International Journal of Pure and Applied Mathematics
Volume 21 No. 3 2005, 75-84

ON HIGHER ORDER LIFTS ON EXTENDED JET BUNDLES

Cansel Aycan Erke

Abstract: The aim of this paper is build up the extension of jet bundles. Besides, by using lift method, we extend the basic geometric structures on the first order jet bundle $J^k\pi$ to m-th extended jet bundle $J^k\pi^m$.

AMS Subject Classification: 28A
Key Words: extended jet bundle, jet vector field, jet 1-form, vertical lift, complete lift

1. Introduction and Notations

The theory of Jet manifolds is an important topic in modern differential geometry. Besides the lift method has an important role in differential geometry, since it permits to extend differentiable structures. As well, contributions to the study of lifts of differentiable structures on jet manifolds were provided in [1]. In the following, by using exact sequence of vector bundles,we recollect the extended jet bundle sequence and by using lifts, we shall extend the basic geometric structures of the first order jet bundle $J^k\pi$ to m-th extended jet bundle $J^k\pi^m$. Troughout the paper, all the mappings and manifolds will be assumed to be C^∞ and the sum will be taken over repeated indices. Also, v^m and c^m will denote the m-order vertical and complete lifts to $J^k\pi^m$ of geometric structures of $J^k\pi$, respectively.

First, we consider the following sequence of $C^\infty-$ manifolds on the manifold M , where ${}^0\pi$, ${}^1\pi$,${}^2\pi$ be $C^\infty-$ functions and ${}^0M$, ${}^1M$, ${}^2M$ be $C^\infty-$ manifolds:

Received: April 13, 2005

$$ {}^{0}M \underset{{}^{0}\pi}{\longleftarrow} {}^{1}M \underset{{}^{1}\pi}{\longleftarrow} {}^{2}M \underset{{}^{2}\pi}{\longleftarrow} \dots \tag{1.1} $$

The (1.1) sequence called an *exact sequence* of the manifold M; if the following sequences are the short exact sequences, namely, for
$\forall i \in \mathbb{Z}^{+}$, $D({}^{i-1}\pi) = R({}^{i}\pi)$,

$$ {}^{i-1}M \underset{{}^{i-1}\pi}{\longleftarrow} {}^{i}M \underset{{}^{i}\pi}{\longleftarrow} {}^{i+1}M $$

Definition 1.1. Let M be a manifold and the length of an exact sequence of M be $m \in \mathbb{N}$.

i) The diagram

$$ \begin{array}{ccccc} & {}^{k-1}\pi & & \\ {}^{k}M & \rightarrow & {}^{k-1}M & \\ {}^{k-1}I & \ddots & \uparrow & {}^{k-1}T_{\pi} \\ & & T({}^{k-1}M) & \end{array} $$

is commutative and ${}^{k-1}I : {}^{k}M \longrightarrow T({}^{k-1}M)$ be imbeddings
$(1 \leq k \leq m)$;

ii) For $1 \leq k \langle m$, the diagram

$$ \begin{array}{ccccc} & {}^{k-1}\pi_{*} & & \\ T({}^{k}M) & \rightarrow & T({}^{k-1}M) & \\ {}^{k}T_{\pi} & \ddots & \uparrow & {}^{k-1}I \\ & & {}^{k}M & \end{array} $$

is completely commutative on ${}^{k}I({}^{k+1}M)$,where ${}^{k}I : {}^{k+1}M \longrightarrow T({}^{k}M)$. So that, we called the sequence of manifolds $\left\{{}^{0}M, {}^{1}M, ..., {}^{k}M\right\}$, an *extended sequence* of the manifold M. The manifold ${}^{k}M$ called the *k. extension* of M.

