_h^m|^2 + \sum_{n=0}^{m-1} |e_h^{n+1} - e_h^n|^2\right] + (1/2)\sum_{n=0}^{m-1} \Delta t|\varepsilon_h^{n+1}|^2$$

$$+\frac{\lambda}{2}\left[|\varepsilon^m_{hu_h^n,\lambda}|^2 + \sum_{n=0}^{m-1} |\varepsilon^{n+1}_{hu_h^n,\lambda} - \varepsilon^n_{hu_h^n,\lambda}|^2\right] + 2\alpha(1-\alpha)\sum_{n=0}^{m-1} \Delta t|D(e_h^{n+1})|^2$$

$$+(\delta_0/4)\sum_{n=0}^{m-1} \Delta t|\lambda u_h^n.\nabla\varepsilon_h^{n+1}|^2$$

$$\leq$$

$$\alpha \text{Re}\,(C_{12}\Delta t + C_{17}h^3)M \max_{0\leq n\leq m-1} |e_h^{n+1}| + \lambda C_{14} M\Delta t h^2 \max_{0\leq n\leq m-1} |\varepsilon^{n+1}_{hu_h^n,\lambda}|$$

$$+\left\{\lambda(1+\sqrt{\delta_0})\{C_9(\mathcal{TC})[M^2 + MC_0 + M^2\Delta t h^{-1}] + \lambda C_{10}(C_0\mathcal{TC})^2 h^{-1}\}\right.$$

$$+\lambda C_{11}[M\Delta t + \delta(h^{-1}(\mathcal{TC}) + M)](1+\sqrt{\delta_0}) + \alpha C_{13} M h^2 + C_{15}\delta M h(\mathcal{TC})$$

$$+C_{16}\{\delta\lambda^{-1}M\frac{h^2}{\sqrt{\delta_0}} + M\delta\sqrt{\delta_0}(\mathcal{TC}))\}C_{18}Mh^2\max(1,\sqrt{\delta_0}) + \lambda C_{23}[\Delta t M^2 + M(\mathcal{TC})]$$

$$\left.+\lambda C_{24}[\delta_0 M^2\Delta t + M\frac{h^2}{\sqrt{\delta_0}} + M^2 h\sqrt{\delta_0} + \delta_0 M h(\sqrt{h}(1+h) + \frac{(\mathcal{TC})}{\sqrt{h}})]\right\}\times$$

$$\left[\left(\alpha(1-\alpha)\sum_{n=0}^{m-1}\Delta t|D(e_h^{n+1})|^2\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{m-1}\Delta t|\varepsilon_h^{n+1}|^2\right)^{\frac{1}{2}} + \left(\sum_{n=0}^{m-1}\Delta t\delta_0|\lambda u_h^n.\nabla\varepsilon_h^{n+1}|^2\right)^{\frac{1}{2}}\right]. \tag{28}$$

From the Lemma 8 and using the Young's inequality, we get the existence of $\varrho > 0$ and C_{26} constant independent of $(h, \Delta t)$ satisfying the following,

$$\alpha \text{Re}\left[|e_h^m|^2 + \sum_{n=0}^{m-1} |e_h^{n+1} - e_h^n|^2\right] + (1/2)\sum_{n=0}^{m-1} \Delta t|\varepsilon_h^{n+1}|^2$$
$$+\frac{\lambda}{2}\left[|\varepsilon^m_{hu_h^n,\lambda}|^2 + \sum_{n=0}^{m-1} |\varepsilon^{n+1}_{hu_h^n,\lambda} - \varepsilon^n_{hu_h^n,\lambda}|^2\right] + 2\alpha(1-\alpha)\sum_{n=0}^{m-1} \Delta t|D(e_h^{n+1})|^2$$
$$+(\delta_0/4)\sum_{n=0}^{m-1} \Delta t|\lambda u_h^n.\nabla\varepsilon_h^{n+1}|^2$$

$$\leq \frac{C_{26}^2}{\varrho}\left\{\alpha \mathrm{Re}\,(\Delta t+h^3)M[\mathcal{TC}+Mh^3+(\Delta t+h^3)M]+\lambda M\Delta t h^2[(1+\Delta t)Mh^2\right.$$

$$+\lambda\delta_0(\mathcal{TC}+Mh^3+M^2h)+M\Delta t h^2]+\left\{\lambda(1+\sqrt{\delta_0})[(\mathcal{TC})[M^2+MC_0+M^2\Delta t h^{-1}]\right.$$

