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CAUCHY ESTIMATES FOR THE OPERATIONAL CALCULUS

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Abstract: Let T be a (linear, not necessarily bounded) operator on a Banach space X, whose resolvent contains the open unit disc Ω (or the set $\{z; |z| > 1\}$), and whose resolvent operator $R(\cdot)$ satisfies the inequality $||(1-|z|)R(z)|| \leq M$ for all $z \in \Omega$ (or $|z| > 1$, respectively). We show that $||[(1-|z|)R(z)]^n|| < Men$ for all $n \in \mathbb{N}$ and $z \in \Omega$ ($|z| > 1$, respectively). In case X is Hilbert space, and T is a contraction satisfying $||(1-|z|)R(z)|| \leq M$ in Ω, one has $||[(1-|z|)R(z)]^n|| = O(1)$ for all n and $|z| \neq 1$. These resolvent estimates imply Cauchy-type estimates for $f^{(n)}(T)$.

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Let T be a (linear, not necessarily bounded) operator on the Banach space X. Denote $R(z) := R(z;T) := (zI-T)^{-1}$ and $\Omega = \{z \in \mathbb{C}; |z| < 1\}$.

Theorem 1. *(a) Suppose the resolvent set $\rho(T)$ contains Ω, and*

$$||(1-|z|)R(z)|| \leq M \qquad (z \in \Omega). \tag{1}$$

Then

$$||[(1-|z|)R(z)]^n|| < Men \qquad (z \in \Omega; n \in \mathbb{N}). \tag{2}$$

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(b) Similarly, if $\rho(T)$ contains the set $\{z \in \mathbb{C}; |z| > 1\}$ and T satisfies (1) in $\{z; |z| > 1\}$, then (2) is valid for all $n \in \mathbb{N}$ and $|z| > 1$.

(c) If X is a Hilbert space and T is a contraction satisfying (1) in Ω, then

$$||[(1-|z|)R(z)]^n|| = O(1) \qquad (|z| \neq 1; n \in \mathbb{N}). \tag{3}$$

Proof. (a) Fix $n \in \mathbb{N}, n \geq 2$, $z \in \Omega$, and $0 < t < 1$. The (positively oriented) circle Γ_t with centre at z and radius $t(1-|z|)$ and its interior are contained in Ω. By the Cauchy formulas for the derivatives of the analytic function $R(\cdot)$ in Ω, we have

$$R(z)^n = (-1)^{n-1}\frac{R(z)^{(n-1)}}{(n-1)!} = \frac{(-1)^{n-1}}{2\pi i}\int_{\Gamma_t}\frac{R(\zeta)}{(\zeta - z)^n}d\zeta. \tag{4}$$

For $\zeta \in \Gamma_t$, $||R(\zeta)|| \leq M/(1-|\zeta|)$ by (1), and

$$1 - |\zeta| \geq 1 - [|z| + t(1-|z|)] = (1-t)(1-|z|).$$

Therefore, by (4),

$$||R(z)^n|| \leq \frac{M}{(1-|z|)^n(1-t)t^{n-1}}. \tag{5}$$

We choose $t = 1 - (1/n)$ in (5), in order to maximize $(1-t)t^{n-1}$. This yields the estimate

$$||[(1-|z|)R(z)]^n|| \leq \frac{Mn}{[1-(1/n)]^{n-1}}, \tag{6}$$

and (2) follows from (6) and the elementary inequality $[1-(1/n)]^{n-1} > 1/e$ (for $n \geq 2$).

(b) Analogous calculations yield Part (b).

(c) In case X is a Hilbert space and T is a contraction satisfying (1), then by a theorem of Gohberg and Krein (cf. [3], p. 20), T is similar to a unitary operator U. Let then Q be a bounded non-singular operator such that $T = Q^{-1}UQ$, and let E be the resolution of the identity for U. Then for $|z| \neq 1$,

$$R(z)^n = Q^{-1}\int_{\Gamma}\frac{E(d\zeta)}{(z-\zeta)^n}Q, \tag{7}$$

where Γ denotes the unit circle.