When first manifold of the sequence accept ${}^{0}M = M$, we get to *the canonical extension* of M. The $(k+1)-$*canonical extension* of M is

$$ {}^{k+1}M = \left\{H : H \in T({}^{k}M),\ ({}^{k-1}\pi)_{*}(H) = {}^{k-1}I \circ {}^{k}T_{\pi}(H)\right\} \subset T({}^{k}M)\,. $$

The $(k+1)-$*lenght canonical extended sequence* of M is

$$ M = {}^{0}M \underset{{}^{0}\pi}{\longleftarrow} {}^{1}M \underset{{}^{1}\pi}{\longleftarrow} \dots \underset{{}^{k}\pi}{\longleftarrow} {}^{k+1}M \ . $$

Let (E, π, M) be a bundle and

$$ E = {}^{0}E \underset{{}^{0}\rho}{\longleftarrow} {}^{1}E \underset{{}^{1}\rho}{\longleftarrow} {}^{2}E \underset{{}^{2}\rho}{\longleftarrow} \dots $$

$$M = {}^0M \underset{{}^0\pi}{\longleftarrow} {}^1M \underset{{}^1\pi}{\longleftarrow} {}^2M \underset{{}^2\pi}{\longleftarrow} ...$$

be the canonical extended sequences of manifolds E, M, respectively. Therefore,

$$\pi^i \underset{({}^i\rho,{}^i\pi)}{\longleftarrow} \pi^{i+1} \underset{({}^{i+1}\rho,{}^{i+1}\pi)}{\longleftarrow} \pi^{i+2}$$

bundle sequence are a short exact sequence; where $\pi^i : {}^iE \longrightarrow {}^iM$ submersions, $\pi^i = ({}^iE, \pi^i, {}^iM)$ bundles, $({}^i\rho, {}^i\pi)$ bundle morphisms and $D({}^i\rho, {}^i\pi) = R({}^{i+1}\rho, {}^{i+1}\pi)$. Also the following sequence is an exact sequence of the π bundle.

$$\pi = \pi^0 \underset{({}^0\rho,{}^0\pi)}{\longleftarrow} \pi^1 \underset{({}^1\rho,{}^1\pi)}{\longleftarrow} {}^2\pi \underset{({}^2\rho,{}^2\pi)}{\longleftarrow} ... \tag{1.2}$$

When first manifolds of the sequence accept ${}^0M = M$ and ${}^0E = E$, namely ${}^0\pi = \pi$, we get to *the canonical extension* of π.

Definition 1.2. $\forall i \in \mathbb{Z}^+ \cup \{0\}$, $\pi^i = ({}^iE, \pi^i, {}^iM)$ be bundles and $J^k\pi^i$ be k. jet manifolds on π^i bundles. The sequence

$$\begin{array}{ccccccccc} J^k\pi & = & J^k\pi^0 \overset{{}^0h}{\longleftarrow} & J^k\pi^1 \overset{{}^1h}{\longleftarrow} & J^k\pi^2 \overset{{}^2h}{\longleftarrow} ... & \overset{{}^{k-1}h}{\longleftarrow} J^k\pi^k \longleftarrow ... \\ & & \downarrow \pi_k^0 & \downarrow \pi_k^1 & \downarrow \pi_k^2 & \downarrow \pi_k^k \\ & & {}^0M \overset{{}^0\pi}{\longleftarrow} & {}^1M \overset{{}^1\pi}{\longleftarrow} & {}^2M \overset{{}^2\pi}{\longleftarrow} ... & \overset{{}^{k-1}\pi}{\longleftarrow} {}^kM \longleftarrow ... \end{array}$$

is showed as following for shortness:

$$J^k\pi^0 \longleftarrow J^k\pi^1 \longleftarrow J^k\pi^2 \longleftarrow ...J^k\pi^k \longleftarrow ... \tag{1.3}$$

(1.3) sequence called as *the extended jet bundle sequence.*

At the same time, when we constitute k. jet manifolds on the extended sequence π^i, we get the extended jet sequence $J^k(\pi^i)$. The m-lenght extended sequence of the bundle $J^k\left(\pi^0\right) = J^k(\pi)$ is $(J^k\pi^0)^i$. Also the bundles $J^k(\pi^i)$ and $(J^k\pi^0)^i$ are equivalent. Troughout the paper, the extended jet bundle sequence showed as $J^k\pi^i$ (without parenthesis) and the m-order extension of the bundle $J^k(\pi)$ is $J^k\pi^m$. The local coordinates on the extended jet bundles are given with below table.