$$+\lambda(C_0\mathcal{TC})^2h^{-1}]+\lambda[M\Delta t+\delta(h^{-1}(\mathcal{TC})+M)](1+\sqrt{\delta_0})+\alpha Mh^2+\delta Mh(\mathcal{TC})$$

$$+[\delta\lambda^{-1}M\frac{h^2}{\sqrt{\delta_0}}+M\delta\sqrt{\delta_0}(\mathcal{TC}))]Mh^2\max(1,\sqrt{\delta_0})+\lambda(\Delta t M^2+M(\mathcal{TC}))$$

$$\left.\left.+\lambda[\delta_0M^2\Delta t+M\frac{h^2}{\sqrt{\delta_0}}+M^2h\sqrt{\delta_0}+\delta_0Mh(\sqrt{h}(1+h)+\frac{(\mathcal{TC})}{\sqrt{h}})]\right\}\right\}^2$$

$$+\varrho\left\{\alpha\mathrm{Re}\,|e_h^m|^2+\lambda|\varepsilon^m_{hu_h^n,\lambda}|^2+\sum_{n=0}^{m-1}\Delta t[\alpha(1-\alpha)|D(e_h^{n+1})|^2+|\varepsilon_h^{n+1}|^2+\delta_0|\lambda u_h^n.\nabla\varepsilon_h^{n+1}|^2\right\}.$$

Hence we obtain,

$$|e_h^m|^2+|\varepsilon^m_{hu_h^n,\lambda}|^2+\sum_{n=0}^{m-1}\Delta t[|\varepsilon_h^{n+1}|^2+|D(e_h^{n+1})|^2+\delta_0|\lambda u_h^n.\nabla\varepsilon_h^{n+1}|^2]$$

$$\leq C_{27}^2\left\{\alpha\mathrm{Re}\,(1+h)M[\mathcal{TC}+Mh^3+(\Delta t+h^3)M]\right.$$

$$+\lambda Mh^2[(1+\Delta t)Mh^2+\lambda\delta_0(\mathcal{TC}+Mh^3+M^2h)+M\Delta t h^2]$$

$$+\left\{\lambda(1+\sqrt{\delta_0})[(M^2+MC_0+M^2\Delta t h^{-1})+\lambda C_0^2(\mathcal{TC})h^{-1}]\right. \tag{29}$$

$$+\lambda[M+(h^{-1}(\mathcal{TC})+M)](1+\sqrt{\delta_0})+\alpha M+Mh(\mathcal{TC})$$

$$+[\lambda^{-1}M\frac{h^2}{\sqrt{\delta_0}}+M\sqrt{\delta_0}(\mathcal{TC}))]Mh^2\max(1,\sqrt{\delta_0})+\lambda M(M+1)$$

$$\left.\left.+\lambda[\delta_0M^2+M(1+M)+M\sqrt{\delta_0}(\sqrt{h}(1+h)+\frac{(\mathcal{TC})}{\sqrt{h}})]\right\}\right\}^2(\mathcal{TC})^2,$$

where $C_{27}=\dfrac{C_{26}}{\sqrt{\varrho(1-\varrho)\min\{\alpha\mathrm{Re}\,,(\lambda/2),\alpha(1-\alpha)\}}}$. Writing

$$|u_h^m-u^m|\leq|\eta_h^m|+|e_h^m| \text{ and } |D(u_h^n-u^n)|\leq|D(\eta_h^n)|+|D(e_h^n)|,$$

we can see from the theorem hypothesis, inequality (2) and (29) that there exists a constant C_{28} independent of $(h, \Delta t)$ such that,