For all $x, y \in X$, the complex Borel measure $(E(\cdot)x, y)$ on Γ has total variation $\leq ||x||\,||y||$. Since $|z - \zeta| \geq \Big|1 - |z|\Big|$ for $\zeta \in \Gamma$, we obtain from (7)

$$||[(1-|z|)R(z)]^n|| \leq ||Q^{-1}||||Q||$$

for all z, $|z| \neq 1$, and $n \in \mathbb{N}$, as wanted. □

Corollary 2. Let T be a contraction in the Banach space X, such that $||(|z|-1)R(z)|| \leq M$ for $|z| > 1$. Let $R > 1$. Suppose f is analytic and $|f(z)| \leq K$ in $|z| < R$. Then

$$||f^{(n)}(T)|| \leq MKe\frac{(n+1)!R}{(R-1)^{n+1}}. \tag{8}$$

If X is a Hilbert space and the contraction T has spectrum on the unit circle and satisfies (1), then one has the stronger estimate

$$||f^{(n)}(T)|| \leq \text{const.}\frac{n!R}{(R-1)^{n+1}}. \tag{9}$$

Proof. Let $1 < r < R$. We have

$$f^{(n)}(T) = \frac{n!}{2\pi i}\int_{|\zeta|=r} f(\zeta)R(\zeta)^{n+1}d\zeta \tag{10}$$

(cf. [1], p. 591). By Theorem 1, the integrand has norm $\leq MKe\frac{n+1}{(r-1)^{n+1}}$ on $|\zeta| = r$. Therefore $||f^{(n)}(T)|| \leq MKe\frac{(n+1)!r}{(r-1)^{n+1}}$, and (8) follows by letting $r \to R$.

If X is a Hilbert space, and T is a contraction with spectrum on the unit circle (and satisfies (1)), it follows from (10) and Theorem 1(c) that $||f^{(n)}(T)|| \leq \text{const.}\frac{n!r}{(r-1)^{n+1}}$, and (9) follows by letting $r \to R$. □

Example 3. The following example shows that the factor e may be omitted in (2) for special operators.

Let X be the Lebesgue space $L^p(0,1)$ for any p, $1 \leq p < \infty$, or the space of continuous functions $C([0,1])$. Let S be the multiplication operator $S : h(t) \to th(t)$, and let V be the classical Volterra operator $V : h(t) \to \int_0^t h(s)ds$. Fix a real number $c > 1$, and let

$$T = icI + S + V.$$

The spectrum of S is the interval $[0,1]$, and $[S,V] = V^2$ (where $[S,V]$ denotes the Lie product). By Corollary 5.23 in [2], the spectrum of $S+V$ coincides with the spectrum of S. The spectrum of T is then $ic + [0,1]$. Since $c > 1$, we have in particular $\Omega \subset \rho(T)$, as desired. By Corollary 5.21(c) in [2] with $f(\zeta) = (z - ic - \zeta)^{-n}$, we have

$$\begin{aligned} R(z;T)^n &= R(z-ic;S+V)^n = f(S+V) = f(S) + f'(S)V \\ &= R(z-ic;S)^n + nR(z-ic;S)^{n+1}V \qquad (z \notin [0,1]+ic). \end{aligned} \tag{11}$$

Clearly,

$$R(z-ic;S)^n : h(t) \to (z-ic-t)^{-n}h(t) \qquad (t \in [0,1]). \tag{12}$$

For $z \in \Omega$ and $t \in [0,1]$, we have $|z-ic-t| \geq |t+ic|-|z| > 1-|z|$ (since $c > 1$), and also $|z-ic-t| \geq c-|z| > c-1$. It then follows from (11) and (12) that for all $z \in \Omega$ and $n \in \mathbb{N}$,

$$||R(z;T)^n|| \leq (1-|z|)^{-n}\Big[1+\frac{n||V||}{c-1}\Big].$$

Thus T satisfies (1) with $M = 1+||V||/(c-1)$, and since $1+n||V||/(c-1) \leq Mn$, the inequality (2) is valid with the factor e omitted.

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