The extended jet bundles	Local coordinates	
$\pi^0 = \pi$	$(x^{0i}, u^{0\alpha})$	$(1 \leq i \leq m,\ 1 \leq \alpha \leq n)$
$J^k\pi^0$	$(x^{0i}, u^{0\alpha}, u^{0\alpha}_{ri})$	$(r = 1, 2, ..., k)$
$\pi^1 = T\pi$	$(x^{0i}, u^{0\alpha}, x^{1i}, u^{1\alpha})$	
$J^k\pi^1$	$(x^{0i}, u^{0\alpha}, x^{1i}, u^{1\alpha}, u^{0\alpha}_{ri}, u^{1\alpha}_{ri})$	
π^2	$(x^{0i}, u^{0\alpha}, x^{1i}, u^{1\alpha}, x^{2i}, u^{2\alpha})$	
$J^k\pi^2$	$(x^{0i}, u^{0\alpha}, x^{1i}, u^{1\alpha}, x^{2i}, u^{2\alpha}, u^{0\alpha}_{ri}, u^{1\alpha}_{ri}, u^{2\alpha}_{ri})$	
....	...	
π^m	$(x^{0i}, u^{0\alpha}, x^{1i}, u^{1\alpha}, ..., x^{mi}, u^{m\alpha})$	
$J^k\pi^m$	$(x^{0i}, u^{0\alpha}, x^{1i}, u^{1\alpha}, ..., x^{mi}, u^{m\alpha}, u^{0\alpha}_{ri}, u^{1\alpha}_{ri}, ..., u^{m\alpha}_{ri})$	

Table 1.1. The local coordinates of the extended jet bundles

2. The Higher Order Lifts on The Extended Jet Bundles

Let (E, π, M) be a bundle and $({}^mE, \pi^m, {}^mM)$ the m. extended bundle of π. Besides the jet bundle of π be $J^k\pi$ and the m. extended jet bundle of π^m be $J^k\pi^m$. The basic lifts of geometric objects from $J^k\pi$ to $J^k\pi^m$ are described below.

2.1. The Lift of Functions

The *m-order vertical lift* of a function $f \in \tau^0_0(J^k\pi) \equiv \cdot\cdot(J^k\pi)$ to $J^k\pi^m$ is the function $f^{v^m} \in \cdot\cdot(J^k\pi^m)$ given by

$$f^{v^m} = f^{v^{m-1}} \circ \tau_{J^k\pi^{m-1}} = f \circ \tau_{J^k\pi} \circ \tau_{J^k\pi^1} \circ .... \circ \tau_{J^k\pi^{m-1}}$$

where $\tau_{J^k\pi^{m-1}} : J^k\pi^m \longrightarrow J^k\pi^{m-1}$ is the canonical projection.

The *m-order complete lift* of the function $f \in \cdot\cdot(J^k\pi)$ to $J^k\pi^m$ is the function $f^{c^m} \in \cdot\cdot(J^k\pi^m)$ given by

$$\begin{aligned} f^{c^m} &= \imath_m(df^{c^{m-1}}) \\ &= \sum_{s=0}^{m-1}\left(x^{s+1i}\left(\frac{\partial f^{c^{m-1}}}{\partial x^{si}}\right)^v + u^{s+1\alpha}\left(\frac{\partial f^{c^{m-1}}}{\partial u^{s\alpha}}\right)^v + \sum_{r=1}^{k} u^{s+1\alpha}_{ri}\left(\frac{\partial f^{c^{m-1}}}{\partial u^{s\alpha}_{ri}}\right)^v\right) \end{aligned}$$