$$
\begin{aligned}
&|u_h^m - u^m|^2 + \sum_{n=0}^{m-1} \Delta t |D(u_h^n - u^n)|^2 \\
&\leq C_{28}^2 \left[\left\{ \alpha \mathrm{Re}\,(1+h)M \left[\mathcal{TC} + Mh^3 + (\Delta t + h^3)M \right] \right.\right. \\
&\quad + \lambda M h^2 \left[(1+\Delta t)Mh^2 + \lambda\delta_0(\mathcal{TC} + Mh^3 + M^2 h) + M\Delta t h^2 \right] \\
&+ \left\{ \lambda(1+\sqrt{\delta_0}) \left[(M^2 + MC_0 + M^2\Delta t h^{-1}) + \lambda(\mathcal{TC})h^{-1}) \right] \right. \\
&\quad + \lambda(M + (\mathcal{TC})h^{-1})(1+\sqrt{\delta_0}) + \alpha M + Mh(\mathcal{TC}) \\
&\quad + \left[\lambda^{-1} M \frac{h^2}{\sqrt{\delta_0}} + M\sqrt{\delta_0}(\mathcal{TC})) \right] Mh^2 \max(1, \sqrt{\delta_0}) + \lambda M(M+1) \\
&\quad \left.\left.\left. + \lambda \left[\delta_0 M^2 + M(1+M) + M\sqrt{\delta_0}(\sqrt{h}(1+h) + \frac{(\mathcal{TC})}{\sqrt{h}}) \right] \right\} \right\}^2 + M^2(h^2+1) \right] (\mathcal{TC})^2.
\end{aligned}
$$

Now for a given C_0, one can choose M, h small enough to have that

$$
\begin{aligned}
&C_{28}^2[\{\alpha \mathrm{Re}\,(1+h)M[\mathcal{TC} + Mh^3 + (\Delta t + h^3)M] + \lambda M h^2[(1+\Delta t)Mh^2 + \\
&\lambda\delta_0(\mathcal{TC} + Mh^3 + M^2 h) + M\Delta t h^2] + \{\lambda(1+\sqrt{\delta_0})[(M^2 + MC_0 + M^2\Delta t h^{-1}) + \\
&\lambda(\mathcal{TC})h^{-1})] + \lambda(M + (\mathcal{TC})h^{-1})(1+\sqrt{\delta_0}) + \alpha M + Mh(\mathcal{TC}) + [\lambda^{-1} M \frac{h^2}{\sqrt{\delta_0}} + \\
&M\sqrt{\delta_0}(\mathcal{TC}))]Mh^2 \max(1, \sqrt{\delta_0}) + \lambda M(M+1) + \lambda[\delta_0 M^2 + M(1+M) + M\sqrt{\delta_0}(\sqrt{h}(1+ \\
&h) + \frac{(\mathcal{TC})}{\sqrt{h}})]\}\}^2 + M^2(h^2+1)] \\
&\leq C_0^2,
\end{aligned}
$$

so one can choose M_0, h_0 small enough to ensure that for $M \leq M_0$ and $h \leq h_0$, the last inequality holds and therefore,

$$
|u_h^m - u^m| + \left(\sum_{n=0}^{m-1} \Delta t |D(u_h^n - u^n)|^2 \right)^{\frac{1}{2}} \leq C_0(\mathcal{TC}).
$$

A device similar to the one used to establish the last Lemma yields:

$$
|\xi^m_{h,u_h^{m-1}}| \leq C_3 M h^2 [1 + \lambda(M + (\mathcal{TC})],
$$

hence writing $|(\sigma_h^m - \sigma(t_m))_{u_h^{m-1},\lambda}| \leq |\xi^m_{h,u_h^{m-1}}| + |\varepsilon^m_{h,u_h^{m-1}}|$, we can choose M_0, h_0 small enough to ensure that for $M \leq M_0$ and $h \leq h_0$, we have

$$\left|(\sigma_h^m - \sigma(t_m))_{u_h^{m-1},\lambda}\right| + \left(\sum_{n=0}^{n=m-1} \Delta t |\sigma_h^m - \sigma(t_m)|^2\right)^{\frac{1}{2}} \leq C_0(\mathcal{TC}).$$

This prove that for $M \leq M_0$, $h \leq h_0$: $(\sigma_h^n, u_h^n)_{0\leq n\leq m} \in B^m_{h,\Delta t}$ and consequently $\forall m/0 \leq m \leq N$, $(\sigma_h^n, u_h^n)_{0\leq n\leq m} \in B^m_{h,\Delta t}$. Therefore, the induction hypotheses hold for all $0 \leq m \leq N$, which gives the theorem result.

6. Conclusion

With the judicious choice of stabilization coefficients, as presented in Remark 3, our method is as precise as the DG one, as showed by Baranger et al [2]. When the time discretization constraint is concerned, our scheme is found to be less restrictive.

The proof given here can be extended:

— To the more realistic rheological PTT model.

— To a quadrilateral FE approximation, following Baranger et al [1] and to the higher finite element methods($P_k, k \geq 1$).

— The use of an uncoupled fractional step scheme would be cheaper, following Esselaoui et al [10]. The numerical analysis of such a method is currently in progress.

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