where $df^{c^{m-1}}$ is the differential of the function $f^{c^{m-1}}$ on $J^k\pi^{m-1}$ jet bundle and $\imath_m$ is the linear map $\imath_m : \ \chi^*(J^k\pi^{m-1}) \longrightarrow \cdot\cdot(J^k\pi^m)$. Here we denote by $(x^{s+1i}, u^{s+1\alpha}, u^{s+1\alpha}_{ri})$ the local coordinates of $J^k\pi^m$ jet bundle and $\imath_m(dx^{si}) = x^{s+1i}$, $\imath_m(du^{s\alpha}) = u^{s+1\alpha}$, $\imath_m(du^{s\alpha}_{ri}) = u^{s+1\alpha}_{ri}$.

2.2. The Lift of Vector Fields

The *m-order vertical lift* of a vector field $X \in \chi(J^k\pi)$ to $J^k\pi^m$ is the vector field $X^{v^m} \in \chi(J^k\pi^m)$ given by

$$X^{v^m}\left(f^{c^m}\right) = (Xf)^{v^m}, \forall f \in \cdot\cdot(J^k\pi),$$

and the *m-order complete lift* of vector field $X \in \chi(J^k\pi)$ to $J^k\pi^m$ is the vector field $X^{c^m} \in \chi(J^k\pi^m)$ given by

$$X^{c^m}\left(f^{c^m}\right) = (Xf)^{c^m}, \forall f \in \cdot\cdot(J^k\pi).$$

In the following are described m-order vertical and complete lifts of a jet vector field on $J^k\pi$ jet bundle via the basic lift properties.

Proposition 1. Where a jet vector field $X \in \chi(J^k\pi)$ is given by

$$X = X^{0i}\frac{\partial}{\partial x^{0i}} + X^{0\alpha}\frac{\partial}{\partial u^{0\alpha}} + \sum_{r=1}^{k} X_{ri}^{0\alpha}\frac{\partial}{\partial u_{ri}^{0\alpha}}, \tag{2.1}$$

the m-order vertical lift of $X \in \chi(J^k\pi)$ to $J^k\pi^m$ jet vector bundle is

$$X^{v^m} = (X^{0i})^{v^m}\frac{\partial}{\partial x^{mi}} + \left(X^{0\alpha}\right)^{v^m}\frac{\partial}{\partial u^{m\alpha}} + \sum_{r=1}^{k}(X_{ri}^{0\alpha})^{v^m}\frac{\partial}{\partial u_{ri}^{m\alpha}} \ .$$

Proof. For $\forall f \in \cdot\cdot(J^k\pi)$,

$$\begin{aligned}
(Xf)^{v^m} &= (X^{0i}\frac{\partial f}{\partial x^{0i}} + X^{0\alpha}\frac{\partial f}{\partial u^{0\alpha}} + \sum_{r=1}^{k} X_{ri}^{0\alpha}\frac{\partial f}{\partial u_{ri}^{0\alpha}})^{v^m} \\
&= (X^{0i})^{v^m}(\frac{\partial}{\partial x^{0i}})^{v^m}(f)^{c^m} + (X^{0\alpha})^{v^m}(\frac{\partial}{\partial u^{0\alpha}})^{v^m}(f)^{c^m} \\
&\quad + \sum_{r=1}^{k}(X_{ri}^{0\alpha})^{v^m}(\frac{\partial}{\partial u_{ri}^{0\alpha}})^{v^m}(f)^{c^m} \\
&= \left((X^{0i})^{v^m}\frac{\partial}{\partial x^{mi}} + (X^{0\alpha})^{v^m}\frac{\partial}{\partial u^{m\alpha}} + \sum_{r=1}^{k}(X_{ri}^{0\alpha})^{v^m}\frac{\partial}{\partial u_{ri}^{m\alpha}}\right)(f)^{c^m}
\end{aligned}$$

and $X^{v^m}\left(f^{c^m}\right) = (Xf)^{v^m}, \forall f \in \cdot\cdot(J^k\pi)$ is providing,

$$X^{v^m} = (X^{0i})^{v^m}\frac{\partial}{\partial x^{mi}} + \left(X^{0\alpha}\right)^{v^m}\frac{\partial}{\partial u^{m\alpha}} + \sum_{r=1}^{k}(X_{ri}^{0\alpha})^{v^m}\frac{\partial}{\partial u_{ri}^{m\alpha}} \ .$$

Corollary 2. Let be $Sp\left\{ \frac{\partial}{\partial x^{0i}}, \frac{\partial}{\partial u^{0\alpha}}, \frac{\partial}{\partial u^{0\alpha}_{ri}} : 1 \le r \le k \right\} = \chi(J^k\pi)$, also

$$\left(\frac{\partial}{\partial x^{0i}}\right)^{v^m} = \frac{\partial}{\partial x^{mi}}, \quad \left(\frac{\partial}{\partial u^{0\alpha}}\right)^{v^m} = \frac{\partial}{\partial u^{m\alpha}}, \quad \left(\frac{\partial}{\partial u^{0\alpha}_{ri}}\right)^{v^m} = \frac{\partial}{\partial u^{m\alpha}_{ri}}$$

and $Sp\left\{ \frac{\partial}{\partial x^{si}}, \frac{\partial}{\partial u^{s\alpha}}, \frac{\partial}{\partial u^{s\alpha}_{ri}} : 0 \le s \le m,\ 1 \le r \le k \right\} = \chi(J^k\pi^m)$.

Similarly, one may prove

Proposition 2. Where a jet vector field $X \in \chi(J^k\pi)$ is given by (2.1), the m-order complete lift of $X \in \chi(J^k\pi)$ to $J^k\pi^m$ jet vector bundle is

$$\begin{aligned} X^{c^m} = & \sum_{s=0}^{m} \Big\{ \tbinom{m}{s} (X^{0i})^{v^{m-s}c^s} \frac{\partial}{\partial x^{si}} + \binom{m}{s} \left(X^{0\alpha}\right)^{v^{m-s}c^s} \frac{\partial}{\partial u^{s\alpha}} \\ & + \sum_{r=1}^{k} \tbinom{m}{s} \left(X^{0\alpha}_{ri}\right)^{v^{m-s}c^s} \frac{\partial}{\partial u^{s\alpha}_{ri}} \Big\}. \end{aligned}$$

Proof. For $\forall f \in \cdot\cdot (J^k\pi)$,

$$\begin{aligned} (Xf)^{c^m} &= (X^{0i}\frac{\partial f}{\partial x^{0i}} + X^{0\alpha}\frac{\partial f}{\partial u^{0\alpha}} + \sum_{r=1}^{k} X^{0\alpha}_{ri}\frac{\partial f}{\partial u^{0\alpha}_{ri}})^{c^m} \\ &= \sum_{s=0}^{m} \tbinom{m}{s} \Big\{ (X^{0i})^{v^{m-s}c^s} (\frac{\partial f}{\partial x^{0i}})^{v^s c^{m-s}} + (X^{0\alpha})^{v^{m-s}c^s} (\frac{\partial f}{\partial u^{0\alpha}})^{v^s c^{m-s}} \\ &\quad + \sum_{r=1}^{k} (X^{0\alpha}_{ri})^{v^{m-s}c^s} (\frac{\partial f}{\partial u^{0\alpha}_{ri}})^{v^s c^{m-s}} \Big\} \\ &= \sum_{s=0}^{m} \tbinom{m}{s} \Big\{ (X^{0i})^{v^{m-s}c^s} (\frac{\partial f^{c^m}}{\partial x^{si}}) + (X^{0\alpha})^{v^{m-s}c^s} (\frac{\partial f^{c^m}}{\partial u^{s\alpha}}) \\ &\quad + \sum_{r=1}^{k} (X^{0\alpha}_{ri})^{v^{m-s}c^s} (\frac{\partial f^{c^m}}{\partial u^{s\alpha}_{ri}}) \Big\} \\ &= \sum_{s=0}^{m} \tbinom{m}{s} \Big\{ (X^{0i})^{v^{m-s}c^s} \frac{\partial}{\partial x^{si}} + (X^{0\alpha})^{v^{m-s}c^s} \frac{\partial}{\partial u^{s\alpha}} \\ &\quad + \sum_{r=1}^{k} (X^{0\alpha}_{ri})^{v^{m-s}c^s} \frac{\partial}{\partial u^{s\alpha}_{ri}} \Big\} (f)^{c^m} \end{aligned}$$

and $X^{c^m}\left(f^{c^m}\right) = (Xf)^{c^m}, \forall f \in \cdot\cdot (J^k\pi)$ is providing,

$$\begin{aligned} X^{c^m} = & \sum_{s=0}^{m} \Big\{ \tbinom{m}{s} (X^{0i})^{v^{m-s}c^s} \frac{\partial}{\partial x^{si}} + \binom{m}{s} \left(X^{0\alpha}\right)^{v^{m-s}c^s} \frac{\partial}{\partial u^{s\alpha}} \\ & + \sum_{r=1}^{k} \tbinom{m}{s} \left(X^{0\alpha}_{ri}\right)^{v^{m-s}c^s} \frac{\partial}{\partial u^{s\alpha}_{ri}} \Big\}. \end{aligned}$$

Corollary 3. Let be $Sp\left\{\frac{\partial}{\partial x^{0i}}, \frac{\partial}{\partial u^{0\alpha}}, \frac{\partial}{\partial u^{0\alpha}_{ri}} : 1 \leq r \leq k\right\} = \chi(J^k\pi^m)$, also

$$\left(\frac{\partial}{\partial x^{0i}}\right)^{c^{m-s}v^s} = \frac{\partial}{\partial x^{si}}, \ \left(\frac{\partial}{\partial u^{0\alpha}}\right)^{c^{m-s}v^s} = \frac{\partial}{\partial u^{s\alpha}}, \ \left(\frac{\partial}{\partial u^{0\alpha}_{ri}}\right)^{c^{m-s}v^s} = \frac{\partial}{\partial u^{s\alpha}_{ri}}$$

and $Sp\left\{\frac{\partial}{\partial x^{si}}, \frac{\partial}{\partial u^{s\alpha}}, \frac{\partial}{\partial u^{s\alpha}_{ri}} : 0 \leq s \leq m, \ 1 \leq r \leq k\right\} = \chi(J^k\pi^m)$.

2.3. The Lift of 1-Forms

The *m-order vertical lift* of a 1-form $\omega \in \chi^*(J^k\pi)$ to $J^k\pi^m$ is the 1-form $\omega^{v^m} \in \chi^*(J^k\pi^m)$ given by

$$\omega^{v^m}(X^{c^m}) = (\omega X)^{v^m}, \forall X \in \chi(J^k\pi)$$

and the *m-order complete lift* of a 1-form $\omega \in \chi^*(J^k\pi)$ to $J^k\pi^m$ is the 1-form $\omega^{c^m} \in \chi^*(J^k\pi^m)$ given by

$$\omega^{c^m}\left(X^{c^m}\right) = (\omega X)^{c^m}, \forall X \in \chi(J^k\pi) \ .$$

Analogously, m-order vertical and complete lifts of a jet 1-form on $J^k\pi$ jet bundle can prove the following

Proposition 3. Where a jet 1-form $\omega \in \chi^*(J^k\pi)$ is given by

$$\omega = \omega_{0i}dx^{0i} + \omega_{0\alpha}du^{0\alpha} + \sum_{r=1}^{k}\omega^{ri}_{0\alpha}du^{0\alpha}_{ri} \ , \tag{2.2}$$

the m-order vertical lift of $\omega \in \chi^*(J^k\pi)$ to $J^k\pi^m$ jet vector bundle is

$$\omega^{v^m} = (\omega_{0i})^{v^m} dx^{0i} + (\omega_{0\alpha})^{v^m} du^{0\alpha} + \sum_{r=1}^{k}(\omega^{ri}_{0\alpha})^{v^m} du^{0\alpha}_{ri} \ .$$

Proof. Where a jet vector field $X \in \chi(J^k\pi)$ is given by (2.1), also

$$
\begin{aligned}
(\omega X)^{v^m} &= \Big[(\omega_{0i}dx^{0i} + \omega_{0\alpha}du^{0\alpha} + \sum_{r=1}^{k} \omega_{0\alpha}^{ri} du_{ri}^{0\alpha}) \\
&\quad (X^{0i}\frac{\partial}{\partial x^{0i}} + X^{0\alpha}\frac{\partial}{\partial u^{0\alpha}} + \sum_{r=1}^{k} X_{ri}^{0\alpha}\frac{\partial}{\partial u_{ri}^{0\alpha}})\Big]^{v^m} \\
&= (\omega_{0i}X^{0i}\frac{\partial}{\partial x^{0i}}\left[x^{0i}\right])^{v^m} + (\omega_{0\alpha}X^{0\alpha}\frac{\partial}{\partial u^{0\alpha}}\left[u^{0\alpha}\right])^{v^m} \\
&\quad + \sum_{r=1}^{k}(\omega_{0\alpha}^{ri}X_{ri}^{0\alpha}\frac{\partial}{\partial u_{ri}^{0\alpha}}\left[u_{ri}^{0\alpha}\right])^{v^m}
\end{aligned}
$$

$$
\begin{aligned}
&= \Bigg[\Big((\omega_{0i})^{v^m} dx^{0i} + (\omega_{0\alpha})^{v^m} du^{0\alpha} + \textstyle\sum_{r=1}^{k}(\omega_{0\alpha}^{ri})^{v^m} du_{ri}^{0\alpha}\Big) \\
&\quad \left(\sum_{s=0}^{m}\left(\tbinom{m}{s}(X^{0i})^{v^{m-s}c^s}\frac{\partial}{\partial x^{si}} + \binom{m}{s}\left(X^{0\alpha}\right)^{v^{m-s}c^s}\frac{\partial}{\partial u^{s\alpha}}\right.\right. \\
&\quad \left.\left. + \sum_{r=1}^{k}\tbinom{m}{s}\left(X_{ri}^{0\alpha}\right)^{v^{m-s}c^s}\frac{\partial}{\partial u_{ri}^{s\alpha}}\right)\right)\Bigg] \\
&= \left((\omega_{0i})^{v^m} dx^{0i} + (\omega_{0\alpha})^{v^m} du^{0\alpha} + \sum_{r=1}^{k}(\omega_{0\alpha}^{ri})^{v^m} du_{ri}^{0\alpha}\right)(X)^{c^m}
\end{aligned}
$$

and $\omega^{v^m}(X^{c^m}) = (\omega X)^{v^m}, \forall X \in \chi(J^k\pi)$ is providing,

$$
\omega^{v^m} = (\omega_{0i})^{v^m} dx^{0i} + (\omega_{0\alpha})^{v^m} du^{0\alpha} + \sum_{r=1}^{k}(\omega_{0\alpha}^{ri})^{v^m} du_{ri}^{0\alpha} \quad .
$$

Proposition 4. Where a jet 1-form $\omega \in \chi^*(J^k\pi)$ is given by (2.2), the m-order complete lift of $\omega \in \chi^*(J^k\pi)$ to $J^k\pi^m$ jet vector bundle is

$$
\begin{aligned}
\omega^{c^m} &= \sum_{s=0}^{m}\tbinom{m}{s}(\omega_{0i})^{v^{m-s}c^s}(dx^{0i})^{c^{m-s}v^s} + \tbinom{m}{s}(\omega_{0\alpha})^{v^{m-s}c^s}\left(du^{0\alpha}\right)^{c^{m-s}v^s} \\
&\quad + \tbinom{m}{s}(\omega_{0\alpha}^{ri})^{v^{m-s}c^s}(du_{ri}^{0\alpha})^{c^{m-s}v^s} \quad .
\end{aligned}
$$

Proof. Similar to the proof of Proposition 2 and Proposition 3.

Corollary 4. Let be $Sp\left\{dx^{0i}, du^{0\alpha}, du_{ri}^{0\alpha} : 1 \le r \le k\right\} = \chi^*(J^k\pi)$,also

$$
(dx^{0i})^{v^m} = dx^{0i}, \left(du^{0\alpha}\right)^{v^m} = du^{0\alpha}, (du_{ri}^{0\alpha})^{v^m} = du_{ri}^{0\alpha}
$$

and $Sp\left\{dx^{si}, du^{s\alpha}, du_{ri}^{s\alpha} : 0 \le s \le m,\ 1 \le r \le k\right\} = \chi^*(J^k\pi^m)$.

Corollary 5. Let be $Sp\left\{dx^{0i}, du^{0\alpha}, du^{0\alpha}_{ri} : 1 \leq r \leq k\right\} = \chi^*(J^k\pi)$,also

$$(dx^{0i})^{v^{m-s}c^s} = dx^{si},\ \left(du^{0\alpha}\right)^{v^{m-s}c^s} = du^{s\alpha},\ (du^{0\alpha}_{ri})^{v^{m-s}c^s} = du^{s\alpha}_{ri}$$

and

$$Sp\left\{dx^{si}, du^{s\alpha}, du^{s\alpha}_{ri}\ :\ 0 \leq s \leq m, 1 \leq r \leq k\right\} = \chi^*(J^k\pi^m)\ .$$

2.4. The Lift of Tensor Fields of Type (1,1)

The *m-order vertical lift* of jet tensor field $F \in \tau^1_1(J^k\pi)$ to $J^k\pi^m$ is the jet tensor field $F^{v^m} \in \tau^1_1(J^k\pi^m)$ given by

$$F^{v^m}(Z^{c^m}) = (FZ)^{v^m},\ \forall Z \in \chi(J^k\pi)$$

and the *m-order complete lift* of jet tensor field $F \in \tau^1_1(J^k\pi)$ to $J^k\pi^m$ is the jet tensor field $F^{c^m} \in \tau^1_1(J^k\pi^m)$ given by

$$F^{c^m}(Z^{c^m}) = (FZ)^{c^m},\ \forall Z \in \chi(J^k\pi).$$

References

[1] C. Aycan, *Lifts on Jet Bundles and Prolongation Relations*, M.Sc. Thesis, Pamukkale Univ., Denizli (1998).

[2] C. Aycan, *The Lifts of Euler-Lagrange and Hamilton Equations on the Extended Jet Bundles*, D.Sc. Thesis, Osmangazi Univ., Eskişehir, (2003).

[3] C. Aycan, Ş. Civelek, The lifts of jet vector fields, *Algebras Groups and Geometries*, To Appear.

[4] R.H. Bowman, On differentiable extensions, *Tensor N.S.*, **21** (1970), 139-150.

[5] L.A. Cordero, M. De Leon, Lifts of tensor fields to the frame bundle, *Rend. Circ. Mat. Palermo*, **32**, No. 2 (1983), 236-271.

[6] Ş. Civelek , The extended vector bundles, *J. Geom.*, **62** (1998), 40-47.

[7] M. De Leon, M. Salgodo, Diagonal lifts of tensor fields to the frame bundle of second order, *Acta Sci. Math.*, **50** (1986), 67-86.

[8] D.J. Saunders, *The Geometry of Jet Bundles*, Cambridge Univ. Press., Cambridge (1989).

[9] K. Yano, S. Ishihara, *Tangent and Cotangent Bundles*, Marcel Dekker Inc., New York (1973).